International Journal of Modern Nonlinear Theory and Application, 2023, 12, 68-80
https://www.scirp.org/journal/ijmnta
ISSN Online: 2167-9487
ISSN Print: 2167-9479

Galerkin Method for Numerical Solution of Volterra Integro-Differential Equations with Certain Orthogonal Basis Function

Omotayo Adebayo Taiwo[1], Liman Kibokun Alhassan[1], Olutunde Samuel Odetunde[2], Olatayo Olusegun Alabi[3]

[1]Department of Mathematics, University of Ilorin, Ilorin, Nigeria
[2]Department of Mathematical Sciences, Olabisi Onabanjo University, Ago-Iwoye, Nigeria
[3]Department of Statistics, Federal University of Technology, Akure, Nigeria
Email: tayo@unilorin.edu.ng, limanalhassan@icloud.com, tuned.odetunde@oouagoiwoye.edu.ng, alabioo@futa.edu.ng

How to cite this paper: Taiwo, O.A., Alhassan, L.K., Odetunde, O.S. and Alabi, O.O. (2023) Galerkin Method for Numerical Solution of Volterra Integro-Differential Equations with Certain Orthogonal Basis Function. *International Journal of Modern Nonlinear Theory and Application*, **12**, 68-80.
https://doi.org/10.4236/ijmnta.2023.122005

Received: April 4, 2023
Accepted: June 27, 2023
Published: June 30, 2023

Abstract

This paper concerns the implementation of the orthogonal polynomials using the Galerkin method for solving Volterra integro-differential and Fredholm integro-differential equations. The constructed orthogonal polynomials are used as basis functions in the assumed solution employed. Numerical examples for some selected problems are provided and the results obtained show that the Galerkin method with orthogonal polynomials as basis functions performed creditably well in terms of absolute errors obtained.

Keywords

Galerkin Method, Integro-Differential Equation, Orthogonal Polynomials, Basis Function, Approximate Solution

1. Introduction

Integro-differential equations (IDEs) have attracted growing interest over the years because of the mathematical application in real life problems. Mathematical modeling of real life problems usually resulted in fractional equations. Many mathematical formulations of physical phenomena contain integro-differential equations. These equations arise in many fields like Physics, Astronomy, Potential theory, Fluid dynamics, Biological models and Chemical kinetics. Integro-differential equations contain both integral and differential operators. The derivatives of the unknown functions may appear to any order (see [1] and [2]). [3] obtained solution of an integro-differential equation arising in oscillating magnetic field using Homotopy perturbation method. Galerkin method is a powerful tool for solving many kinds of equations in various fields of science and engineering. It is one of the most important weighted residual methods inverted by Russians mathematicians Boris Grigoryevrich Galerkin. Recently, various Galerkin algoriyhm have been applied in numerical solution of integral and integro-differential equations. The following methods that are based on the Galerkin ideas, includes Galerkin Finite Element [4], iterative Galerkin with hybrid functions [5], Crank-Nicolson least squares Galerkin [6], and Legendre Galerkin [7]. [8] published a note on three numerical procedures to solve Volterra integro-differential equations on structural analysis.

2. Problem Considered

We consider the higher order linear integro-differential equation as follows:

$$\sum_{i=0}^{n} P_i y^{(i)} + \lambda \int_{g(x)}^{h(x)} k(x,t)\, y(t)\,\mathrm{d}t = f(x) \tag{1}$$

Subject to the following conditions

$$y^{(k)}(a) = \alpha_k, k = 1, 2, \cdots, n \tag{2}$$

where $\alpha_k (k \geq 0)$ are constant coefficients, $g(x)$ and $h(x)$ are lower and upper limits of integration, λ is a constant parameter and $k(x,t)$ is a function of two variables x and t called the kernel, $f(x)$ is a known function and $y(x)$ is the unknown function to be determined.

3. Definitions

Integro-differential equation

An integro-differential equation is an equation involving both integral and derivatives of a function. Example of such equation is stated below:

$$a_2 y''(x) + a_1 y'(x) + a_0 y(x) + \lambda \int_a^b H(x,t)\, y(t)\,\mathrm{d}t = f(x) \tag{3}$$

Galerkin method

Galerkin method is a method of determining coefficient a_k in a power series solution of the form:

$$y(x) \cong y_0(x) + \sum_{k=0}^{n} a_k y_k(x) \tag{4}$$

of the ordinary differential equation $L[y(x)] = 0$ so that $L[y(x)]$, the result of applying the ordinary differential operator to $y(x)$, is orthogonal to every $y_k(x)$ for $k = 1, 2, \cdots, n$

Chebyshev Polynomial

The Chebyshev polynomials of the first kind are a set of orthogonal polynomials defined as the solutions to the Chebyshev differential equation and denoted by $T_n(x)$. The Chebyshev polynomial of the first kind denoted by $T_n(x)$ is defined by the contour integral

$$T_n(x) = \frac{1}{4\pi i} \oint \frac{(1-t^2)\, t^{-n-1}}{(1-2tz+t^2)} \mathrm{d}t$$

Where the contour encloses the origin and is traversed in a counter clockwise direction.

