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Synthesis of an Optimal Control for Linear Stationary Discrete Dynamical Systems

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Abstract

In this paper, an algorithm designed by the author is used to construct the general solution to difference equations with constant coefficients. It is worth noting that the algorithm does not require any information on the multiple roots of the characteristic equation. This means one does not need to reconfigure the algorithm when changing the multiplicity groups. It is for this reason that the algorithm is called “universal”. In the present study, we solve the task of finding a linear optimal control for linear stationary discrete one- and higher-dimensional systems with scalar control. Moreover, we give analytical expressions for the control that minimize the quadratic criterion and ensure the asymptotic stability of the closed system. The obtained optimal control depends only on the parameters of the initial system and the roots of the characteristic equation.

Keywords

Difference Equations, Multiple Roots, Optimal Control

1. Statement and Solution of the Problem

As in [1]-[3], we assume that the processes in the open part of the system (excluding the controller) are described by the equation of order k

$$y[n+k]+b_1 y[n+k-1]+\cdots+b_k y[n]=\alpha U[n], \tag{1.1}$$

in which the coefficients α and $b_j, j=\overline{1,k}$ are constants. Let us express Equation (1.1) as a system of first-order difference equations:

$$x_j[n+1]=x_{j+1}[n],\ \ j=\overline{1,k-1}, \tag{1.2}$$

$$x_k[n+1]=-\sum_{i=1}^{k} b_{k-i+1}x_i[n]+\alpha U[n].$$

If $n=0$, the system state is determined by

$$x_i[0]=x_{i0},\ i=\overline{1,k},\ \ U[0]=U_0. \tag{1.3}$$

We need to find an analytical expression for the control

$$U[n]=\varphi(x_1[n],\cdots,x_k[n]), \tag{1.4}$$

which transfers the system from any given point in the region (1.3) to the origin

$$U[\infty]=0,\ \ x_i[\infty]=0,\ \ i=\overline{1,k} \tag{1.5}$$

while attaining the minimum value of the quadratic functional

$$I[U]=\frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{\nu=1}^{k}a_\nu x_\nu^2[n]+cU^2[n]\right), \tag{1.6}$$

where $a_\nu, \nu=\overline{1,k}$, and c are given numbers.

To solve the task, we introduce the auxiliary functional (see [4])

$$I^*[U]=\sum_{n=0}^{\infty}F^*[n], \tag{1.7}$$

$$\begin{aligned}F^*[n]=&\frac{1}{2}\sum_{\nu=1}^{k}a_\nu x_\nu^2[n]+\frac{1}{2}cU^2[n]+\sum_{j=1}^{k-1}\lambda_j[n](x_j[n+1]-x_{j+1}[n])\\&+\lambda_k[n]\left(x_k[n+1]+\sum_{i=1}^{k}b_{k-i+1}x_i[n]-\alpha U[n]\right).\end{aligned} \tag{1.8}$$

The necessary conditions of extremum for the functional (1.8) are (see [4])

$$\frac{\partial F^*[n]}{\partial x_i[n]}+\left.\frac{F^*[n]}{\partial x_i[n+1]}\right|_{n=n-1}=0,\ \ i=\overline{1,k},$$

$$\frac{\partial F^*[n]}{\partial u[n]}=0. \tag{1.9}$$

We rewrite the optimality conditions (1.2) and (1.9) as follows:

$$x_j[n+1]=x_{j+1}[n],\ \ j=\overline{1,k-1},$$

$$x_k[n+1]=-\sum_{i=1}^{k}b_{k-i+1}x_i[n]+\alpha U[n], \tag{1.10}$$

$$a_1x_1[n]+b_k\lambda_k[n]+\lambda_1[n-1]=0, \tag{1.11}$$

$$a_jx_j[n]-\lambda_{j-1}[n]+b_{k-j+1}[n]+\lambda_j[n-1]=0,\ \ j=\overline{2,k},$$

$$U[n]=\frac{\alpha}{c}\lambda_k[n]. \tag{1.12}$$

2. Synthesis of a Control That Transfers the System from an Arbitrary Point in Open Space to the Origin

We need to obtain the expressions for the variables of the system of Equations (1.10)-(1.12).

First, we write the system's characteristic equation. To do this, we exclude $U[n]$ from the system (1.10) using expression (1.12). As a result, we obtain a system of order k:

$$x_i[n+1] = x_{i+1}[n],\ i = \overline{1, k-1},$$

$$x_k[n+1] = -\sum_{i=1}^{k} b_{k-i+1} x_i[n] + \frac{\alpha^2}{c}\lambda_k[n], \tag{2.1}$$

$$a_i x_1[n] + b_k \lambda_k[n] + \lambda_1[n-1] = 0,$$

$$a_j x_j[n] - \lambda_{j-1}[n] + b_{k-j+1}\lambda_k[n] + \lambda_j[n-1] = 0,\ \ j = \overline{2, k}. \tag{2.2}$$

Let us write the characteristic equations of system (2.1), (2.2):

$$\Delta(z) = \begin{vmatrix} z & -1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & z^{-1} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 0 & 0 & \cdots & 0 \\ b_k & b_{k-1} & \cdots & b_1 + z & 0 & 0 & \cdots & \alpha^2/c \\ a_1 & 0 & \cdots & 0 & z^{-1} & 0 & \cdots & b_k \\ 0 & a_2 & \cdots & 0 & -1 & z^{-1} & \cdots & b_{k-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_k & 0 & 0 & \cdots & b_1 + z^{-1} \end{vmatrix} = 0. \tag{2.3}$$

Assume that none of the roots of this equation in the root plane lies on the unit circle with the center at the origin. Furthermore, Equation (2.3) is self-adjoint [4], which means that its roots have the property $z_{k+i} = z_i^{-1}, i = \overline{1, k}$. We assume that $r_\alpha, \alpha = \overline{1, k}$ are roots with a modulus less than 1 and, correspondingly, $r_{k+\alpha}$ are those with a modulus exceeding 1.

