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New Probability Distributions in Astrophysics: X. Truncation and Mass-Luminosity Relationship for the Frèchet Distribution

Lorenzo Zaninetti

Physics Department, Turin, Italy
Email: l.zaninetti@alice.it

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Abstract

The Frèchet distribution has aided the modelling of scientific data in many contexts. We demonstrate how it can be adapted to model astrophysical data. We analyze the truncated version of the Frèchet distribution deriving the probability density function (PDF), the distribution function, the average value, the *r*th moment about the origin, the median, the random generation of values and the maximum likelihood estimator, which allows us to derive the two unknown parameters. This first PDF in the regular and truncated version is then applied to model the mass of the stars. A canonical transformation from the mass to the luminosity allows us to derive a new PDF, which is derived in its regular and truncated version. Finally, we apply this new PDF model on the distribution in luminosity of NGC 2362.

Keywords

Stars: Normal, Stars: Luminosity Function, Mass Function Stars: Statistics

1. Introduction

The Frècet distribution, after [1], was first applied for the particle size distribution in powdered coal [2]. We report some efforts, among others, to derive the parameters of the Frècet distribution: [3] analyzed a quick estimator that differs from the matching moments method and the maximum likelihood estimator (MLE); [4] analyzed the MLE and the probability weighted moment estimation; and [5] explored the MLE, the method of matching moments, the percentile estimators, the L-moments, the ordinary and weighted least squares, the maximum product of spacing and the maximum goodness-of-fit estimators. The applications cover inter-facial damage in microelectronic packages and the material properties of constituent particles in an aluminum alloy [6]; the series of annual 1-day maximum rainfall [7]; and the total monthly rainfall [5]. The case of the Frècet distribution truncated at the right was introduced by [8] and that of the double truncation was carefully analyzed in [9].

The rest of this paper is structured as follows. It first reviews the two parameters of the Frècet distribution in the interval $[0,\infty]$, see Section 2, and then explores the bi-truncated case in Section 3. Section 4 transforms the standard and the truncated Frèchet distribution in mass into distributions in luminosity according to the well-known mass-luminosity relationship. Finally, the astrophysical applications to mass and luminosity for stars are reported in Section 5.

2. Regular Case

Let X be a random variable defined in $[0,\infty]$; the two parameter Frècet distribution function (DF), $F(x)$, is

$$F(x;b,\alpha) = \mathrm{e}^{-\left(\frac{x}{b}\right)^{-\alpha}}, \tag{1}$$

where b and α, both positive, are the scale and the shape parameters, respectively, see [1]. The probability density function (PDF), $f(x)$, is

$$f(x;b,\alpha) = \frac{\left(\frac{x}{b}\right)^{-\alpha}\alpha\,\mathrm{e}^{-\left(\frac{x}{b}\right)^{-\alpha}}}{x}. \tag{2}$$

We now introduce

$$GAMMA_r = \Gamma\left(\frac{\alpha - r}{\alpha}\right), \tag{3}$$

where r is an integer and $\Gamma(z)$ is the gamma function, which is defined as

$$\Gamma(z) = \int_0^\infty \mathrm{e}^{-t} t^{z-1} \mathrm{d}t. \tag{4}$$

The average value or mean, μ, is defined for $\alpha > 1$

$$\mu(b,\alpha) = b\Gamma_1, \tag{5}$$

the variance, σ^2, is defined for $\alpha > 2$

$$\sigma^2(b,\alpha) = b^2\left(-\Gamma_1^2 + \Gamma_2\right), \tag{6}$$

the skewness is defined for $\alpha > 3$

$$\mathrm{skewness}(b,\alpha) = \frac{2\Gamma_1^3 - 3\Gamma_2\Gamma_1 + \Gamma_3}{\left(-\Gamma_1^2 + \Gamma_2\right)^{\frac{3}{2}}}, \tag{7}$$

the kurtosis is defined for $\alpha > 4$

$$\mathrm{kurtosis}(b,\alpha) = \frac{-3\Gamma_1^4 + 6\Gamma_1^2\Gamma_2 - 4\Gamma_1\Gamma_3 + \Gamma_4}{\left(-\Gamma_1^2 + \Gamma_2\right)^2}, \tag{8}$$

and the rth moment about the origin, μ_r', is defined for $\alpha > r$

$$\mu_r'(b,\alpha) = b^r\Gamma_r. \tag{9}$$

The median, $q_{1/2}$, is at

$$q_{1/2}(b,\alpha) = \ln(2)^{-\frac{1}{\alpha}} b, \tag{10}$$

and the mode is at

$$\mathrm{mode}(b,\alpha) = (1+\alpha)^{-\frac{1}{\alpha}} \alpha^{\frac{1}{\alpha}} b. \tag{11}$$