Orthogonal over a set

A set of function $\{\phi_r(x)\}$ is said to be orthogonal over a set of points $\{x_i\}$ with respect to the weight function $w(x)$, if

$$\sum_{i=0}^{N} w(x_i)\phi_j(x_i)\phi_k(x_i) = 0,\ i \neq k$$

Orthogonal over an interval

A set of functions $\{\phi_r(x)\}$ is said to orthogonal on an interval $[a,b]$ with respect to the weight function $w(x)$, if

$$\int_a^b w(x)\phi_i(x)\phi_j(x)\mathrm{d}x = 0,\ i \neq j$$

Approximate solution

Approximate solution is used for the expression obtained after the unknown constants have been generated and substituting back into the assumed solution. It is hereby call approximate solution since it is a reasonable approximation to the exact solution.

4. Construction of Orthogonal Polynomials

In this section, we constructed orthogonal polynomials $f_i(x)$, valid on the interval $[a,b]$ with the leading term x^i

Then, starting with

$$f_0(x) = 1, \tag{5}$$

Thus, we find the linear polynomials $f_1(x)$, with leading term *x*, is written as

$$f_1(x) = x + k_{1,0} f_0(x), \tag{6}$$

where, $k_{1,0}$ is a constant to be determined. Since $f_0(x)$ and $f_1(x)$ are orthogonal, we have,

$$\int_a^b w(x) f_0(x) f_1(x)\mathrm{d}x = 0 = \int_a^b x w(x) f_0(x)\mathrm{d}x + k_{1,0}\int_a^b w(x)(f_0(x))^2\mathrm{d}x$$

using (5) and (6).

From the above, we have,

$$k_{1,0} = -\frac{\int_a^b x w(x) f_0(x)}{\int_a^b w(x)(f_0(x))^2}\mathrm{d}x$$

Hence, (6) gives,

$$f_1(x) = x - \frac{\int_a^b x w(x) f_0(x)}{\int_a^b w(x)(f_0(x))^2}\mathrm{d}x$$

Now, the polynomials $f_2(x)$, of degree 2 and the leading term x^2 is written as

$$f_2(x) = x^2 + k_{2,0} f_0(x) + k_{2,1} f_1(x) \tag{7}$$

where the constants $k_{2,0}$ and $k_{2,1}$ are determined by using orthogonality conditions

$$\int_a^b w(x) f_p(x) f_q(x) \mathrm{d}x = \begin{cases} 0, & p \neq q \\ \int_a^b w(x) f_p^2(x) \mathrm{d}x, & p = q \end{cases} \tag{8}$$

Since $f_2(x)$ is orthogonal to $f_0(x)$, we have

$$\int_a^b w(x) f_0(x) \left[x^2 + k_{2,0} f_0(x) + k_{2,1} f_1(x) \right] \mathrm{d}x = 0 \tag{9}$$

Since,

$$\int_a^b w(x) f_0(x) f_1(x) \mathrm{d}x = 0$$

The above equation gives

$$k_{2,0} = -\frac{\int_a^b x^2 w(x) f_0(x)}{\int_a^b w(x) \left(f_0(x)\right)^2} \mathrm{d}x = -\frac{\int_a^b x^2 w(x) \mathrm{d}x}{\int_a^b w(x) \mathrm{d}x} \tag{10}$$

Again, since $f_2(x)$ is orthogonal to $f_1(x)$, we have

$$\int_a^b w(x) f_1(x) \left[x^2 + k_{2,0} f_0(x) + k_{2,1} f_1(x) \right] \mathrm{d}x = 0$$

Thus, using (7), we obtain

$$k_{2,1} = -\frac{\int_a^b x^2 w(x) f_1(x)}{\int_a^b w(x) \left(f_1(x)\right)^2} \mathrm{d}x \tag{11}$$

Since $k_{2,1}$ and $k_{2,0}$ are known, (7) determines $f_2(x)$. Proceeding in the same way, the method is generalized and we have,

$$f_j(x) = x^j + k_{j,0} f_0(x) + k_{j,1} f_1(x) + \cdots + k_{j,j-1} \tag{12}$$

where the constants $k_{j,i}$ and so chosen that $f_j(x)$ is orthogonal to

$$f_0(x), f_1(x), \cdots, f_{j-1}(x)$$

These conditions yield,

$$k_{j,i} = -\frac{\int_a^b x^j w(x) f_i(x)}{\int_a^b w(x) \left(f_i(x)\right)^2} \mathrm{d}x \tag{13}$$

Few terms of orthogonal polynomials valid in the interval [−1, 1] are given below.

$$\begin{aligned} f_0(x) &= 1 \\ f_1(x) &= x \\ f_2(x) &= x^2 - \frac{1}{3} \\ f_3(x) &= x^3 - \frac{3}{5} x \\ f_4(x) &= x^4 - \frac{6}{7} x^2 + \frac{3}{35} \\ &\vdots \end{aligned}$$

etc.