To write the solution of systems (2.1), (2.2), we will use the source (see [5])

$$x_i[n] = \sum_{m=1}^{k}\left[C_m S_{1,m}^{n-m+i} + C_{k+m} S_{1,k+m}^{n-k-m+i}\right],\ i = \overline{1, k}, \tag{2.4}$$

where C_m and C_{k+m} are arbitrary constants, $S_{1,m}^{n-m+i}$ and $S_{1,k+m}^{n-k-m+i}$ are combinations with repetition [6] on the roots $r_1, \cdots, r_m$ and $r_1, \cdots, r_{k+m}$, consisting of $n-m+i$ and $n-k-m+i$ roots, correspondingly. For instance,

$$S_{1,3}^2 = r_1^2 + r_1 r_2 + r_1 r_3 + r_2^2 + r_2 r_3 + r_3^2.$$

From system (2.1) and solutions (2.4), it follows that

$$\begin{aligned} \lambda_k[n] = \frac{c}{\alpha^2}\sum_{m=1}^{k}\Bigg\{ & C_m\left[S_{1,m}^{n-m+k+1} + \sum_{i=1}^{k} b_{k-i+1} S_{1,m}^{n-m+i}\right] \\ & + C_{k+m}\left[S_{1,k+m}^{n-m+1} + \sum_{i=1}^{k} b_{k-i+1} S_{1,k+m}^{n-k-m+i}\right]\Bigg\}. \end{aligned} \tag{2.5}$$

According to the boundary conditions, the variables $x_i[n], i = \overline{1, k}$, and $U[n]$ must tend to zero as $n \to \infty$. This is only possible if $C_{k+m} = 0, m = \overline{1, k}$. Therefore, according to solutions (2.4) and (2.5), we have

$$x_i[n] = \sum_{m=1}^{k} C_m S_{1,m}^{n-m+i},\ i = \overline{1, k}, \tag{2.6}$$

$$\lambda_k[n]=\frac{c}{\alpha^2}\sum_{m=1}^{k}C_m\left(S_{1,m}^{n-m+k+1}+\sum_{i=1}^{k}b_{k-i+1}S_{1,m}^{n-m+i}\right). \tag{2.7}$$

Note that solutions (2.6) and (2.7) do not require information about the multiplicity of the roots. From the resulting equations, we exclude C_m and obtain

$$\lambda_k[n]=\frac{c}{\alpha^2\Delta^k}\sum_{m=1}^{k}\Delta_m^k\left[S_{1,m}^{n-m+k+1}+\sum_{i=1}^{k}b_{k-i+1}S_{1,m}^{n-m+i}\right], \tag{2.8}$$

where the determinant

$$\Delta^k=\begin{vmatrix} S_{1,1}^{n} & S_{1,2}^{n-1} & \cdots & S_{1,k}^{n-k+1}\\ S_{1,1}^{n+1} & S_{1,2}^{n} & \cdots & S_{1,k}^{n-k+2}\\ \vdots & \vdots & \ddots & \vdots\\ S_{1,1}^{n+k-1} & S_{1,2}^{n+k-2} & \cdots & S_{1,k}^{n} \end{vmatrix}, \tag{2.9}$$

and Δ_m^k is the determinant obtained from Δ^k by replacing the *m*th column with the vector $x_i[n]$. Let us expand the determinant Δ_m^k along the column $x_i[n]$. The result is

$$\lambda_k[n]=\frac{c}{\alpha^2\Delta^k}\left[\sum_{m=1}^{k}\sum_{i=1}^{k}x_i[n]\Delta_{i,m}^k S_{1,m}^{n-m+k+1}+\sum_{m=1}^{k}\sum_{i=1}^{k}x_i[n]\Delta_{i,m}^k\sum_{\beta=1}^{k}b_{k-\beta+1}S_{1,m}^{n-m+\beta}\right], \tag{2.10}$$

where $\Delta_{i,m}^k$ is the algebraic complement (cofactor) of the entry in the *i*th row and *m*th column of determinant Δ^k. In Equation (2.10), we have

$$\sum_{m=1}^{k}\sum_{i=1}^{k}x_i[n]\Delta_{i,m}^k\sum_{\beta=1}^{k}b_{k-\beta+1}S_{1,m}^{n-m+\beta}=\sum_{i=1}^{k}x_i[n]\sum_{\beta=1}^{k}b_{k-\beta+1}\sum_{m=1}^{k}\Delta_{i,m}^k S_{1,m}^{k-m+\beta}. \tag{2.11}$$

Moreover, in Equation (2.11), we have

$$\sum_{m=1}^{k}\Delta_{i,m}^k S_{1,m}^{n-m+\beta}=\begin{cases}0 & \text{if } \beta\neq i,\\ \Delta^k & \text{if } \beta=i\end{cases} \tag{2.12}$$

since this expression is the sum of the entries in the *i*th row of determinant Δ^k (see Equation (2.9)) multiplied by the cofactors of the *β*th row of the same determinant. Accordingly, we can write Equation (2.10) as

$$\lambda_k[n]=\frac{c}{\alpha^2}\sum_{i=1}^{k}x_i[n]\left[b_{k-i+1}+\frac{1}{\Delta^k}\sum_{m=1}^{k}\Delta_{i,m}^k S_{1,m}^{n-m+k+1}\right]. \tag{2.13}$$

Further transformations of expression (2.13) are performed in the **Appendix**.

According to Equations (A.11), (2.12), and (1.12), the desired control becomes

$$U[n]=\frac{1}{\alpha}\sum_{i=1}^{k}x_i[n]\left(b_{k-i+1}+(-1)^{k-i}\theta_{1,k}^{k-i+1}\right). \tag{2.14}$$

Let us introduce Vieta's numbers, defined as indicated below:

$$V_j=(-1)^j\theta_{1,k}^j,\quad j=\overline{1,k}. \tag{2.15}$$

Now, we can rewrite expression (2.14) as

$$U[n]=\frac{1}{\alpha}\sum_{i=1}^{k}x_i[n]\left(b_{k-i+1}-V_{k-i+1}\right). \tag{2.16}$$

Under control Equation (2.16), process Equation (1.2) acquires either the form

$$x_j[n+1] = x_{j+1}[n], \quad j = \overline{1,k-1},$$

$$x_k[n+1] = -\sum_{i=1}^{k} V_{k-i+1} x_i[n] \tag{2.17}$$

or the form

$$x_k[n+k] + \sum_{i=1}^{k} V_i x_k[n+k-i] = 0. \tag{2.18}$$

According to Equation (2.6), we obtain

$$x_i[n] = \sum_{m=1}^{k} C_m S_{1,m}^{n-m+i}, \quad i = \overline{1,k}, \tag{2.19}$$

and $x_i[n] \to 0$ as $n \to 0$ since the combinations $S_{1,m}^{n-m+i}$ correspond to roots with a modulus less than 1. The constants C_m are computed from conditions (1.3).

