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Exploration of Traversable Wormholes Sustained by an Extra Spatial Dimension

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Abstract

The main goal of this paper is to determine the effect of an extra dimension on a traversable wormhole. Here an earlier study by the author [Phys. Rev. D **98**, 064041 (2018)] is extended in several significant ways. To begin with, the extra spatial dimension is assumed to be time dependent, while the redshift and shape functions, as well as the extra dimension, are functions of both r and l, the respective radial and extra coordinates; the last of these is therefore a function of r, l, and t. The main objective is to determine the conditions that allow the throat of the wormhole to be threaded with ordinary matter (by respecting the null energy condition) and that the same conditions lead to a violation of the null energy condition in the fifth dimension, which is therefore responsible for sustaining the wormhole. The dependence of the extra dimension on l and t is subject to additional conditions that are subsequently analyzed in this paper. Finally, the extra dimension may be extremely small or even curled up.

Keywords

Traversable Wormholes, Extra Spatial Dimension, Time Dependence

1. Introduction

While wormholes are as good a prediction of Einstein's theory as black holes, such structures can only be maintained by violating the null energy condition, requiring the use of "exotic matter" [1]. Such wormholes would be subject to extreme fine-tuning of the metric coefficients in order to exist [2]. Another possibility for the theoretical construction is the use of phantom dark energy, which is known to violate the null energy condition [3] [4].

Other departures from classical relativity have been suggested for dealing with the energy violation. Refs. [5] and [6], as well as the present paper, rely on an extra spatial dimension, discussed further below. Part of the motivation comes from Lobo and Oliveira [7] in the context of $f(R)$ modified gravity: in principle, a wormhole could be constructed from ordinary matter, while the unavoidable violation of the null energy condition can be attributed to the effect of the modified gravitational theory. In analogous fashion, Refs. [8] [9] show that ordinary matter may be allowed since the violation of the null energy condition is due to the noncommutative-geometry background.

A suitable model for a five- or higher-dimensional wormhole is suggested by the following classical line element in Schwarzschild coordinates:

$$\mathrm{d}s^2 = -\mathrm{e}^{2\Phi(r)}\mathrm{d}t^2 + \mathrm{e}^{2\lambda(r)}\mathrm{d}r^2 + r^2 S_2^2, \tag{1}$$

where $S_2^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2$. In a seemingly natural extension, the two-sphere S_2^2 is replaced by the n-sphere S_n^2, where

$$\mathrm{d}S_n^2 = \mathrm{d}\theta_1^2 + \sin^2\theta_1\mathrm{d}\theta_2 + \sin^2\theta_1\sin^2\theta_2\mathrm{d}\theta_3 + \cdots + \prod_{i=1}^{n-1}\sin^2\theta_i\mathrm{d}\theta_n^2 \tag{2}$$

[10] [11]. For a 3-sphere, we can use the familiar notation

$$S_3^2 = \mathrm{d}\theta^2 + \sin^2\theta\left(\mathrm{d}\phi^2 + \sin^2\phi\mathrm{d}\alpha^2\right). \tag{3}$$

This form arises in the classical Friedmann-Lemaître-Robertson-Walker (FLRW) model

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \left[a(t)\right]^2\left[\frac{\mathrm{d}r^2}{1-kr^2} + \mathrm{d}\theta^2 + \sin^2\theta\left(\mathrm{d}\phi^2 + \sin^2\phi\mathrm{d}\alpha^2\right)\right]. \tag{4}$$

Wormhole solutions based on line element (2) are discussed in Ref. [8], given a noncommutative-geometry background.

Since Morris-Thorne wormholes do not normally require a cosmological setting, it seems more natural to consider the line element proposed in Ref. [5]:

$$\mathrm{d}s^2 = -\mathrm{e}^{2\Phi(r)}\mathrm{d}t^2 + \mathrm{e}^{2\lambda(r)}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right) + \mathrm{e}^{2\mu(r,l)}\mathrm{d}l^2, \tag{5}$$

where l is the extra coordinate. (We are using units in which $c = G = 1$.)

The idea of an extra spatial dimension had its origin in the Kaluza-Klein theory and eventually led to the compactified extra dimensions in string theory. That an extra dimension may be macroscopic was proposed by Paul Wesson in a series of papers [12] [13] [14] [15], leading to the *induced-matter theory*: the field equations for a five-dimensional totally flat space yield the Einstein field equations in four dimensions containing matter. In this paper, we assume that the extra dimension may be time dependent and that no *a priori* restriction is to be placed on the size.

Of particular interest to us is the form of the line element in Ref. [12]:

$$\mathrm{d}s^2 = \left(l/L^2\right)g_{\alpha\beta}\left(x^\gamma, l\right)\mathrm{d}x^\alpha\mathrm{d}x^\beta \pm \mathrm{d}l^2. \tag{6}$$

Here $\mathrm{d}l^2$ can take on the more general form $A\left(x^\gamma, l\right)\mathrm{d}l^2$. To emphasize the dependence of the metric coefficients on l, we will employ the following line element for a wormhole based on an extension of Equation (5):

$$ds^2 = -e^{2\Phi(r,l)}dt^2 + e^{2\lambda(r,l)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) + e^{2\mu(r,l,t)}dl^2. \tag{7}$$

(To make our results as general as possible, we also assume that μ may be time dependent.)