5. Demonstration of Orthogonal Galerkin Method on General Problem Considered

In this section, we considered (1) and (2).

Here we assumed an approximate solution of the form

$$u(x) \cong u_N(x) = \sum_{i=0}^{N} a_i f_i(x), \quad -1 \le x \le 1 \tag{14}$$

where $f_i(x)(i \ge 0)$ are the orthogonal polynomial constructed and valid in the interval [−1, 1].

Thus, differentiating (14)/with respect to x, n times, we have

$$u^{(n)}(x) \cong u_N^{(n)}(x) = \sum_{i=0}^{N} a_i f_i^{(n)}(x) \tag{15}$$

Substituting (14) and (15) into (1), we obtain

$$\sum_{k=0}^{n}\sum_{i=0}^{N} P_k a_i f_i^{(n)}(x) = f(x) + \lambda \sum_{i=0}^{N} a_i \int_0^x k(x,t) \sum_{i=0}^{N} f_i(t)\,\mathrm{d}t \tag{16}$$

We determined the unknown coefficients a_i using the Galerkin idea by multiplying both sides of (16) by $f_j(x)$ and then integrating with respect to x from −1 to 1.

Thus, we obtain

$$\begin{aligned} &\sum_{k=0}^{n}\sum_{i=0}^{N} P_k a_i \int_{-1}^{1} f_i^{(n)}(x) f_j(x)\,\mathrm{d}x \\ &= \int_{-1}^{1} f_j(x) f(x)\,\mathrm{d}x + \lambda \sum_{i=0}^{N} a_i \int_0^x \int_{-1}^{1} k(x,t) \sum_{i=0}^{N} f_i(t) f_j(x)\,\mathrm{d}t\mathrm{d}x, \ \ j = 0,1,\cdots,N \end{aligned} \tag{17}$$

This process generates a system of linear equations for the unknown $\{a_i\}_{i=0}^{N}$ together with the conditions

$$\sum_{i=0}^{N} a_i f_i^{(j)}(a) = \alpha_j, \quad j = 1,2,\cdots,n \tag{18}$$

for the same number of equations in the linear system.

The unknown parameters are determined by solving the system (17) and (18). The values of the constants obtained are then substituted back into (14) to get the required approximate solution for the appropriate order.

6. Numerical Experiments

In this section, we consider four selected problems for experimenting and compare our results with existing results.

Numerical example 1

We consider the Volterra integro-differential equations of the second kind of the form:

$$y'(x) = 1 - 2x\sin x + \int_0^x y(t)\,\mathrm{d}t \tag{19}$$

together with the condition given as

$$y(0)=0 \tag{20}$$

The exact solution is given as

$$y(x)=x\cos x$$

Here we solved example 1 for case $N=4$.

Thus, Equation (14) becomes

$$y_4(x)=\sum_{i=0}^{4}a_i f_i(x) \tag{21}$$

Substituting the values of $f_i(x), 0\le i\le 4$, we obtain

$$\begin{aligned} y_4(x)=a_0+(2x-1)a_1+\left((2x-1)^2-\frac{1}{3}\right)a_2+\left((2x-1)^3-\frac{3}{5}(2x-1)\right)a_3 \\ +\left((2x-1)^4-\frac{6}{7}(2x-1)^2+\frac{3}{35}\right)a_4 \end{aligned} \tag{22}$$

and,

$$y_4'(x)=2a_1+(8x-4)a_2+\left(6(2x-1)^2-\frac{6}{5}\right)a_3+\left(8(2x-1)^3-\frac{48}{7}x+\frac{24}{35}\right)a_4 \tag{23}$$

Substituting (23) into (19) for case $N=4$, we obtain

$$\begin{aligned} &2a_1+(8x-4)a_2+\left(6(2x-1)^2-\frac{6}{5}\right)a_3+\left(8(2x-1)^3-\frac{48}{7}x+\frac{24}{35}\right)a_4 \\ &-\int_a^x\left\{a_0+(2t-1)a_1+\left((2t-1)^2-\frac{1}{2}\right)a_2+\left((2t-1)^3-\frac{3}{5}(2t-1)\right)a_3\right. \\ &\left.+\left((2t-1)^4-\frac{6}{7}(2t-1)^2+\frac{3}{35}\right)a_4\right\}\mathrm{d}t=1-2x\sin x \end{aligned} \tag{24}$$