Thus, when solving a specific problem, find the roots $r_j, j = \overline{1,k}$, of Equation (2.3). Among these roots, select those whose modulus is less than 1. Then, construct a control Equation (2.16). Compute the constants $r_i, i = \overline{1,k}$. Obtain the transient process Equation (2.19).

Example 1. Let us suppose that $k = 2$. In this case, system (1.2) has the form

$$x_1[n+1] = x_2[n],$$

$$x_2[n+1] = -\sum_{i=1}^{2} b_{2-i+1} x_i[n] + \alpha U[n].$$

Characteristic Equation (2.3) is

$$(b_1 + z)(b_1 + z^{-1}) + b_2\left(z(b_1 + z) + z^{-1}(b_1 + z^{-1})\right) - \frac{\alpha^2}{c}(a_1 + a_2) + b_2^2 = 0. \tag{2.20}$$

The equation does not change if we substitute the variable z with z^{-1}. This indicates that it is self-adjoint, *i.e.*, the roots satisfy the condition $z_{2+i} = z_i^{-1}, i = \overline{1,2}$. Assume that the roots $r_\alpha, \alpha = 1,2$, do not lie on the unit circle with the center at the origin and their moduli are less than 1. The desired control Equation (2.16) is written in this case as

$$U[n] = \frac{1}{\alpha}\sum_{i=1}^{2} x_i[n] b_{3-i} - \frac{1}{\alpha}\sum_{i=1}^{2} V_{3-i} x_i[n], \tag{2.21}$$

with

$$V_1 = -(r_1 + r_2) \text{ and } V_2 = r_1 r_2.$$

The controlled process (2.18) is

$$x_3[n+2] + V_1 x_3[n+1] + V_2 x_3[n] = 0. \tag{2.22}$$

From Equation (2.6), we obtain

$$x_1[n] = C_1 S_{1,1}^{n} + C_2 S_{1,2}^{n-1}, \quad x_2[n] = C_1 S_{1,1}^{n+1} + C_2 S_{1,2}^{n}, \tag{2.23}$$

where

$$S_{1,1}^{n} = r_1^{n}, \quad S_{1,1}^{n+1} = r_1^{n+1},$$

$$S_{1,2}^{n-1} = r_1^{n-1} + r_{1,1}^{n-2} r_2 + \cdots + r_2^{n-1}, \quad S_{1,2}^{n} = r_1^{n} + r_1^{n-1} r_2 + \cdots + r_2^{n}.$$

According to Equation (2.23), we have the initial conditions

$$x_1[0] = C_1, \; x_2[0] = C_1 r_1 + C_2. \tag{2.24}$$

3. Systems of Difference Equations

Let us consider a system described by a system of difference equations of order k with constant coefficients

$$y[n+1] = Ay[n] + BU[n], \tag{3.1}$$

where

$$y[n] = [y_1[n], \cdots, y_k[n]]^{\mathrm{T}}, \; B^{\mathrm{T}} = [0_{k-1}^{\mathrm{T}}, \alpha], \; A = [\alpha_{i,j}], \; i, j = \overline{1,k}, \tag{3.2}$$

the control $U[n]$ is a scalar, 0_{k-1} is the $(k-1)$-dimensional zero vector, and α is a constant parameter. We need to solve problem Equations (1.4)-(1.6).

Let us write system (3.1) in the form Equation (1.2) (see 5):

$$x_1[n+k] + \varnothing_1 x_1[n+k-1] + \cdots + \varnothing_k x_1[n] = A_g \tilde{A}^{k-2} \left[\frac{0_{k-2}}{1} \right] \alpha U[n], \tag{3.3}$$

$$\begin{aligned}
\varnothing_1 &= A_g \tilde{A}^{k-1} M^{-1} N_{.k}^* - \alpha_{1,1}, \\
\varnothing_2 &= A_g \left(\tilde{A}^{k-1} M^{-1} N_{.(k-1)}^* - A_v \right), \\
\varnothing_3 &= A_g \left(\tilde{A}^{k-1} M^{-1} N_{.(k-2)}^* - \tilde{A} A_v \right), \\
&\vdots \\
\varnothing_k &= A_g \left(\tilde{A}^{k-1} M^{-1} N_{.1}^* - \tilde{A}^{k-2} A_v \right).
\end{aligned} \tag{3.4}$$

According to Equation (3.2), we have in Equation (3.4)

$$A_g = [\alpha_{1,2}, \cdots, \alpha_{1,k}], \; A_v = [\alpha_{2,1}, \cdots, \alpha_{k,1}], \; \tilde{A} = [\alpha_{ij}], \; i, j = \overline{2,k}. \tag{3.5}$$

The $(k-1) \times (k-1)$ matrix

$$M^{\mathrm{T}} = \left[A_g^{\mathrm{T}}, (A_g \tilde{A})^{\mathrm{T}}, \cdots, (A_g \tilde{A}^{k-2})^{\mathrm{T}} \right]. \tag{3.6}$$

We assume that $|M| \neq 0$.

The $(k-1) \times k$ matrix

$$N^* = \begin{bmatrix}
\alpha_{1,1} & -1 & \cdots & 0 & 0 \\
A_g A_v & \alpha_{1,1} & \cdots & 0 & 0 \\
A_g \tilde{A} A_v & A_g A_v & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
A_g \tilde{A}^{k-4} A_v & A_g \tilde{A}^{k-3} A_v & \cdots & -1 & 0 \\
A_g \tilde{A}^{k-3} A_v & A_g \tilde{A}^{k-4} A_v & \cdots & \alpha_{1,1} & -1
\end{bmatrix}$$

and $N_{.r}^*, r = \overline{1,k}$ are the columns of this matrix.