Before continuing, let us recall the basic properties of a Morris-Thorne wormhole. Letting $e^{2\lambda(r,l)} = 1 - b(r,l)/r$, the line element becomes

$$ds^2 = -e^{2\Phi(r,l)}dt^2 + \frac{dr^2}{1 - b(r,l)/r} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) + e^{2\mu(r,l,t)}dl^2. \tag{8}$$

Here $\Phi = \Phi(r,l)$ is called the *redshift function*, which must be finite everywhere to prevent the occurrence of an event horizon; $b = b(r,l)$ is called the *shape function*. The spherical surface $r = r_0$ is called the *throat* of the wormhole. The shape function must satisfy the following conditions: $b(r_0,l) = r_0$ and $\partial b(r_0,l)/\partial r < 1$, called the *flare-out condition* in Ref. [1]. These conditions can only be met by violating the null energy condition (NEC), which states that for the energy momentum tensor $T_{\alpha\beta}$,

$$T_{\alpha\beta}k^{\alpha}k^{\beta} \geq 0 \text{ for all null vectors } k^{\alpha}. \tag{9}$$

To see the significance in a wormhole setting, we assume an orthonormal frame and note that $T_{00} = \rho$ is the energy density and $T_{11} = p_r$ is the radial pressure. Now the outgoing radial null vector $(1,1,0,0)$ yields $\rho + p_r < 0$ whenever the NEC is violated.

Our new model, Equation (7), greatly extends the discussion in Ref. [5], which assumes that the metric coefficients are independent of *l*. According to Ref. [12], the dependence on *l* is associated with the presence of matter, whose nature can be determined by reducing the 5D Ricci-flat equation $R_{AB} = 0$ to the 4D Einstein equation $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$.

As noted earlier, our main goal is to determine the conditions that allow the throat of the wormhole to be threaded with ordinary matter and that the same conditions result in a violation of the NEC in the fifth dimension. The conditions discussed include the effect of the time dependence.

2. The Solution

In this section we obtain the general solution from line element (8) and study the consequences in subsequent sections. We choose an orthonormal basis $\{e_{\hat{\alpha}}\}$ which is dual to the following 1-form basis:

$$\begin{gathered}\theta^0 = e^{\Phi(r,l)}dt, \quad \theta^1 = \left[1 - \frac{b(r,l)}{r}\right]^{-1/2} dr, \quad \theta^2 = r\,d\theta, \\ \theta^3 = r\sin\theta\,d\phi, \quad \theta^4 = e^{\mu(r,l,t)}dl.\end{gathered} \tag{10}$$

Alternatively,

$$\begin{gathered}dt = e^{-\Phi(r,l)}\theta^0, \quad dr = \left[1 - \frac{b(r,l)}{r}\right]^{1/2}\theta^1, \quad d\theta = \frac{1}{r}\theta^2, \\ d\phi = \frac{1}{r\sin\theta}\theta^3, \quad dl = e^{-\mu(r,l,t)}\theta^4.\end{gathered} \tag{11}$$

Following Ref. [16], we use the method of differential forms to obtain the connection 1-forms, the curvature 2-forms, and the resulting components of the Riemann curvature tensor. To that end, we calculate the exterior derivatives in terms of the basis $\{\theta^i\}$:

$$\mathrm{d}\theta^0 = \frac{\partial\Phi(r,l)}{\partial r}\left[1-\frac{b(r,l)}{r}\right]^{1/2}\theta^1\wedge\theta^0 + \frac{\partial\Phi(r,l)}{\partial l}\mathrm{e}^{-\mu(r,l,t)}\theta^4\wedge\theta^0, \tag{12}$$

$$\mathrm{d}\theta^1 = \frac{1}{2r}\left[1-\frac{b(r,l)}{r}\right]^{-1}\frac{\partial b(r,l)}{\partial l}\mathrm{e}^{-\mu(r,l,t)}\theta^4\wedge\theta^1, \tag{13}$$

$$\mathrm{d}\theta^2 = \frac{1}{r}\left[1-\frac{b(r,l)}{r}\right]^{1/2}\theta^1\wedge\theta^2, \tag{14}$$

$$\mathrm{d}\theta^3 = \frac{1}{r}\left[1-\frac{b(r,l)}{r}\right]^{1/2}\theta^1\wedge\theta^3 + \frac{1}{r}\cot\theta\,\theta^2\wedge\theta^3, \tag{15}$$

$$\mathrm{d}\theta^4 = \frac{\partial\mu(r,l,t)}{\partial r}\left[1-\frac{b(r,l)}{r}\right]^{1/2}\theta^1\wedge\theta^4 + \mathrm{e}^{-\Phi(r,l)}\frac{\partial\mu(r,l,t)}{\partial t}\theta^0\wedge\theta^4. \tag{16}$$

The connection 1-forms ω^i_k have the symmetry $\omega^0_i = \omega^i_0\ (i=1,2,3,4)$ and $\omega^i_j = -\omega^j_i\ (i,j=1,2,3,4, i\neq j)$, and are related to the basis $\{\theta^i\}$ by

$$\mathrm{d}\theta^i = -\omega^i_k\wedge\theta^k. \tag{17}$$

The solution of this system is

$$\omega^0_1 = \frac{\mathrm{d}\Phi(r,l)}{\mathrm{d}r}\left[1-\frac{b(r,l)}{r}\right]^{1/2}\theta^0, \tag{18}$$

$$\omega^0_4 = \frac{\partial\Phi(r,l)}{\partial l}\mathrm{e}^{-\mu(r,l,t)}\theta^0 + \mathrm{e}^{-\Phi(r,l)}\frac{\partial\mu(r,l,t)}{\partial t}\theta^4, \tag{19}$$

$$\omega^2_1 = \frac{1}{r}\left[1-\frac{b(r,l)}{r}\right]^{1/2}\theta^2, \tag{20}$$