Random generation of the Frècet variate X is given by

$$X : b,\alpha \approx (-\ln(R))^{-\frac{1}{\alpha}} b, \tag{12}$$

where R is the unit rectangular variate.

The two parameters b and α can be derived by the numerical solution of the two following equations, which arise from the maximum likelihood estimator (MLE),

$$\frac{\alpha\left(-\left(\sum_{i=1}^{n}\left(\frac{x_i}{b}\right)^{-\alpha}\right)+n\right)}{b} = 0, \tag{13a}$$

$$n\ln(b) + \frac{n}{\alpha} + \sum_{i=1}^{n}\left(\left(\frac{x_i}{b}\right)^{-\alpha} \ln\left(\frac{x_i}{b}\right) - \ln(x_i)\right) = 0, \tag{13b}$$

where x_i are the elements of the experimental sample with i varying between 1 and n.

3. The Truncated Frècet Distribution

Let X be a random variable defined in $[x_l, x_u]$; the truncated two-parameter Frècet DF, $F_T(x)$, is

$$F_T(x;b,\alpha,x_l,x_u) = \frac{-\mathrm{e}^{-x^{-\alpha}b^{\alpha}} + \mathrm{e}^{-x_l^{-\alpha}b^{\alpha}}}{-\mathrm{e}^{-x_u^{-\alpha}b^{\alpha}} + \mathrm{e}^{-x_l^{-\alpha}b^{\alpha}}}, \tag{14}$$

and the PDF, $f_T(x)$, is

$$f_T(x;b,\alpha,x_l,x_u) = \frac{\left(\frac{x}{b}\right)^{-\alpha} \alpha \mathrm{e}^{-\left(\frac{x}{b}\right)^{-\alpha}}}{x\left(\mathrm{e}^{-\left(\frac{x_u}{b}\right)^{-\alpha}} - \mathrm{e}^{-\left(\frac{x_l}{b}\right)^{-\alpha}}\right)}. \tag{15}$$

We now present two different formulae for the rth moment about the origin, μ_r': the first is

$$\mu_r'(b,\alpha,x_l,x_u) = \frac{b^r\left(\Gamma\left(\frac{\alpha-r}{\alpha},\left(\frac{x_l}{b}\right)^{-\alpha}\right) - \Gamma\left(\frac{\alpha-r}{\alpha},\left(\frac{x_u}{b}\right)^{-\alpha}\right)\right)}{-\mathrm{e}^{-x_u^{-\alpha}b^{\alpha}} + \mathrm{e}^{-x_l^{-\alpha}b^{\alpha}}}, \tag{16}$$

where

$$\Gamma(a,z) = \int_z^{\infty} t^{a-1}\mathrm{e}^{-t}\,\mathrm{d}t, \tag{17}$$

is the upper incomplete gamma function, see formula (8) in [9] and the second formula is