In Equation (3.3), we have

$$A_g \tilde{A}^{k-2} \left[\frac{0_{k-2}}{1} \right] = \begin{cases} a_{1,2} & \text{if } k = 2, \\ A_g \tilde{A}^{k-3} = A_g \tilde{A}^{k-3} \tilde{a} & \text{if } k \geq 3, \end{cases} \tag{3.7}$$

where

$$\tilde{a} = \left[\alpha_{2,k}, \cdots, \alpha_{k,k}\right]^{\mathrm{T}}. \tag{3.8}$$

Let us rewrite Equation (3.3) in the form

$$x_1[n+k] + \varnothing_1 x_1[n+k-1] + \cdots + \varnothing_k x_1[n] = \alpha \tilde{U}[n],$$

with scalar equation

$$\tilde{U}[n] = A_g \tilde{A}^{k-2} \begin{bmatrix} 0_{k-2} \\ \hline 1 \end{bmatrix} U[n]. \tag{3.9}$$

We have obtained the problem Equations (1.4)-(1.6). Let us write down the determinant Equation (2.3) after replacing the coefficients $b_i, i = \overline{1,k}$, by $\varnothing_i, i = \overline{1,k}$, as expressed in Equation (3.4). Among the $2k$ roots of Equation (2.3), we select k roots whose modulus is less than 1. Using these roots, we construct the numbers

$$V_j = (-1)^j \theta_{1,k}^j, \quad j = \overline{1,k},$$

where $\theta_{1,k}^j$ are j-combinations of the roots $\tau_1, \cdots, \tau_k$. Thus, control Equation (3.9) becomes

$$\tilde{U}[n] = \frac{1}{\alpha} \sum_{i=1}^{k} x_i[n] \left(\varnothing_{k-i+1} - V_{k-i+1}\right). \tag{3.10}$$

According to Equation (3.9), the control $U[n]$ for problem Equation (3.3) is

$$U[n] = \left(A_g \tilde{A}^{k-3} \tilde{a}\right)^{-1} \alpha^{-1} \sum_{i=1}^{k} x_i[n] \left(\varnothing_{k-i+1} - V_{k-i+1}\right). \tag{3.11}$$

Example 2. Let us consider system (3.1) when $k = 2$:

$$y_1[n+1] = \alpha_{1,1} y_1[n] + \alpha_{1,2} y_2[n], \tag{3.12}$$

$$y_2[n+1] = \alpha_{2,1} y_1[n] + \alpha_{2,2} y_2[n] + \alpha U[n].$$

The remarks we made in Example 1 are also valid in this case.

For system (3.12), we have

$$A_g = [\alpha_{1,2}], \ A_v = [\alpha_{2,1}], \ \tilde{A} = [\alpha_{2,2}], \ M = [\alpha_{1,2}], \ M^{-1} = \alpha_{1,2}^{-1}, \ \alpha_{12} \neq 0.$$

Let us write Equation (3.12) in the form

$$y_1[n+2] + \varnothing_1 y_1[n+1] + \varnothing_2 y_1[n] = A_g \alpha U[n], \tag{3.13}$$

where, according to Equation (3.4), we have

$$\varnothing_1 = -(\alpha_{2,2} + \alpha_{1,1}), \ \varnothing_2 = \alpha_{1,1}\alpha_{2,2} - \alpha_{1,2}\alpha_{2,1}.$$

The characteristic equation of the process is

$$(\varnothing_1 + z)(\varnothing_1 + z^{-1}) + \varnothing_2 \left(z(\varnothing_1 + z) + z^{-1}(\varnothing_1 + z^{-1})\right) - \frac{\alpha^2}{C}(a_1 + a_2) + \phi_2^2 = 0. \tag{3.14}$$

The moduli of r_1 and r_2 are less than 1. With these roots, we construct the numbers

$$V_1 = -(r_1 + r_2) \text{ and } V_2 = r_1 r_2. \tag{3.15}$$

Further, according to Equation (3.11), we obtain the desired control:

$$U[n] = \frac{1}{\alpha}\left(y_1[n](\varnothing_2 - V_2) + y_2[n](\varnothing_1 - V_1)\right). \tag{3.16}$$

It follows from expressions (3.13) and (3.16) that the controlled process has the form

$$y_1[n+2] + y_1[n+1]V_1 + y_1[n]V_2 = 0.$$

Example 3. Stabilization of a rocket's rotation angle relative to its longitudinal axis.

The equation of a rocket's rotation relative to its longitudinal axis has the form

$$J\frac{\mathrm{d}^2\varphi(t)}{\mathrm{d}t^2} = k_1\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} + k_2\psi(\varphi(t)) + \alpha U(t), \tag{3.17}$$

where J and $\varphi(t)$ are, respectively, the moment of inertia and the absolute angle of rotation of the rocket relative to its longitudinal axis, $\alpha U(t)$ is the control torque, and $k_2\psi(\varphi(t))$ is a known moment of resistance depending only on the angle $\varphi(t)$. Let us write Equation (3.12) in difference form:

$$\frac{J}{T}\left(\varphi_2(t+T) - \varphi_2(t)\right) = k_1\left(\varphi_1(t+T) - \varphi_1(t)\right)/T + k_2\psi(\varphi_1(t)) + \alpha U(t).$$

For the discrete time $t = nT$, we can write the same equation in matrix form as

$$y[n+1] = Ay[n] + B\tilde{U}[n], \tag{3.18}$$

where, using the notations $y_1(t) = \varphi_1(t)$ and $y_2(t) = \varphi_2(t)$, we have

$$y[n] = [y_1[n], y_2[n]],\ A = [\alpha_{i,j}],\ i, j = 1,2,\ \alpha_{1,1} = 1,\ \alpha_{1,2} = T,$$
$$\alpha_{2,1} = 0,\ \alpha_{2,2} = 1 + k_1T/J,$$
$$B = [0\ \ \alpha]^{\mathrm{T}},\ \tilde{U}[n] = T\left(U[n] + k\psi(y_1[n]/\alpha)\right).$$

We have obtained the problem from Example 2.