$$\omega^3_1 = \frac{1}{r}\left[1-\frac{b(r,l)}{r}\right]^{1/2}\theta^3, \tag{21}$$

$$\omega^3_2 = \frac{1}{r}\cot\theta\,\theta^3, \tag{22}$$

$$\omega^4_1 = \frac{\partial\mu(r,l,t)}{\partial r}\left[1-\frac{b(r,l)}{r}\right]^{1/2}\theta^4 - \frac{1}{2r}\left[1-\frac{b(r,l)}{r}\right]^{-1}\frac{\partial b(r,l)}{\partial l}\mathrm{e}^{-\mu(r,l,t)}\theta^1, \tag{23}$$

$$\omega^0_2 = \omega^0_3 = \omega^2_4 = \omega^3_4 = 0. \tag{24}$$

The curvature 2-forms Ω^i_j are calculated directly from the Cartan structural equations

$$\Omega^i_j = \mathrm{d}\omega^i_j + \omega^i_k\wedge\omega^k_j. \tag{25}$$

and are listed next. (To shorten the expressions, we will denote $\Phi(r,l)$ by Φ, $b(r,l)$ by b, and $\mu(r,l,t)$ by μ.)

$$\begin{aligned}\Omega_1^0 =& \left[-\frac{\partial^2\Phi}{\partial r^2}\left(1-\frac{b}{r}\right)+\frac{1}{2r^2}\frac{\partial\Phi}{\partial r}\left(r\frac{\partial b}{\partial r}-b\right)-\left(\frac{\partial\Phi}{\partial r}\right)^2\left(1-\frac{b}{r}\right)\right.\\ &\left.-\frac{1}{2r}\mathrm{e}^{-2\mu}\frac{\partial\Phi}{\partial l}\frac{\partial b}{\partial l}\left(1-\frac{b}{r}\right)^{-1}\right]\theta^0\wedge\theta^1\\ &+\mathrm{e}^{-\mu}\left[-\frac{\partial^2\Phi}{\partial r\partial l}\left(1-\frac{b}{r}\right)^{1/2}+\frac{1}{2r}\frac{\partial\Phi}{\partial r}\frac{\partial b}{\partial l}\left(1-\frac{b}{r}\right)^{-1/2}-\frac{\partial\Phi}{\partial r}\frac{\partial\Phi}{\partial l}\left(1-\frac{b}{r}\right)^{1/2}\right.\\ &\left.+\frac{\partial\Phi}{\partial l}\frac{\partial\mu}{\partial r}\left(1-\frac{b}{r}\right)^{1/2}\right]\theta^0\wedge\theta^4+\frac{1}{2r}\mathrm{e}^{-\Phi}\mathrm{e}^{-\mu}\frac{\partial b}{\partial l}\frac{\partial\mu}{\partial t}\left(1-\frac{b}{r}\right)^{-1}\theta^1\wedge\theta^4,\end{aligned} \tag{26}$$

$$\Omega_2^0 = -\frac{1}{r}\frac{\partial\Phi}{\partial r}\left(1-\frac{b}{r}\right)\theta^0\wedge\theta^2, \tag{27}$$

$$\Omega_3^0 = -\frac{1}{r}\frac{\partial\Phi}{\partial r}\left(1-\frac{b}{r}\right)\theta^0\wedge\theta^3, \tag{28}$$

$$\begin{aligned}\Omega_4^0 =& \mathrm{e}^{-\mu}\left(1-\frac{b}{r}\right)^{1/2}\left(-\frac{\partial^2\Phi}{\partial r\partial l}+\frac{\partial\Phi}{\partial l}\frac{\partial\mu}{\partial r}-\frac{\partial\Phi}{\partial r}\frac{\partial\Phi}{\partial l}\right)\theta^0\wedge\theta^1\\ &+\frac{1}{2r}\mathrm{e}^{-\mu}\frac{\partial\Phi}{\partial r}\frac{\partial b}{\partial l}\left(1-\frac{b}{r}\right)^{-1/2}\theta^0\wedge\theta^1\\ &+\left[-\mathrm{e}^{-2\mu}\frac{\partial^2\Phi}{\partial l^2}+\mathrm{e}^{-2\mu}\frac{\partial\Phi}{\partial l}\frac{\partial\mu}{\partial l}-\mathrm{e}^{-2\mu}\left(\frac{\partial\Phi}{\partial l}\right)^2+\mathrm{e}^{-2\Phi}\frac{\partial^2\mu}{\partial t^2}\right.\\ &\left.+\mathrm{e}^{-2\Phi}\left(\frac{\partial\mu}{\partial t}\right)^2-\frac{\partial\Phi}{\partial r}\frac{\partial\mu}{\partial r}\left(1-\frac{b}{r}\right)\right]\theta^0\wedge\theta^4\\ &+\mathrm{e}^{-\Phi}\left(1-\frac{b}{r}\right)^{1/2}\left(-\frac{\partial\Phi}{\partial r}\frac{\partial\mu}{\partial t}+\frac{\partial^2\mu}{\partial r\partial t}+\frac{\partial\mu}{\partial r}\frac{\partial\mu}{\partial t}\right)\theta^1\wedge\theta^4,\end{aligned} \tag{29}$$

$$\Omega_2^1 = \frac{1}{2r^3}\left(r\frac{\partial b}{\partial r}-b\right)\theta^1\wedge\theta^2-\frac{1}{2r^2}\mathrm{e}^{-\mu}\frac{\partial b}{\partial l}\left(1-\frac{b}{r}\right)^{-1/2}\theta^2\wedge\theta^4, \tag{30}$$