Thus, evaluating the integral in (24) and simplifying, we obtain

$$\begin{aligned} &-xa_0+\left(2+x-x^2\right)a_1+\left(\frac{15}{2}x+2x^2-\frac{4}{3}x^3+4\right)a_2 \\ &+\left(6(2x-1)^2+\frac{2}{3}x-2x^4+4x^3-\frac{12}{5}x^2-\frac{6}{5}\right)a_3 \\ &+\left(8(2x-1)^3-\frac{248}{35}x-\frac{16}{5}x^5+8x^4-\frac{48}{7}x^3+\frac{16}{7}x^2+\frac{24}{35}\right)a_4=1-2x\sin x \end{aligned} \tag{25}$$

The unknown coefficients $a_i(i\le 4)$ are determined using the Galerkin idea by multiplying both sides of (25) by $f_j(2x-1)$ and then integrating the resulted equation between $x=-1$ to $x=1$.

For case $j=1$, we multiplied both sides of (25) by $(2x-1)$ and then integrating the resulted equation between $x=-1$ to $x=1$, to obtain

$$-\frac{4}{3}a_0-2a_1-\frac{2}{5}a_2-\frac{460}{9}a_3+\frac{6032}{35}a_4=-0.7953 \tag{26}$$

For case $j=2$, we multiplied both sides of (25) by $(2x-1)^2-\frac{1}{3}$ and then integrating the resulted equation between $x=-1$ to $x=1$, to obtain

$$\frac{8}{3}a_0+\frac{148}{45}a_1+\frac{20}{9}a_2+\frac{183272}{1575}a_3-\frac{128032}{315}a_4=0.363 \tag{27}$$

For case $j = 3$, we multiplied both sides of (25) by $(2x-1)^3-\frac{3}{5}(2x-1)$ and then integrating the resulted equation between $x = -1$ to $x = 1$, to obtain

$$-\frac{32}{5}a_0-\frac{92}{15}a_1-\frac{1544}{525}a_2-\frac{20776}{75}a_3+\frac{11053408}{11025}a_4=0.18 \tag{28}$$

For case $j = 4$, we multiplied both sides of (25) by $(2x-1)^4-\frac{6}{7}(2x-1)^2+\frac{3}{35}$ and then integrating the resulted equation between $x = -1$ to $x = 1$, to obtain

$$\frac{1664}{105}a_0+\frac{432}{35}a_1+\frac{416}{105}a_2+\frac{2518688}{3675}a_3-\frac{124288}{49}a_4=-2.3 \tag{29}$$

Now, using the condition given in (22), we obtain

$$a_0-a_1+\frac{2}{3}a_2-\frac{2}{5}a_3+\frac{8}{25}a_4=0 \tag{30}$$

Hence, (26)-(30) are then solved to obtain the unknown constants $a_i\,(i=0,1,2,3,4)$ which are then substituted to the approximate Equation (22).

Again, we solved (1) and (2) for case $N = 6$ by re-writing (21) as:

$$y_6(x)=\sum_{i=0}^{6}a_i f_i(x) \tag{31}$$

Hence, (31) becomes

$$\begin{aligned} y_6(x)&=a_0+(2x-1)a_1+\left((2x-1)^2-\frac{1}{3}\right)a_2\\ &+\left((2x-1)^3-\frac{3}{5}(2x-1)\right)a_3\\ &+\left((2x-1)^4-\frac{6}{7}(2x-1)^2+\frac{3}{35}\right)a_4\\ &+\left((2x-1)^5-\frac{10}{9}(2x-1)^3+\frac{5}{21}(2x-1)\right)a_5\\ &+\left((2x-1)^6-\frac{15}{11}(2x-1)^4+\frac{5}{11}(2x-1)^2-\frac{5}{231}\right)a_6 \end{aligned} \tag{32}$$

And,

$$\begin{aligned} y_6'(x)&=2a_1+(8x-4)a_2+\left(6(2x-1)^2-\frac{6}{5}\right)a_3\\ &+\left(8(2x-1)^3-\frac{48}{7}x+\frac{24}{35}\right)a_4\\ &+\left(10(2x-1)^4-\frac{20(2x-1)^3}{3}+\frac{10}{21}\right)a_5\\ &+\left(12(2x-1)^5-\frac{120(2x-1)^3}{11}+\frac{40x}{11}-\frac{20}{11}\right)a_6 \end{aligned} \tag{33}$$