$$
\begin{aligned}
\mu_r'(b,\alpha,x_l,x_u) = {} & \frac{1}{(\alpha-r)(2\alpha-r)(3\alpha-r)\left(\mathrm{e}^{-x_u^{-\alpha}b^{\alpha}}-\mathrm{e}^{-x_l^{-\alpha}b^{\alpha}}\right)} \\
& \times\Bigg(-\alpha\Bigg(-b^{-\alpha+\frac{r}{2}}x_l^{\frac{r}{2}+\alpha}\mathrm{e}^{-\frac{x_l^{-\alpha}b^{\alpha}}{2}}(-2\alpha+r)^2 M_{\frac{2\alpha-r}{2\alpha},\frac{3\alpha-r}{2\alpha}}\left(x_l^{-\alpha}b^{\alpha}\right) \\
& +b^{-\alpha+\frac{r}{2}}x_u^{\frac{r}{2}+\alpha}\mathrm{e}^{-\frac{x_u^{-\alpha}b^{\alpha}}{2}}(-2\alpha+r)^2 M_{\frac{2\alpha-r}{2\alpha},\frac{3\alpha-r}{2\alpha}}\left(x_u^{-\alpha}b^{\alpha}\right) \\
& +\alpha\Bigg(\mathrm{e}^{-\frac{x_l^{-\alpha}b^{\alpha}}{2}}\left(x_l^{\frac{r}{2}+\alpha}(-2\alpha+r)b^{-\alpha+\frac{r}{2}}-\alpha b^{\frac{r}{2}}x_l^{\frac{r}{2}}\right)M_{-\frac{r}{2\alpha},\frac{3\alpha-r}{2\alpha}}\left(x_l^{-\alpha}b^{\alpha}\right) \\
& +\mathrm{e}^{-\frac{x_u^{-\alpha}b^{\alpha}}{2}}M_{-\frac{r}{2\alpha},\frac{3\alpha-r}{2\alpha}}\left(x_u^{-\alpha}b^{\alpha}\right)\left(-x_u^{\frac{r}{2}+\alpha}(-2\alpha+r)b^{-\alpha+\frac{r}{2}}+x_u^{\frac{r}{2}}\alpha b^{\frac{r}{2}}\right)\Bigg)\Bigg)\Bigg),
\end{aligned}
\tag{18}
$$

where $M_{\mu,\nu}(z)$ is the Whittaker M function, see [10]. The variance can be evaluated with the usual formula

$$\sigma^2(b,\alpha,x_l,x_u)=\mu_2'(b,\alpha,x_l,x_u)-\left(\mu_1'(b,\alpha,x_l,x_u)\right)^2, \tag{19}$$

the median is at

$$q_{1/2}(b,\alpha,x_l,x_u)=b\left(-\ln\left(\frac{\mathrm{e}^{-x_l^{-\alpha}b^{\alpha}}}{2}+\frac{\mathrm{e}^{-x_u^{-\alpha}b^{\alpha}}}{2}\right)\right)^{-\frac{1}{\alpha}}, \tag{20}$$

and the mode is at the same position as the regular case, see Equation (11). In the truncated case the mean and the variance are defined for $\alpha>0$, see Figure 1 and Figure 2.

The random generation of the truncated Frècet variate X is given by

$$X:b,\alpha,x_l,x_u\approx b\left(-\ln\left(-R\mathrm{e}^{-x_l^{-\alpha}b^{\alpha}}+R\mathrm{e}^{-x_u^{-\alpha}b^{\alpha}}+\mathrm{e}^{-x_l^{-\alpha}b^{\alpha}}\right)\right)^{-\frac{1}{\alpha}}. \tag{21}$$

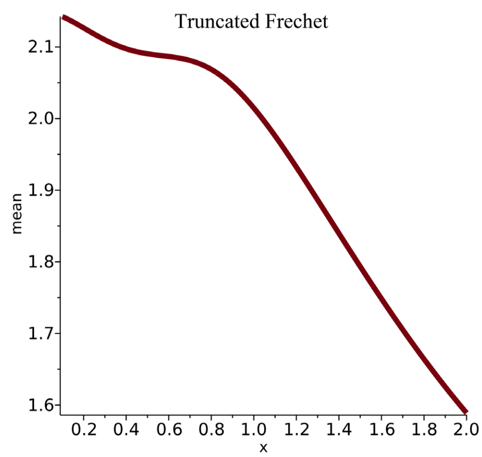

Figure 1. Mean of the truncated Frèchet distribution as function of α when $b=1$, $x_l=0.1$ and $x_u=10$.