$$\Omega_3^1 = \frac{1}{2r^3}\left(r\frac{\partial b}{\partial r}-b\right)\theta^1\wedge\theta^3-\frac{1}{2r^2}\mathrm{e}^{-\mu}\frac{\partial b}{\partial l}\left(1-\frac{b}{r}\right)^{-1/2}\theta^3\wedge\theta^4, \tag{31}$$

$$\begin{aligned}\Omega_4^1 =& \left[-\frac{\partial^2\mu}{\partial r^2}\left(1-\frac{b}{r}\right)+\frac{1}{2r^2}\frac{\partial\mu}{\partial r}\left(r\frac{\partial b}{\partial r}-b\right)-\left(\frac{\partial\mu}{\partial r}\right)^2\left(1-\frac{b}{r}\right)\right.\\ &-\frac{3}{4r^2}\mathrm{e}^{-2\mu}\left(1-\frac{b}{r}\right)^{-2}\left(\frac{\partial b}{\partial l}\right)^2-\frac{1}{2r}\mathrm{e}^{-2\mu}\left(1-\frac{b}{r}\right)^{-1}\frac{\partial^2 b}{\partial l^2}\\ &\left.+\frac{1}{2r}\mathrm{e}^{-2\mu}\left(1-\frac{b}{r}\right)^{-1}\frac{\partial b}{\partial l}\frac{\partial\mu}{\partial l}\right]\theta^1\wedge\theta^4+\left[-\mathrm{e}^{-\Phi}\frac{\partial^2\mu}{\partial r\partial t}\left(1-\frac{b}{r}\right)^{1/2}\right.\\ &\left.-\mathrm{e}^{-\Phi}\frac{\partial\mu}{\partial r}\frac{\partial\mu}{\partial t}\left(1-\frac{b}{r}\right)^{1/2}+\mathrm{e}^{-\Phi}\frac{\partial\Phi}{\partial r}\frac{\partial\mu}{\partial t}\left(1-\frac{b}{r}\right)^{1/2}\right]\theta^0\wedge\theta^4\\ &-\frac{1}{2r}\mathrm{e}^{-\Phi}\mathrm{e}^{-\mu}\frac{\partial\mu}{\partial t}\frac{\partial b}{\partial l}\left(1-\frac{b}{r}\right)^{-1}\theta^0\wedge\theta^1,\end{aligned} \tag{32}$$

$$\Omega_3^2 = \frac{b}{r^3}\theta^2\wedge\theta^3, \tag{33}$$

$$\Omega_4^2 = -\frac{1}{r}\frac{\partial\mu}{\partial r}\left(1-\frac{b}{r}\right)\theta^2\wedge\theta^4 - \frac{1}{2r^2}\mathrm{e}^{-\mu}\left(1-\frac{b}{r}\right)^{-1/2}\frac{\partial b}{\partial l}\theta^1\wedge\theta^2, \tag{34}$$

$$\Omega_4^3 = -\frac{1}{r}\frac{\partial\mu}{\partial r}\left(1-\frac{b}{r}\right)\theta^3\wedge\theta^4 - \frac{1}{2r^2}\mathrm{e}^{-\mu}\left(1-\frac{b}{r}\right)^{-1/2}\frac{\partial b}{\partial l}\theta^1\wedge\theta^3. \tag{35}$$

The components of the Riemann curvature tensor can be read off directly from the form

$$\Omega_j^i = -\frac{1}{2}R_{mnj}^i\theta^m\wedge\theta^n \tag{36}$$

and are listed next:

$$\begin{aligned} R_{011}^0 &= \frac{\partial^2\Phi(r,l)}{\partial r^2}\left(1-\frac{b(r,l)}{r}\right) - \frac{1}{2r^2}\frac{\partial\Phi(r,l)}{\partial r}\left(r\frac{\partial b(r,l)}{\partial r}-b(r,l)\right) \\ &\quad + \left(\frac{\partial\Phi(r,l)}{\partial r}\right)^2\left(1-\frac{b(r,l)}{r}\right) + \frac{1}{2r}\mathrm{e}^{-2\mu(r,l,t)}\frac{\partial\Phi(r,l)}{\partial l}\left(1-\frac{b(r,l)}{r}\right)^{-1}\frac{\partial b(r,l)}{\partial l}, \end{aligned} \tag{37}$$

$$R_{022}^0 = R_{033}^0 = \frac{1}{r}\frac{\partial\Phi(r,l)}{\partial r}\left(1-\frac{b(r,l)}{r}\right), \tag{38}$$

$$\begin{aligned} R_{044}^0 &= \mathrm{e}^{-2\mu(r,l,t)}\left[\frac{\partial^2\Phi(r,l)}{\partial l^2} - \frac{\partial\Phi(r,l)}{\partial l}\frac{\partial\mu(r,l,t)}{\partial l} + \left(\frac{\partial\Phi(r,l)}{\partial l}\right)^2\right] \\ &\quad - \mathrm{e}^{-2\Phi(r,l)}\left[\frac{\partial^2\mu(r,l,t)}{\partial t^2} + \left(\frac{\partial\mu(r,l,t)}{\partial t}\right)^2\right] + \frac{\partial\Phi(r,l)}{\partial r}\frac{\partial\mu(r,l,t)}{\partial r}\left(1-\frac{b(r,l)}{r}\right), \end{aligned} \tag{39}$$

$$R_{122}^1 = R_{133}^1 = -\frac{1}{2r^3}\left(r\frac{\partial b(r,l)}{\partial r}-b(r,l)\right), \tag{40}$$