4. Conclusion

We obtained analytical expressions for optimal controls depending only on the parameters of the original system and the roots of the characteristic equations of the accompanying variational problems. The roots should not lie on the unit circle with the center at the origin. If at least one root lies on that circle, then the control problem does not have a solution. Self-oscillations arise in the system.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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List of Mathematical Notations

n : a discrete quantity, $n = 0,1,\cdots$;

α, c, a_i, b_i : constant quantities;

$x[n], y[n], \lambda[n], U[n], F^*[n], I^*[n], \Delta(z)$: functions;

$C_m, m = \overline{1,2k}$: constants;

$\theta_{1,m}^{\gamma}, S_{1,m}^{\gamma}$: γ-combinations of the roots $r_1,\cdots,r_m$, $\gamma = 1,2,\cdots$;

Δ^k : a determinant of order k;

A, B : matrices.

Appendix

Transformation of expression (2.13). The algebraic complement (cofactor) $\Delta_{i,m}^k$ of the determinant Equation (2.9) can be written as

$$(-1)^{i+m}\Delta_{i,m}^k = \begin{vmatrix} S_{1,1}^n & \cdots & S_{1,m-1}^{n-m+2} & S_{1,m+1}^{n-m} & \cdots & S_{1,k}^{n-k+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ S_{1,1}^{n+i-2} & \cdots & S_{1,m-1}^{n+i-m} & S_{1,m+1}^{n+i-m-2} & \cdots & S_{1,k}^{n+i-k-1} \\ S_{1,1}^{n+i} & \cdots & S_{1,m-1}^{n+i-m+2} & S_{1,m+1}^{n+i-m} & \cdots & S_{1,k}^{n+i-k+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ S_{1,1}^{n+k-1} & \cdots & S_{1,m-1}^{n+k-m+1} & S_{1,m+1}^{n+k-m-1} & \cdots & S_{1,k}^{n} \end{vmatrix}. \quad (A.1)$$

Let us transform the determinant in Equation (A.1). Multiply row $(k-2)$ by r_1 and subtract the result from row $(k-1)$, and so on; multiply row i by r_1 and subtract the result from row $(i+1)$; multiply row $(i-2)$ by r_1 and subtract the result from row $(i-1)$; and so on; multiply the first row by r_1 and subtract the result from the second row. Apply the following formula to all obtained differences [2]:

$$S_{1,j}^q = \begin{cases} r_1 S_{1,j}^{q-1} + S_{2,j}^q, & j > 1, \\ r_1 S_{1,j}^{q-1}. \end{cases} \quad (A.2)$$

Multiply row $(i-1)$ by r_1^2 and subtract the result from row i. Then apply the following formula to the obtained expressions [2]:

$$S_{1,j}^q = r_1^2 S_{1,j}^{q-2} + r_1 S_{2,j}^{q-1} + S_{2,j}^q, \quad j > 1. \quad (A.3)$$

From Equation (A.1), we obtain

$$(-1)^{i+m}\Delta_{i,m}^k = \begin{vmatrix} S_{1,1}^n & S_{1,2}^{n-1} & \cdots & S_{2,m-1}^{n-m+2} & S_{2,m+1}^{n-m} & \cdots & S_{2,k}^{n-k+1} \\ 0 & S_{2,2}^n & \cdots & S_{2,m-1}^{n-m+3} & S_{2,m+1}^{n-m+1} & \cdots & S_{2,k}^{n-k+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & S_{2,2}^{n+i-3} & \cdots & S_{2,m-1}^{n+i-m} & S_{2,m+1}^{n+i-m-2} & \cdots & S_{2,k}^{n+i-k-1} \\ 0 & r_1 S_{2,2}^{n+i-2} + S_{2,2}^{n+i-1} & \cdots & r_1 S_{2,m-1}^{n+i-m+1} + S_{2,m-1}^{n+i-m+2} & r_1 S_{2,m+1}^{n+i-m-1} + S_{2,m+1}^{n+i-m} & \cdots & r_1 S_{2,k}^{n+i-k} + S_{2,k}^{n+i-k+1} \\ 0 & S_{2,2}^{n+i} & \cdots & S_{2,m-1}^{n+i-m+3} & S_{2,m+1}^{n+i-m+1} & \cdots & S_{2,k}^{n+i-k+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & S_{2,2}^{n+k-2} & \cdots & S_{2,m-1}^{n+k-m+1} & S_{2,m+1}^{n+k-m+1} & \cdots & S_{2,k}^{n} \end{vmatrix} \begin{matrix} (1) \\ (2) \\ \vdots \\ (i-2) \\ (i-1) \\ (i) \\ \vdots \\ (k-2) \end{matrix} \quad (A.4)$$

$$\Delta_{i,m}^{k}=(-1)^{i+m}S_{1,1}^{n}\times\begin{vmatrix} S_{2,2}^{n} & \cdots & S_{2,m-1}^{n-m+3} & S_{2,m+1}^{n-m+1} & \cdots & S_{2,k}^{n-k+2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ S_{2,2}^{n+i-3} & \cdots & S_{2,m-1}^{n+i-m} & S_{2,m+1}^{n+i-m-2} & \cdots & S_{2,k}^{n+i-k-1} \\ r_1S_{2,2}^{n+i-2}+S_{2,2}^{n+i-1} & \cdots & r_1S_{2,m-1}^{n+i-m+1}+S_{2,m-1}^{n+i-m+2} & r_1S_{2,m+1}^{n+i-m+1}+S_{2,m+1}^{n+i-m} & \cdots & r_1S_{2,k}^{n+i-k}+S_{2,k}^{n+i-k+1} \\ S_{2,2}^{n+i} & \cdots & S_{2,m-1}^{n+i-m+3} & S_{2,m+1}^{n+i-m+1} & \cdots & S_{2,k}^{n+i-k+2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ S_{2,2}^{n+k-2} & \cdots & S_{2,m-1}^{n+k-m+1} & S_{2,m+1}^{n+k-m+1} & \cdots & S_{2,k}^{n} \end{vmatrix}\begin{matrix}(1)\\ \vdots \\ (i-2)\\ (i-1)\\ (i)\\ \vdots \\ (k-2)\end{matrix} \quad (A.5)$$