Thus substituting (32) and (33) into (19), we obtain

$$
\begin{aligned}
&2a_1+(8x-4)a_2+\left(6(2x-1)^2-\frac{6}{5}\right)a_3+\left(8(2x-1)^3-\frac{48}{7}x+\frac{24}{35}\right)a_4\\
&+\left(10(2x-1)^4-\frac{20(2x-1)^3}{3}+\frac{10}{21}\right)a_5+\left(12(2x-1)^5-\frac{120(2x-1)^3}{11}+\frac{40x}{11}\right.\\
&\left.-\frac{20}{11}\right)a_6-\int_0^x\left\{a_0+(2t-1)a_1+\left((2t-1)^2-\frac{1}{2}\right)a_2+\left((2t-1)^3-\frac{3}{5}(2t-1)\right)a_3\right.\\
&+\left((2t-1)^4-\frac{6}{7}(2t-1)^2+\frac{3}{35}\right)a_4+\left(10(2t-1)^4-\frac{20(2t-1)^3}{3}+\frac{10}{21}\right)a_5\\
&\left.+\left(12(2t-1)^5-\frac{120(2t-1)^3}{11}+\frac{40t}{11}-\frac{20}{11}\right)a_6\right\}\mathrm{d}t=1-2x\sin x
\end{aligned}
\tag{34}
$$

Thus, evaluating the integral in (34) and simplifying, we obtain

$$
\begin{aligned}
&-xa_0+\left(2+x-x^2\right)a_1+\left(\frac{15}{2}x+2x^2-\frac{4}{3}x^3+4\right)a_2\\
&+\left(6(2x-1)^2+\frac{2}{3}x-2x^4+4x^3-\frac{12}{5}x^2-\frac{6}{5}\right)a_3\\
&+\left(8(2x-1)^3-\frac{248}{35}x-\frac{16}{5}x^5+8x^4-\frac{48}{7}x^3+\frac{16}{7}x^2+\frac{24}{35}\right)a_4\\
&+\left(10(2x-1)^4-\frac{20}{3}(2x-1)^3+\frac{1200}{147}x-32x^5+\frac{280}{3}x^4-\frac{320}{3}x^3+60x^2\right)a_5\\
&+\left(12(2x-1)^5-\frac{120}{11}(2x-1)^3+\frac{72}{11}x-\frac{1280}{11}x^6+192x^5-\frac{2400}{11}x^4\right.\\
&\left.+\frac{1280}{11}x^3-\frac{320}{11}x^2\right)a_6=1-2x\sin x
\end{aligned}
\tag{35}
$$

The unknown coefficients $a_i(i\le 4)$ are determined using the Galerkin idea by multiplying both sides of (35) by $f_j(2x-1)$ and then integrating the resulted equation between $x=-1$ to $x=1$.

For case $j=1$, we multiplied both sides of (35) by $(2x-1)$ and then integrating the resulted equation between $x=-1$ to $x=1$, to obtain

$$-\frac{4}{3}a_0-2a_1-\frac{2}{5}a_2-\frac{460}{9}a_3+\frac{6032}{35}a_4-\frac{97264}{189}a_5+\frac{360160}{231}a_6=-0.7953 \tag{36}$$

For case $j=2$, we multiplied both sides of (35) by $(2x-1)^2-\frac{1}{3}$ and then integrating the resulted equation between $x=-1$ to $x=1$, to obtain

$$\frac{8}{3}a_0+\frac{148}{45}a_1+\frac{20}{9}a_2+\frac{183272}{1575}a_3-\frac{128032}{315}a_4+\frac{238816}{189}a_5-\frac{1507904}{385}a_6=0.363 \tag{37}$$

For case $j=3$, we multiplied both sides of (35) by $(2x-1)^3-\frac{3}{5}(2x-1)$ and then integrating the resulted equation between $x=-1$ to $x=1$, to obtain

$$
\begin{aligned}
&-\frac{32}{5}a_0-\frac{92}{15}a_1-\frac{1544}{525}a_2-\frac{20776}{75}a_3+\frac{11053408}{11025}a_4-\frac{1007648}{315}a_5\\
&+\frac{2330560}{231}a_6=0.18
\end{aligned}
\tag{38}
$$

For case j = 4, we multiplied both sides of (35) by $(2x-1)^4-\frac{6}{7}(2x-1)^2+\frac{3}{35}$ and then integrating the resulted equation between x = −1 to x = 1, to obtain

$$\frac{1664}{105}a_0+\frac{432}{35}a_1+\frac{416}{105}a_2+\frac{2518688}{3675}a_3-\frac{124288}{49}a_4+\frac{360321152}{47659}a_5 -\frac{63937952}{24255}a_6=-2.3 \quad (39)$$

For case j = 5, we multiplied both sides of (25) by $(2x-1)^5-\frac{10}{9}(2x-1)^3+\frac{5}{21}(2x-1)$ and then integrating the resulted equation between x = −1 to x = 1, to obtain

$$-\frac{2528}{63}a_0-\frac{4976}{180}a_1+\frac{2720}{7}a_2-\frac{550112}{315}a_3+\frac{1396705664}{218295}a_4-\frac{85672064}{3969}a_5 +\frac{48377661184}{693693}a_6=-\frac{3152}{63}-\frac{2002592}{63}\cos(1)+\frac{144}{7} \quad (40)$$