$$\begin{aligned} R_{144}^1 &= \frac{\partial^2\mu(r,l,t)}{\partial r^2}\left(1-\frac{b(r,l)}{r}\right) + \left(\frac{\partial\mu(r,l,t)}{\partial r}\right)^2\left(1-\frac{b(r,l)}{r}\right) \\ &\quad - \frac{1}{2r^2}\frac{\partial\mu(r,l,t)}{\partial r}\left(r\frac{\partial b(r,l)}{\partial r}-b(r,l)\right) \\ &\quad + \frac{3}{4r^2}\mathrm{e}^{-2\mu(r,l,t)}\left(1-\frac{b(r,l)}{r}\right)^{-2}\left(\frac{\partial b(r,l)}{\partial l}\right)^2 \\ &\quad + \frac{1}{2r}\mathrm{e}^{-2\mu(r,l,t)}\left(1-\frac{b(r,l)}{r}\right)^{-1}\frac{\partial^2 b(r,l)}{\partial l^2} \\ &\quad - \frac{1}{2r}\mathrm{e}^{-2\mu(r,l,t)}\left(1-\frac{b(r,l)}{r}\right)^{-1}\frac{\partial b(r,l)}{\partial l}\frac{\partial\mu(r,l,t)}{\partial l}, \end{aligned} \tag{41}$$

$$R_{233}^2 = -\frac{b(r,l)}{r^3}, \tag{42}$$

$$R_{244}^2 = R_{344}^3 = \frac{1}{r}\frac{\partial\mu(r,l,t)}{\partial r}\left(1-\frac{b(r,l)}{r}\right), \tag{43}$$

$$R_{041}^0 = \mathrm{e}^{-\mu(r,l,t)}\left(1-\frac{b(r,l)}{r}\right)^{1/2}\left[\frac{\partial^2\Phi(r,l)}{\partial r\partial l} + \frac{\partial\Phi(r,l)}{\partial r}\frac{\partial\Phi(r,l)}{\partial l}\right.$$

$$-\frac{\partial\Phi(r,l)}{\partial l}\frac{\partial\mu(r,l,t)}{\partial r}\Bigg]-\frac{1}{2r}\mathrm{e}^{-\mu(r,l,t)}\frac{\partial\Phi(r,l)}{\partial r}\left(1-\frac{b(r,l)}{r}\right)^{-1/2}\frac{\partial b(r,l)}{\partial l}, \tag{44}$$

$$R^{0}_{411}=\frac{1}{2r}\mathrm{e}^{-\Phi(r,l)}\mathrm{e}^{-\mu(r,l,t)}\frac{\partial\mu(r,l,t)}{\partial t}\left(1-\frac{b(r,l)}{r}\right)^{-1}\frac{\partial b(r,l)}{\partial l}, \tag{45}$$

$$R^{1}_{044}=\mathrm{e}^{-\Phi(r,l)}\left(1-\frac{b(r,l)}{r}\right)^{1/2}\left[-\frac{\partial\Phi(r,l)}{\partial r}\frac{\partial\mu(r,l,t)}{\partial t}+\frac{\partial^2\mu(r,l,t)}{\partial r\partial t}+\frac{\partial\mu(r,l,t)}{\partial r}\frac{\partial\mu(r,l,t)}{\partial t}\right], \tag{46}$$

$$R^{2}_{214}=R^{3}_{314}=-\frac{1}{2r^2}\mathrm{e}^{-\mu(r,l,t)}\left(1-\frac{b(r,l)}{r}\right)^{-1/2}\frac{\partial b(r,l)}{\partial l}. \tag{47}$$

The last form to be derived in this section is the Ricci tensor, which is obtained by a trace on the Riemann curvature tensor:

$$R_{ab}=R^{c}_{acb}. \tag{48}$$

The various components are

$$\begin{aligned}R_{00}=&\frac{\partial^2\Phi(r,l)}{\partial r^2}\left(1-\frac{b(r,l)}{r}\right)-\frac{1}{2r^2}\frac{\partial\Phi(r,l)}{\partial r}\left(r\frac{\partial b(r,l)}{\partial r}-b(r,l)\right)\\&+\left(\frac{\partial\Phi(r,l)}{\partial r}\right)^2\left(1-\frac{b(r,l)}{r}\right)+\frac{1}{2r}\mathrm{e}^{-2\mu(r,l,t)}\frac{\partial\Phi(r,l)}{\partial l}\left(1-\frac{b(r,l)}{r}\right)^{-1}\frac{\partial b(r,l)}{\partial l}\\&+\frac{2}{r}\frac{\partial\Phi(r,l)}{\partial r}\left(1-\frac{b(r,l)}{r}\right)+\frac{\partial^2\Phi(r,l)}{\partial l^2}\mathrm{e}^{-2\mu(r,l,t)}-\frac{\partial\Phi(r,l)}{\partial l}\frac{\partial\mu(r,l,t)}{\partial l}\mathrm{e}^{-2\mu(r,l,t)}\\&+\left(\frac{\partial\Phi(r,l)}{\partial l}\right)^2\mathrm{e}^{-2\mu(r,l,t)}-\mathrm{e}^{-2\Phi(r,l)}\frac{\partial^2\mu(r,l,t)}{\partial t^2}\\&-\mathrm{e}^{-2\Phi(r,l)}\left(\frac{\partial\mu(r,l,t)}{\partial t}\right)^2+\frac{\partial\Phi(r,l)}{\partial r}\frac{\partial\mu(r,l,t)}{\partial r}\left(1-\frac{b(r,l)}{r}\right),\end{aligned} \tag{49}$$