Expand the determinant in Equation (A.5) along row $(i-1)$:

$$\Delta_{i,m}^{k}=S_{1,1}^{n}\left(-r_1\Delta_{i,m-1}^{k-1}+\Delta_{i-1,m-1}^{k-1}\right),\ i=\overline{1,k},\ m=\overline{1,k}, \quad (A.6)$$

where $\Delta_{i,m-1}^{k-1}$ and $\Delta_{i-1,m-1}^{k-1}$ are the algebraic complements of the following determinant of order $k-1$:

$$\Delta^{k-1}=\begin{vmatrix} S_{2,2}^{n} & \cdots & S_{2,m-1}^{n-m+3} & S_{2,m}^{n-m+2} & S_{2,m+1}^{n-m+1} & \cdots & S_{2,k}^{n-k+2} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ S_{2,2}^{n+i-3} & \cdots & S_{2,m-1}^{n+i-m} & S_{2,m}^{n+i-m-1} & S_{2,m+1}^{n+i-m-2} & \cdots & S_{2,k}^{n+i-k-1} \\ S_{2,2}^{n+i-2} & \cdots & S_{2,m}^{n+i-m+1} & S_{2,m}^{n+i-m} & S_{2,m+1}^{n+i-m-1} & \cdots & S_{2,k}^{n+i-k} \\ S_{2,2}^{n+i-1} & \cdots & S_{2,m-1}^{n+i-m+2} & S_{2,m}^{n+i-m+1} & S_{2,m+1}^{n+i-m} & \cdots & S_{2,k}^{n+i-k+1} \\ S_{2,2}^{n+i} & \cdots & S_{2,m-1}^{n+i-m+3} & S_{2,m}^{n+i-m+2} & S_{2,m+1}^{n+i-m+1} & \cdots & S_{2,k}^{n+i-k+2} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ S_{2,2}^{n+k-2} & \cdots & S_{2,m-1}^{n+k-m+1} & S_{2,m}^{n+k-m} & S_{2,m+1}^{n+k-m-1} & \cdots & S_{2,k}^{n} \end{vmatrix}\begin{matrix}(1)\\ \vdots \\ (i-2)\\ (i-1)\\ (i)\\ (i+1)\\ \vdots \\ (k-1)\end{matrix} \quad (A.7)$$

The algebraic complements of this determinant are such that

$$(-1)^{i+m}\Delta_{i,m-1}^{k-1}=\begin{vmatrix} S_{2,2}^{n} & \cdots & S_{2,m-1}^{n-m+3} & S_{2,m+1}^{n-m+1} & \cdots & S_{2,k}^{n-k+2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ S_{2,2}^{n+i-3} & \cdots & S_{2,m-1}^{n+i-m} & S_{2,m+1}^{n+i-m-2} & \cdots & S_{2,k}^{n+i-k-1} \\ S_{2,2}^{n+i-2} & \cdots & S_{2,m-1}^{n+i-m+1} & S_{2,m+1}^{n+i-m-1} & \cdots & S_{2,k}^{n+i-k} \\ S_{2,2}^{n+i} & \cdots & S_{2,m-1}^{n+i-m+3} & S_{2,m+1}^{n+i-m+1} & \cdots & S_{2,k}^{n+i-k+2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ S_{2,2}^{n+k-2} & \cdots & S_{2,m-1}^{n+k-m+1} & S_{2,m+1}^{n+k-m-1} & \cdots & S_{2,k}^{n} \end{vmatrix}\begin{matrix}(1)\\ \vdots \\ (i-2)\\ (i-1)\\ (i)\\ \vdots \\ (k-2)\end{matrix} \quad (A.8)$$

$$(-1)^{i+m}\Delta_{i-1,m-1}^{k-1}=\begin{vmatrix} S_{2,2}^{n} & \cdots & S_{2,m-1}^{n-m+3} & S_{2,m+1}^{n-m+1} & \cdots & S_{2,k}^{n-k+2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ S_{2,2}^{n+i-3} & \cdots & S_{2,m-1}^{n+i-m} & S_{2,m+1}^{n+i-m-2} & \cdots & S_{2,k}^{n+i-k-1} \\ S_{2,2}^{n+i-1} & \cdots & S_{2,m-1}^{n+i-m+2} & S_{2,m+1}^{n+i-m} & \cdots & S_{2,k}^{n+i-k+1} \\ S_{2,2}^{n+i} & \cdots & S_{2,m-1}^{n+i-m+3} & S_{2,m+1}^{n+i-m+1} & \cdots & S_{2,k}^{n+i-k+2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ S_{2,2}^{n+k-2} & \cdots & S_{2,m-1}^{n+k-m+1} & S_{2,m+1}^{n+k-m-1} & \cdots & S_{2,k}^{n} \end{vmatrix}\begin{matrix}(1)\\ \vdots \\ (i-2)\\ (i-1)\\ (i)\\ \vdots \\ (k-2)\end{matrix} \quad (A.9)$$

Notice that if $m=\overline{2,k}$, then we have

$$\Delta_{i,m}^{k}=\begin{cases} -S_{1,1}^{n}r_1\Delta_{i,m-1}^{k-1} & \text{if } i=1, \\ -S_{1,1}^{n}r_1\Delta_{i,m-1}^{k-1}+\Delta_{i-1,m-1}^{k-1} & \text{if } i=\overline{2,k-1}, \\ S_{1,1}^{n}\Delta_{i-1,m-1}^{k-1} & \text{if } i=k. \end{cases} \quad (A.10)$$

We use the Formula (A.10) in the method of complete induction. Next, we will show that the equality

$$\frac{1}{\Delta^k}\sum_{m=1}^{k}\Delta_{i,m}^{k}S_{1,m}^{n-m+k+1}=(-1)^{k-i}\,\theta_{1,k}^{k-i+1},\ \ i=\overline{1,k} \tag{A.11}$$

holds in Equation (2.13).