For case j = 6, we multiplied both sides of (35) by $(2x-1)^6-\frac{15}{11}(2x-1)^4+\frac{5}{11}(2x-1)^2-\frac{5}{231}$ and then integrating the resulted equation between x = −1 to x = 1, to obtain

$$-\frac{2528}{63}a_0-\frac{4976}{180}a_1+\frac{2720}{7}a_2-\frac{550112}{315}a_3+\frac{1396705664}{218295}a_4-\frac{85672064}{3969}a_5 +\frac{48377661184}{693693}a_6=-\frac{1376}{11}-\frac{230568512}{231}\cos(1)+\frac{1290272}{63}\sin(1) \quad (41)$$

Now, using the condition given in (22), we obtain

$$a_0-a_1+\frac{2}{3}a_2-\frac{2}{5}a_3+\frac{8}{25}a_4-\frac{8}{63}a_5+\frac{16}{231}a_6=0 \quad (42)$$

Hence, (36)-(42) are then solved to obtain the unknown constants $a_i\,(i=0,1,2,3,4,5,5,6)$ which are then substituted to the approximate equation (32). More values of N are computed follow the same procedure and the results obtained are tabulated below.

Example 2:

$$y''(x)+xy'(x)-xy(x)=\mathrm{e}^x-2\sin x+\int_{-1}^{1}y(t)\,\mathrm{d}t$$

With the conditions $y(0)=1$ and $y'(0)=1$, The exact solution is $y(x)=\mathrm{e}^x$.

Example 3: Consider the Fredholm integro-differential equation (See [2])

$$y''''(x)=1+\int_0^1 \mathrm{e}^{-x}y^2(t)\,\mathrm{d}t,\quad 0<x<1$$

Together with the conditions $y(0)=y'(0)=1$; $y(1)=\mathrm{e}$; $y'(1)=\mathrm{e}$. The exact solution is $y(x)=\mathrm{e}^x$.

- **Denotes the results are not available for comparison**
- **Denotes Results are not available for comparison**

Example 4: Consider the Fredholm integro-differential equation (See [2])

$$y''''(x)=x+(x+3)\mathrm{e}^x+y(x)-\int_0^x y(t)\,\mathrm{d}t,\quad 0<x<1$$

With the following conditions

$y(0)=1$; $y(1)=1+\mathrm{e}$; $y''(0)=2$; $y''(1)=3\mathrm{e}$. The exact solution is $y(x)=1+x\mathrm{e}^{x}$.

- **Denotes Results are not available for comparison**
- **Denotes Results are not available for comparison**

7. Discussion of Results

The approximate solution obtained by means of Galerkin method is a finite power series which can be in turn expressed in closed form of exact solution as the degree of the approximant increases. The finite series solution is obtained for each problem considered by increasing the value of *N*, which in turn converges to closed form of exact solution, the absolute errors obtained tend to zero and ensures stability of our method (See **Tables 1-8**). Also, from the results obtained by [2], our method proved superior to [2]. As *N* increases, the results obtained in some cases converged. It proves a very efficient method for the problems attempted, for which the form of the solution is known.

Table 1. Numerical results and absolute errors of example 1 for case $N = 4$.

X	Exact solution	Approximate solution	Approximate solution
0	0	0	0
0.1	0.09999984769	0.1007787777	7.7893×10^{-4}
0.2	0.19999871500	0.20007989915	8.0021×10^{-4}
0.3	0.29999588772	0.30082048742	8.2460×10^{-4}
0.4	0.39990252364	0.40085698231	9.5446×10^{-4}
0.5	0.49998096153	0.50093900554	9.5813×10^{-4}
0.6	0.59996710167	0.60128870164	1.2920×10^{-3}
0.7	0.69994775882	0.70214095881	2.1932×10^{-3}
0.8	0.79992201922	0.80415158192	4.2296×10^{-3}
0.9	0.89988896922	0.90630266921	6.4113×10^{-3}
1.0	0.99984769523	1.00008011995	2.3242×10^{-4}

Table 2. Numerical results and absolute errors of example 1 for case $N = 6$.

X	Exact solution	Approximate solution	Approximate solution
0	0	0	0
0.1	0.09999984769	0.1000123167	1.2409×10^{-5}
0.2	0.19999871500	0.2000261375	2.7350×10^{-5}
0.3	0.29999588772	0.3000306197	3.4732×10^{-5}
0.4	0.39990252364	0.4000469833	5.6731×10^{-5}
0.5	0.49998096153	0.5000608685	7.9907×10^{-5}
0.6	0.59996710167	0.6000487216	8.1620×10^{-5}
0.7	0.69994775882	0.7000377598	9.0001×10^{-5}
0.8	0.79992201922	0.8003215592	3.9951×10^{-5}
0.9	0.89988896922	0.9002501792	3.6121×10^{-5}
1.0	0.99984769523	1.0003425534	5.5778×10^{-5}

Table 3. Numerical results and absolute errors of example 2 for case $N = 4$.