$$\begin{aligned}R_{11}=&-\frac{\partial^2\Phi(r,l)}{\partial r^2}\left(1-\frac{b(r,l)}{r}\right)+\frac{1}{2r^2}\frac{\partial\Phi(r,l)}{\partial r}\left(r\frac{\partial b(r,l)}{\partial r}-b(r,l)\right)\\&-\left(\frac{\partial\Phi(r,l)}{\partial r}\right)^2\left(1-\frac{b(r,l)}{r}\right)-\frac{1}{2r}\mathrm{e}^{-2\mu(r,l,t)}\frac{\partial\Phi(r,l)}{\partial l}\left(1-\frac{b(r,l)}{r}\right)^{-1}\frac{\partial b(r,l)}{\partial l}\\&+\frac{1}{r^3}\left(r\frac{\partial b(r,l)}{\partial r}-b(r,l)\right)-\frac{\partial^2\mu(r,l,t)}{\partial r^2}\left(1-\frac{b(r,l)}{r}\right)\\&-\left(\frac{\partial\mu(r,l,t)}{\partial r}\right)^2\left(1-\frac{b(r,l)}{r}\right)+\frac{1}{2r^2}\frac{\partial\mu(r,l,t)}{\partial r}\left(r\frac{\partial b(r,l)}{\partial r}-b(r,l)\right)\\&-\frac{3}{4r^2}\left(1-\frac{b(r,l)}{r}\right)^{-2}\left(\frac{\partial b(r,l)}{\partial l}\right)^2\mathrm{e}^{-2\mu(r,l,t)}\\&-\frac{1}{2r}\left(1-\frac{b(r,l)}{r}\right)^{-1}\frac{\partial^2 b(r,l)}{\partial l^2}\mathrm{e}^{-2\mu(r,l,t)}\end{aligned}$$

$$+\frac{1}{2r}\left(1-\frac{b(r,l)}{r}\right)^{-1}\frac{\partial b(r,l)}{\partial l}\frac{\partial \mu(r,l,t)}{\partial l}\mathrm{e}^{-\mu(r,l,t)}, \tag{50}$$

$$\begin{aligned}R_{22}=R_{33}=&-\frac{1}{r}\frac{\partial \Phi(r,l)}{\partial r}\left(1-\frac{b(r,l)}{r}\right)+\frac{1}{2r^3}\left(r\frac{\partial b(r,l)}{\partial r}-b(r,l)\right)\\&+\frac{b(r,l)}{r^3}-\frac{1}{r}\left(1-\frac{b(r,l)}{r}\right)\frac{\partial \mu(r,l,t)}{\partial r},\end{aligned} \tag{51}$$

$$\begin{aligned}R_{44}=&-\frac{\partial^2\Phi(r,l)}{\partial l^2}\mathrm{e}^{-2\mu(r,l,t)}+\frac{\partial \Phi(r,l)}{\partial l}\frac{\partial \mu(r,l,t)}{\partial l}\mathrm{e}^{-2\mu(r,l,t)}\\&-\left(\frac{\partial \Phi(r,l)}{\partial l}\right)^2\mathrm{e}^{-2\mu(r,l,t)}+\mathrm{e}^{-2\Phi(r,l)}\frac{\partial^2\mu(r,l,t)}{\partial t^2}\\&+\mathrm{e}^{-2\Phi(r,l)}\left(\frac{\partial \mu(r,l,t)}{\partial t}\right)^2-\frac{\partial \Phi(r,l)}{\partial r}\frac{\partial \mu(r,l,t)}{\partial r}\left(1-\frac{b(r,l)}{r}\right)\\&-\frac{\partial^2\mu(r,l,t)}{\partial r^2}\left(1-\frac{b(r,l)}{r}\right)-\left(\frac{\partial \mu(r,l,t)}{\partial r}\right)^2\left(1-\frac{b(r,l)}{r}\right)\\&+\frac{1}{2r^2}\frac{\partial \mu(r,l,t)}{\partial r}\left(r\frac{\partial b(r,l)}{\partial r}-b(r,l)\right)\\&-\frac{3}{4r^2}\left(1-\frac{b(r,l)}{r}\right)^{-2}\left(\frac{\partial b(r,l)}{\partial l}\right)^2\mathrm{e}^{-2\mu(r,l,t)}\\&-\frac{1}{2r}\left(1-\frac{b(r,l)}{r}\right)^{-1}\frac{\partial^2 b(r,l)}{\partial l^2}\mathrm{e}^{-2\mu(r,l,t)}\\&+\frac{1}{2r}\left(1-\frac{b(r,l)}{r}\right)^{-1}\frac{\partial b(r,l)}{\partial l}\frac{\partial \mu(r,l,t)}{\partial l}\mathrm{e}^{-2\mu(r,l,t)}\\&-\frac{2}{r}\left(1-\frac{b(r,l)}{r}\right)\frac{\partial \mu(r,l,t)}{\partial r},\end{aligned} \tag{52}$$

$$\begin{aligned}R_{01}=&\left(1-\frac{b(r,l)}{r}\right)^{1/2}\left[\mathrm{e}^{-\Phi(r,l)}\frac{\partial \Phi(r,l)}{\partial r}\frac{\partial \mu(r,l,t)}{\partial t}-\mathrm{e}^{-\Phi(r,l)}\frac{\partial^2\mu(r,l,t)}{\partial r\partial t}\right.\\&\left.-\mathrm{e}^{-\Phi(r,l)}\frac{\partial \mu(r,l,t)}{\partial r}\frac{\partial \mu(r,l,t)}{\partial t}\right],\end{aligned} \tag{53}$$

$$R_{04}=\frac{1}{2r}\mathrm{e}^{-\Phi(r,l)}\left(1-\frac{b(r,l)}{r}\right)^{-1}\frac{\partial b(r,l)}{\partial l}\frac{\partial \mu(r,l,t)}{\partial t}\mathrm{e}^{-\mu(r,l,t)}. \tag{54}$$

These forms will come into play in the next section.