Case $i=1$. We should prove that

$$\frac{1}{\Delta^k}\sum_{m=1}^{k}\Delta_{1,m}^{k}S_{1,m}^{n-m+k+1}=(-1)^{k-1}\,\theta_{1,k}^{k}. \tag{A.12}$$

Let us prove that Formula (A.12) holds when $k=1$. On the left-hand side of Formula (A.12), we obtain

$$\frac{1}{\Delta^1}\sum_{m=1}^{1}\Delta_{1,m}^{1}S_{1,m}^{n-m+1+1}=\frac{1}{\Delta^1}\Delta_{1,1}^{1}S_{1,1}^{n+1}=\frac{1}{r_1^n}r_1^{n+1}=r_1.$$

The right-hand side in Formula (A.12) also equals r_1 when $i=1$. Consequently, Formula (A.12) holds when $k=1$.

For $i=1$, according to Equation (A.10), we have

$$\Delta_{1,m}^{k}=-r_1^{n+1}\Delta_{1,m-1}^{k-1},\ \ m=\overline{2,k}. \tag{A.13}$$

Plug the expression (A.13) into Equation (A.12):

$$\begin{aligned}L_1&\equiv\frac{1}{\Delta^k}\sum_{m=2}^{k}(-1)r_1^{n+1}\Delta_{1,m-1}^{k-1}S_{1,m}^{n-m+k+1}\\&=-\frac{1}{\Delta^k}r_1^{n+1}\sum_{m=2}^{k}\Delta_{1,m-1}^{k-1}S_{1,m}^{n-m+k+1}.\end{aligned} \tag{A.14}$$

Assume that Equation (A.11) has been proven for $k=k-1$ and $i=1$, that is,

$$\frac{1}{\Delta^{k-1}}\sum_{m=1}^{k-1}\Delta_{1,m}^{k-1}S_{1,m}^{n-m+k}=(-1)^{k-2}\,\theta_{1,k-1}^{k-1}.$$

According to Equation (A.14), in this case we obtain

$$L_1=-\frac{1}{\Delta^k}r_1^{n+1}(-1)^{k-1-1}\,\theta_{2,k}^{k-1-i+1}\Delta^{k-1}=(-1)^{k-1}\,r_1\theta_{2,k}^{k-1}, \tag{A.15}$$

or

$$L_1=(-1)^{k-1}\,\theta_{1,k}^{k}.$$

Thus, Equation (A.12) has been proven for $i=1$.

Let us consider Equation (A.11) for $i=\overline{2,k-1}$. We shall prove that Equation (A.11) holds when $k=1$. We should obtain the equality

$$\frac{1}{\Delta^1}\sum_{m=1}^{1}\Delta_{i,m}^{1}S_{1,m}^{n-m+1+1}=(-1)^{1-i}\,\theta_{1,1}^{1-i+1},$$

which reduces to $r_1=r_1$. This means that Equation (A.11) holds for $k=1$. Let us prove Equation (A.11) for $k=2$ and $i=2$. On the left-hand side, we have

$$\begin{aligned}\frac{1}{\Delta^2}\sum_{m=1}^{2}\Delta_{2,m}^{2}S_{1,m}^{n-m+2+1}&=\frac{1}{\Delta^2}\left(\Delta_{2,1}^{2}S_{1,1}^{n+2}+\Delta_{2,2}^{2}S_{1,2}^{n+1}\right)\\&=\frac{1}{\Delta^2}\left(-S_{1,2}^{n-1}r_1^{n+2}+r_1^nS_{1,2}^{n+1}\right)\\&=\frac{1}{\Delta^2}r_1^2r_2^2\left(r_1+r_2\right).\end{aligned}$$

The right-hand side is $\theta_{1,2}^{1}=r_1+r_2$. Therefore, Equation (A.11) holds when $k=2$ and $i=2$.

In the considered case, *i.e.*, when $i=\overline{2,k-1}$, it follows from Equation (A.10) that

$$\Delta_{i,m}^{k}=r_1^n\left(\Delta_{i-1,m-1}^{k-1}-r_1\Delta_{i,m-1}^{k-1}\right),\ i=\overline{2,k-1},\ m=\overline{2,k}, \tag{A.16}$$

whenever $k\ge 2$. Consequently, on the left-hand side of Equation (A.11), we obtain

$$L_2=\frac{r_1^n}{\Delta^k}\sum_{m=2}^{k}\left(\Delta_{i-1,m-1}^{k-1}-r_1\Delta_{i,m-1}^{k-1}\right)S_{1,m}^{n-m+k+1}. \tag{A.17}$$

Let us now suppose that Equation (A.11) has been proven for $k=k-1$. If this is the case, we can write Equation (A.17) as

$$\begin{aligned}L_2&=\frac{r_1^n}{\Delta^k}\left((-1)^{k-1-(i-1)}\theta_{2,k}^{k-1-(i-1)+1}-r_1(-1)^{k-1-i}\theta_{2,k}^{k-1-i+1}\right)\Delta^{k-1}\\&=(-1)^{k-i}\left(\theta_{2,k}^{k-i+1}+r_1\theta_{2,k}^{k-i}\right),\end{aligned}$$

that is,

$$L_2=(-1)^{k-i}\theta_{1,k}^{k-i+1}. \tag{A.18}$$

Thus, Formula (A.11) has been proven for $i=\overline{2,k-1}$.