X	Exact solution	Approximate solution	Approximate solution
−1	0.36787944	0.37418684	6.3074×10^{-3}
−0.8	0.44932896	0.45641056	7.0816×10^{-3}
−0.6	0.54881164	0.55712374	8.3121×10^{-3}
−0.4	0.67032005	0.68009815	9.7781×10^{-3}
−0.2	0.81873075	0.82014445	1.4137×10^{-2}
0	1.00000000	1.00180376	1.8937×10^{-2}
0.2	1.22140283	1.24357182	2.2169×10^{-3}
0.4	1.47182472	1.49774274	2.5918×10^{-2}
0.6	1.82211881	1.85630581	3.4187×10^{-2}
0.8	2.22551000	2.26616893	4.0928×10^{-2}
1.0	2.71828182	2.78212785	6.3846×10^{-3}

Table 4. Numerical results and absolute errors of example 2 for case $N = 4$.

X	Exact solution	Approximate solution	Approximate solution
−1	0.36787944	0.367966169	8.6729×10^{-5}
−0.8	0.44932896	0.449409094	8.0134×10^{-5}
−0.6	0.54881164	0.548889417	7.7837×10^{-5}
−0.4	0.67032005	0.676389371	6.9321×10^{-5}
−0.2	0.81873075	0.818758949	7.8199×10^{-5}
0	1.00000000	1.000966532	7.6653×10^{-4}
0.2	1.22140283	1.222229894	8.9614×10^{-4}
0.4	1.47182472	1.472514031	6.8933×10^{-4}
0.6	1.82211881	1.822781972	5.9397×10^{-4}
0.8	2.22551000	2.226029824	4.8892×10^{-4}
1.0	2.71828182	2.718738011	4.5621×10^{-4}

Table 5. Numerical results and absolute errors of example 3 for case $N = 4$.

X	Exact	Approximate of [2]	Approx. of Our Method	Absolute errors of [2]	Absolute errors of Our Method
0.0	1.0000000	1.0000000	1.00000000	0	0
0.1	1.105171		1.105173451	*	2.451×10^{-6}
0.2	1.2214027	1.2214	1.221409351	1.0270×10^{-4}	6.651×10^{-6}
0.3	1.349859	*	1.349868872	*	9.872×10^{-6}
0.4	1.4918246	1.4918	1.491856800	1.1246×10^{-3}	3.220×10^{-5}
0.5	1.648721	*	1.648800850	*	7.985×10^{-5}
0.6	1.8221188	1.8221	1.822700800	6.1188×10^{-3}	5.820×10^{-4}
0.7	2.013753	*	2.014370200	*	6.172×10^{-4}
0.8	2.2255409	2.2255	2.228210900	2.0241×10^{-2}	2.670×10^{-3}
0.9	2.459603	*	2.465275000	*	5.672×10^{-3}
1.0	2.71828183	2.7183	2.725281830	5.1282×10^{-2}	7.000×10^{-3}

*Denotes the results are not available for comparison.

Table 6. Numerical results and absolute errors of example 3 for case $N = 10$.

X	Exact	Approximate of [2]	Approx. of Our Method	Absolute errors of [2]	Absolute errors of Our Method
0.0	1.0000000	1.0000	1.00000000000	0	0
0.1	1.105171	*	1.10517109874	*	9.874×10^{-8}
0.2	1.2214027	1.2214	1.22140278125	2.700×10^{-6}	8.125×10^{-8}
0.3	1.349859	*	1.34985906846	*	6.845×10^{-8}
0.4	1.4918246	1.4918	1.49182466533	2.460×10^{-5}	5.329×10^{-8}
0.5	1.648721	*	1.64872104101	*	4.101×10^{-8}
0.6	1.8221188	1.8221	1.82211884674	1.880×10^{-5}	4.674×10^{-8}
0.7	2.013753	*	2.01375304115	*	4.115×10^{-8}
0.8	2.2255409	2.2255	2.22554093985	4.090×10^{-5}	3.985×10^{-8}
0.9	2.459603	*	2.45960302679	*	2.679×10^{-8}
1.0	2.71828183	2.7183	2.71828184068	1.820×10^{-5}	1.068×10^{-8}

*Denotes the results are not available for comparison.

Table 7. Numerical results and absolute errors of example 4 for case $N = 4$.