3. A Possible Curvature Singularity

In discussing the structure of a wormhole, we recall that at the throat, we have $b(r_0,l)=r_0$ for every l, leading to the question of its physical interpretation. The problem takes care of itself in the following sense: consider the Ricci scalar

$$R=R_i^i=-R_{00}+R_{11}+R_{22}+R_{33}+R_{44}. \tag{55}$$

Using Equation (48), this can be written in the following convenient form:

$$\frac{1}{2}R=-R_{011}^{0}-R_{022}^{0}-R_{033}^{0}-R_{044}^{0}-R_{122}^{1}-R_{133}^{1}-R_{144}^{1}-R_{233}^{2}-R_{244}^{2}-R_{344}^{3}. \quad (56)$$

It now becomes apparent that several of these components are undefined at the throat if we allow the shape function to have the form $b=b(r,l)$: since R is a scalar invariant, we have a curvature singularity at the throat. We conclude that b must be a function of r alone; since we now have $\partial b/\partial l\equiv 0$, the singularity disappears. So we return to $b=b(r)$ and $b(r_0)=r_0$. It is interesting to note that the redshift function can retain the proposed form $\Phi=\Phi(r,l)$.

4. The Null Energy Condition (NEC)

Let us recall from the Introduction that the NEC states that for the energy-momentum tensor $T_{\alpha\beta}$, $T_{\alpha\beta}k^{\alpha}k^{\beta}\geq 0$ for all null vectors k^{α}. We also recall that an ordinary Morris-Thorne wormhole can only be held open if this condition is violated, thereby requiring exotic matter. As noted earlier, our goal is to show that thanks to the extra spatial dimension, the wormhole throat can be threaded with ordinary matter, but there is a violation of the NEC in the fifth dimension.

To that end, we start with the four-dimensional null vector $(1,1,0,0)$. Continuing with our orthonormal frame, consider the Einstein field equations

$$G_{\hat{\alpha}\hat{\beta}}=R_{\hat{\alpha}\hat{\beta}}-\frac{1}{2}Rg_{\hat{\alpha}\hat{\beta}}=8\pi T_{\hat{\alpha}\hat{\beta}}, \quad (57)$$

where

$$g_{\hat{\alpha}\hat{\beta}}=\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}. \quad (58)$$

As before, $T_{00}=\rho$ is the energy density and $T_{11}=p_r$ is the radial pressure, but instead of $\rho+p_r$, we need to consider the more general form

$$\begin{aligned}8\pi T_{\alpha\beta}k^{\alpha}k^{\beta}&=G_{00}+G_{11}+2G_{01}\\&=\left[R_{00}-\frac{1}{2}R(-1)\right]+\left[R_{11}-\frac{1}{2}R(1)\right]+2\left[R_{01}-\frac{1}{2}R(0)\right]\\&=R_{00}+R_{11}+2R_{01}.\end{aligned} \quad (59)$$

Since we are primarily interested in the vicinity of the throat, we recall that $1-b(r_0)/r_0=0$. So by Equation (53), $R_{01}=0$ at the throat and we are back to the original $\rho+p_r$. So from Equations (49) and (50), we have (recalling that $\partial b/\partial l\equiv 0$)

$$\begin{aligned}8\pi(\rho+p_r)\big|_{r=r_0}&=\frac{b'(r_0)-1}{r_0^2}\left[1+\frac{r_0}{2}\frac{\partial\mu(r_0,l,t)}{\partial r}\right]+\frac{\partial^2\Phi(r_0,l)}{\partial l^2}e^{-2\mu(r_0,l,t)}\\&\quad-\frac{\partial\Phi(r_0,l)}{\partial l}\frac{\partial\mu(r_0,l,t)}{\partial l}e^{-2\mu(r_0,l,t)}+\left(\frac{\partial\Phi(r_0,l)}{\partial l}\right)^2e^{-2\mu(r_0,l,t)}\end{aligned}$$

$$-\mathrm{e}^{-2\Phi(r_0,l)}\frac{\partial^2\mu(r_0,l,t)}{\partial t^2}-\mathrm{e}^{-2\Phi(r_0,l)}\left(\frac{\partial\mu(r_0,l,t)}{\partial t}\right)^2. \quad (60)$$

To make the analysis tractable, let us assume for now that our wormhole is time independent; so it is convenient to use the notation $\mu=\mu(r_0,l)$. If, in addition, Φ is independent of *l*, then Equation (60) becomes

$$8\pi(\rho+p_r)\big|_{r=r_0}=\frac{b'(r_0)-1}{r_0^2}\left[1+\frac{r_0}{2}\frac{\partial\mu(r_0,l)}{\partial r}\right], \quad (61)$$

which is the condition discussed in Ref. [5], and leads to the conclusion (since $b'(r_0)<1$) that

$$\rho+p_r>0 \text{ at } r=r_0$$

provided that

$$\frac{\partial\mu(r_0,l)}{\partial r}<-\frac{2}{r_0}. \quad (62)$$

The *l*-dependent terms in Equation (60) could strengthen the conclusion if

$$\frac{\partial^2\Phi(r_0,l)}{\partial l^2}>0 \text{ and } \frac{\partial\Phi(r_0,l)}{\partial l}<0. \quad (63)$$