Let us now consider Equation (A.10) for $i=k$. It can be inferred from Equation (A.11) that we need to prove the formula

$$\frac{1}{\Delta^k}\sum_{m=1}^{k}\Delta_{k,m}^{k}S_{1,m}^{n-m}=\theta_{1,k}^{1}. \tag{A.19}$$

Let us transform the sum $\sum_{m=1}^{k}$ in Equation (A.19). We denote this sum by L_4 and rewrite it as a kth order determinant:

$$L_4=\begin{vmatrix}S_{1,1}^{n} & S_{1,2}^{n-1} & \cdots & S_{1,k}^{n-k+1}\\ \vdots & \vdots & \ddots & \vdots\\ S_{1,1}^{n+k-2} & S_{1,2}^{n+k-3} & \cdots & S_{1,k}^{n-1}\\ S_{1,1}^{n+k} & S_{1,2}^{n+k-1} & \cdots & S_{1,k}^{n+1}\end{vmatrix}\begin{matrix}(1)\\ \vdots\\ (k-1)\\ (k)\end{matrix} \tag{A.20}$$

Now multiply row $(k-1)$ by r_1^2 and subtract the result from row k. Next, multiply row $(k-2)$ by r_1 and subtract the result from row $(k-1)$, and so on. Finally, multiply the first row by r_1 and subtract the result from the second row. After all these operations, we obtain

$$L_4=\begin{vmatrix}r_1^n & S_{1,2}^{n-1} & \cdots & S_{1,k}^{n-k+1}\\ 0 & S_{1,2}^{n}-r_1S_{1,2}^{n-1} & \cdots & S_{1,k}^{n-k+2}-r_1S_{1,k}^{n-k+1}\\ 0 & \vdots & \ddots & \vdots\\ 0 & S_{1,2}^{n+k-3}-r_1S_{1,2}^{n+k-4} & \cdots & S_{1,k}^{n-1}-r_1S_{1,k}^{n-2}\\ 0 & S_{1,2}^{n+k-1}-r_1^2S_{1,2}^{n+k-3} & \cdots & S_{1,k}^{n}-r_1^2S_{1,k}^{n-1}\end{vmatrix}\begin{matrix}(1)\\ (2)\\ \vdots\\ (k-1)\\ (k)\end{matrix}$$

Apply Formula (A.2) to rows $1,\cdots,k-1$ and Formula (A.3) to the last row:

$$L_4=\begin{vmatrix}r_1^n & S_{1,2}^{n-1} & \cdots & S_{1,k}^{n-k+1}\\ 0 & S_{2,2}^{n} & \cdots & S_{2,k}^{n-k+2}\\ 0 & \vdots & \ddots & \vdots\\ 0 & S_{2,2}^{n+k-3} & \cdots & S_{2,k}^{n-1}\\ 0 & r_1S_{2,2}^{n+k-2}+S_{2,2}^{n+k-1} & \cdots & r_1S_{2,k}^{n}+S_{2,k}^{n+1}\end{vmatrix}\begin{matrix}(1)\\ (2)\\ \vdots\\ (k-1)\\ (k)\end{matrix} \tag{A.21}$$

Next,

$$L_4 = r_1^n \left(r_1 \begin{vmatrix} S_{2,2}^n & \cdots & S_{2,k}^{n-k+2} \\ \vdots & \ddots & \vdots \\ S_{2,2}^{n+k-2} & \cdots & S_{2,k}^n \end{vmatrix} + \begin{vmatrix} S_{2,2}^n & \cdots & S_{2,k}^{n-k+2} \\ \vdots & \ddots & \vdots \\ S_{2,2}^{n+k-3} & \cdots & S_{2,k}^{n-1} \\ S_{2,2}^{n+k-1} & \cdots & S_{2,k}^{n+1} \end{vmatrix} \begin{matrix} (1) \\ \vdots \\ (k-2) \\ (k-1) \end{matrix} \right). \tag{A.22}$$

The first determinant of order $(k-1)$ on the right-hand side of Equation (A.22) coincides with the determinant Δ^{k-1} from Equation (A.7). Furthermore, the determinants Δ^{k-1} and Δ^k are related by the following formula (see Equation (A.10)):

$$\Delta^k = r_1^n \Delta^{k-1}. \tag{A.23}$$

Let us introduce the following notations:

$$\theta_i = \begin{vmatrix} S_{i,i}^n & S_{i,i+1}^{n-1} & \cdots & S_{i,k}^{n-k+i} \\ S_{i,i}^{n+1} & S_{i,i+1}^n & \cdots & S_{i,k}^{n-k+i+1} \\ \vdots & \vdots & \ddots & \vdots \\ S_{i,i}^{n+k-i-1} & S_{i,i+1}^{n+k-i-2} & \cdots & S_{i,k}^{n-1} \\ S_{i,i}^{n+k-i+1} & S_{i,i+1}^{n+k-i} & \cdots & S_{i,k}^{n+1} \end{vmatrix} \begin{matrix} (1) \\ (2) \\ \vdots \\ (k-i) \\ (k-i+1) \end{matrix} \tag{A.24}$$

Now we can rewrite Equation (A.22) as

$$\theta_1 = r_1^n \left(r_1 \Delta^{k-1} + \theta_2 \right).$$

From Equation (A.23) we therefore obtain

$$\theta_1 = \Delta^k r_1 + r_1^n \theta_2.$$

In a similar manner, we deduce that

$$\begin{aligned} \theta_1 &= \Delta^k r_1 + r_1^n r_2^n \left(r_2 \Delta^{k-2} + \theta_3 \right) \\ &= \Delta^k r_1 + r_1^n r_2 \Delta^{k-1} + r_1^n r_2^n \theta_3 \\ &= \Delta^k r_1 + \Delta^k r_2 + r_1^n r_2^n \theta_3. \end{aligned}$$

Continuing these transformations and using Equation (A.24) and formulae

$$\Delta^\alpha = r_{k-\alpha+1}^n \Delta^{\alpha-1}, \ \alpha = \overline{k,3}, \tag{A.25}$$

which are more general than Equation (A.23), we can write

$$\theta_1 = \Delta^k \left(r_1 + \cdots + r_{k-2} \right) + r_1^n \cdots r_{k-2}^n \theta_{k-1}, \tag{A.26}$$

where

$$\theta_{k-1} = \begin{vmatrix} S_{k-1,k-1}^n & S_{k-1,k}^{n-1} \\ S_{k-1,k-1}^{n+2} & S_{k-1,k}^{n+1} \end{vmatrix} = r_{k-1}^n r_k^n \left(r_{k-1} + r_k \right).$$

Thus, we find from Equation (A.26) that

$$\theta_1 = \Delta^k \theta_{1,k}^1.$$

This concludes the proof of Formula (A.19).