X	Exact	Approx. of [2]	Approx. of Our Method	Absolute errors of [2]	Absolute errors of Our Method
0.0	1.0000000	1.0000	1.0000000000	0	0
0.1	1.110517	*	1.1105179874	*	9.874×10^{-7}
0.2	1.2442805	1.244	1.2442922210	2.8055×10^{-4}	1.172×10^{-6}
0.3	1.404958	*	1.4049590990	*	1.099×10^{-6}
0.4	1.5967298	1.592	1.4967570200	2.7299×10^{-4}	9.722×10^{-5}
0.5	1.824361	*	1.8244327200	*	7.172×10^{-5}
0.6	2.0932712	2.068	2.0933164710	2.5270×10^{-2}	4.527×10^{-5}
0.7	2.409627	*	2.4096387200	*	1.172×10^{-5}
0.8	2.7804327	2.696	2.7805028800	8.4430×10^{-2}	9.018×10^{-4}
0.9	3.213943	*	3.2140147700	*	7.177×10^{-4}
1.0	3.71828183	3.5	2.7183814900	2.1820×10^{-1}	6.966×10^{-4}

*Denotes the results are not available for comparison.

Table 8. Numerical results and absolute errors of example 4 for case $N = 10$.

X	Exact	Approx. of [2]	Approx. of Our Method	Absolute errors of [2]	Absolute errors of Our Method
0.0	1.0000000	1.0000	1.000000000000	0	0
0.1	1.110517	*	1.1105170009231	*	9.231×10^{-10}
0.2	1.2442805	1.2443	1.2442805007638	1.950×10^{-5}	7.638×10^{-10}
0.3	1.404958	*	1.4049580006618	*	6.618×10^{-10}
0.4	1.5967298	1.5967	1.5967298002963	3.000×10^{-10}	2.963×10^{-10}
0.5	1.824361	*	1.8243610001316	*	1.316×10^{-10}
0.6	2.0932712	2.0933	2.0932712009316	1.772×10^{-8}	9.316×10^{-9}
0.7	2.409627	*	2.4096270492700	*	4.927×10^{-8}
0.8	2.7804327	2.7804	2.7804327297800	3.214×10^{-7}	2.978×10^{-8}
0.9	3.213643	*	3.2136430198200	*	1.982×10^{-8}
1.0	3.71828183	3.7184	2.7182827690000	1.820×10^{-5}	9.390×10^{-7}

*Denotes the results are not available for comparison.

8. Conclusion

In this work, we have proposed the Galerkin method for solving both the boundary and initial value problems for a class of higher order linear and nonlinear Volterra and Fredholm integro-differential based on the constructed orthogonal polynomials as basis function. Illustrative examples are included to demonstrate the validity and applicability of the technique and the tables of results presented reveal that the absolute error decreases when the degree of approximation increases. Furthermore, since basis functions constructed are polynomials, the values of the integrals for the nonlinear integro differential equations are calculated as approximately close to the exact solutions.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Atkinson, K. (1997) Numerical Solution of Integral Equation of the Second Kind. Cambridge University Press, Cambridge. https://doi.org/10.1017/CBO9780511626340

[2] Venkatesh, S.G., Ayyaswamy, S.K. and Raja Balachandar, S. (2012) Legendre Approximation Solution for a Class of Higher-Order Volterra Integro-Differntial Equations. *Ain Shams Engineering Journal*, **3**, 417-422. https://doi.org/10.1016/j.asej.2012.04.007

[3] Dehghan, M. and Shakeri, F. (2008) Solution of an Integro-Differential Arising in Oscillating Magnetic Field Using Homotopy Perturbation Method. *Progressive in Electromagnetic Research*, **78**, 361-376. https://doi.org/10.2528/PIER07090403

[4] Jangveladze, T., Kiguradz, Z. and Neto, B. (2011) Galerkin Finite Element Method for One Non-Linear Integro-Differential Model. *Applied Mathematics and Computation*, **217**, 6883-6892. https://doi.org/10.1016/j.amc.2011.01.053

[5] Maleknejad, K. and Tavassoli-Kajani, M. (2011) Solving Linear Integro-Differential Equations System by Galerkin Method with Hybrid Functions. *Applied Mathematics and Computation*, **150**, 603-612. https://doi.org/10.1016/j.amc.2003.10.046

[6] Ghasemi, T., Tavassol-Kajani, M. and Babolian, E. (2007) Numerical Solution of Nonlinear Integro-Differential Equations. Wave-Galerkin Method and Homotopy Perturbation Method. *Applied Mathematics and Computation*, **188**, 450-455. https://doi.org/10.1016/j.amc.2006.10.001

[7] Fakhar-Izadi, F. and Dehghan, M. (2011) An Efficient Pseudo-Spectral Legendre Galarkin Method for Solving a Nonlinear Integro-Differetial Equations Arising in Population Dynamics. *Mathematical Method for Applied Sciences*, **30**, 1485-1511. https://doi.org/10.1002/mma.2698

[8] Tomassiello, S.A. (2011) A Note on Three Numerical Procedures to Solve Volterra Integro-Differential Equations on Structural Analysis. *Applied Mathematics and Computation*, **62**, 3188-3193. https://doi.org/10.1016/j.camwa.2011.08.031