Since Condition (63) depends on *l*, the conclusion goes well beyond Ref. [5] but it may have little practical significance by itself. On the other hand, Φ is a function of both *r* and *l*. So let us return to Inequality (62) and concentrate on the null vector $(1,0,0,0,1)$ in the five-dimensional space:

$$\begin{aligned}8\pi T_{\alpha\beta}k^\alpha k^\beta &= G_{00}+G_{44}+2G_{04}\\ &=\left[R_{00}-\frac{1}{2}R(-1)\right]+\left[R_{44}-\frac{1}{2}R(1)\right]+2\left[R_{04}-\frac{1}{2}R(0)\right]\\ &=R_{00}+R_{44}+2R_{04}.\end{aligned} \quad (64)$$

While $g_{04}=0$ from Equation (58), observe that by Equation (54), R_{04} is also equal to zero since $\partial b/\partial l\equiv 0$. Still assuming time independence, the condition $1-b(r_0)/r_0=0$ then yields

$$8\pi(\rho+p_r)\big|_{r=r_0}=\frac{1}{2}\frac{b'(r_0)-1}{r_0}\left[-\frac{\partial\Phi(r_0,l)}{\partial r}+\frac{\partial\mu(r_0,l)}{\partial r}\right]. \quad (65)$$

Inequality (62), $\partial\mu(r_0,l)/\partial r<-2/r_0$, implies that the second factor on the right-hand side of Equation (65) is positive if

$$\frac{\partial\Phi(r_0,l)}{\partial r}=-A<\frac{\partial\mu(r_0,l)}{\partial r}<-\frac{2}{r_0}, \quad (66)$$

strikingly similar to Inequality (62). Since $b'(r_0)<1$, it follows from Equation (65) that $\rho(r_0)+p_r(r_0)<0$ in the fifth dimension. So the NEC is indeed violated even though the throat of the wormhole is threaded with ordinary matter.

Remark: One physical consequence of the NEC is that it forces the local energy density to be positive. Also, in the four-dimensional case, the NEC must be satisfied for all null vectors if the throat is to be threaded with ordinary matter.

That these requirements can be met has already been shown in Ref. [5] and need not be repeated here.

5. The Violation of the NEC in the Fifth Dimension

In this section we are going to show that a violation of the NEC in the fifth dimension is indeed unavoidable. To see why, let us consider line element (8) with $b = b(r)$ and having the opposite signature (while still assuming time independence):

$$\mathrm{d}s^2 = \mathrm{e}^{2\Phi(r,l)}\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1-b(r)/r} - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right) - \mathrm{e}^{2\mu(r,l)}\mathrm{d}l^2. \tag{67}$$

Because of the symmetry, this would merely change the labels, making $\mu(r,l)$ the redshift function. However, since we are using the orthonormal basis in Section 2, the calculations are not affected. In particular, both Inequalities (62) and (66) remain the same. So the NEC is violated in the fifth dimension, even though the throat is threaded with ordinary matter.

6. The Size of the Extra Dimension and the Time-Dependence

Returning to line element (8), it is interesting to note that apart from the exponential functions, μ itself does not appear as a factor in the components of the Riemann curvature tensor, but only its derivatives. This allows μ to have virtually any magnitude. So if $\mu(r,l,t)$ is negative and large in absolute value, then $\mathrm{e}^{2\mu(r,l,t)}$ is necessarily small and may even be curled up.

Turning now to the time dependence, observe that in Equation (60), the time-dependent terms in the expression for $8\pi(\rho + p_r)$ can be written in the form

$$H = -\mathrm{e}^{-2\Phi(r_0,l)}\left[\frac{\partial^2\mu(r_0,l,t)}{\partial t^2} + \left(\frac{\partial\mu(r_0,l,t)}{\partial t}\right)^2\right]. \tag{68}$$

Suppose that $|\mu(r,l,t)|$ is relatively small. Then Equation (68) may lead to an interesting interpretation: being confined to a small range of values and changing with time, $\mu(r,l,t)$ could undergo a slow oscillation. Then H in Equation (68) is positive near any peak, where $\partial^2\mu(r,l,t)/\partial t^2 < 0$ and $\partial\mu(r,l,t)/\partial t$ is close to zero. This behavior could greatly increase the value of H during certain periods. So judging from Equation (60), there will be periods in which $\rho + p_r > 0$ at $r = r_0$ even if Inequality (62) is weakened.

7. Conclusions

An earlier paper by the author [5] assumes that Φ, b, and μ are functions of the radial coordinate r only. In this paper, we extend this model by assuming that Φ, b, and μ are functions of both r and l and that, in addition, μ is time dependent. It is shown that an unrestricted dependence on the fifth dimension could lead to a curvature singularity at the throat, forcing b to be a function of r only. The main goal is to determine the conditions under which the NEC is met in the four-dimensional setting, thereby allowing the throat of the wormhole to be threaded with ordinary matter. The same conditions lead to a violation of the NEC in the fifth dimension. This violation turns out to be unavoidable and is the key factor in sustaining the wormhole structure.

The dependence of Φ on l could favorably affect the ability to use ordinary matter near the throat. The same is true for the dependence of μ on t during certain periods. Finally, there is no restriction on the size of μ. So if μ is negative and large in absolute value, the extra dimension would be extremely small and may even be curled up.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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