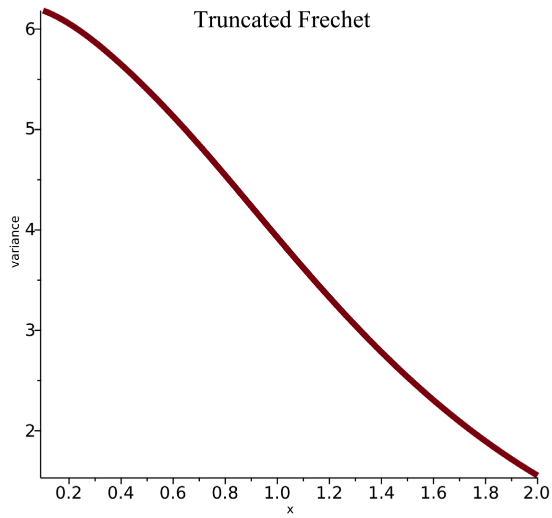

Figure 2. Variance of the truncated Frèchet distribution as function of α when $b=1$, $x_l=0.1$ and $x_u=10$.

The four parameters x_l, x_u, b and α can be obtained in the following way. Consider a sample $\mathcal{X} = x_1, x_2, \cdots, x_n$ and let $x_{(1)} \ge x_{(2)} \ge \cdots \ge x_{(n)}$ denote their order statistics, so that $x_{(1)} = \max(x_1, x_2, \cdots, x_n)$, $x_{(n)} = \min(x_1, x_2, \cdots, x_n)$. The first two parameters x_l and x_u are

$$x_l = x_{(n)}, \quad x_u = x_{(1)}. \tag{22}$$

The MLE allows us to derive the two remaining parameters b and α from the experimental sample

$$\frac{n\alpha}{b} + \frac{n\left(\frac{\left(\frac{x_u}{b}\right)^{-\alpha}\alpha \mathrm{e}^{-\left(\frac{x_u}{b}\right)^{-\alpha}}}{b} - \frac{\left(\frac{x_l}{b}\right)^{-\alpha}\alpha \mathrm{e}^{-\left(\frac{x_l}{b}\right)^{-\alpha}}}{b}\right)\left(\mathrm{e}^{-\left(\frac{x_u}{b}\right)^{-\alpha}} - \mathrm{e}^{-\left(\frac{x_l}{b}\right)^{-\alpha}}\right)}{\left(-\mathrm{e}^{-\left(\frac{x_u}{b}\right)^{-\alpha}} + \mathrm{e}^{-\left(\frac{x_l}{b}\right)^{-\alpha}}\right)^2} \tag{23a}$$

$$+\sum_{i=1}^{n}\left(-\frac{\left(\frac{x_i}{b}\right)^{-\alpha}\alpha}{b}\right) = 0,$$

$$n\ln(b) + \frac{n}{\alpha} + \frac{n\left(-\left(\frac{x_u}{b}\right)^{-\alpha}\ln\left(\frac{x_u}{b}\right)\mathrm{e}^{-\left(\frac{x_u}{b}\right)^{-\alpha}} + \left(\frac{x_l}{b}\right)^{-\alpha}\ln\left(\frac{x_l}{b}\right)\mathrm{e}^{-\left(\frac{x_l}{b}\right)^{-\alpha}}\right)\left(\mathrm{e}^{-\left(\frac{x_u}{b}\right)^{-\alpha}} - \mathrm{e}^{-\left(\frac{x_l}{b}\right)^{-\alpha}}\right)}{\left(-\mathrm{e}^{-\left(\frac{x_u}{b}\right)^{-\alpha}} + \mathrm{e}^{-\left(\frac{x_l}{b}\right)^{-\alpha}}\right)^2} \tag{23b}$$

$$+\sum_{i=1}^{n}\left(-\ln(x_i) + \left(\frac{x_i}{b}\right)^{-\alpha}\ln\left(\frac{x_i}{b}\right)\right) = 0.$$

4. The Mass-Luminosity Relationship

The mass-luminosity relationship for the stars is well established from both a theoretical point of view, $L \propto \mathcal{M}^3$ or $L \propto \mathcal{M}^4$, see [11], and from an observational point of view, $L \propto \mathcal{M}^{3.43}$ in the case of MAIN,V; see [12] for further details. We therefore introduce the following transformation for our PDFs

$$L = c\mathcal{M}^{\beta}, \tag{24}$$

where L is the luminosity of a star, $\mathcal{M}$ is the mass of the star, and c and β are two theoretical parameters. This transformation implies

$$\mathcal{M} = \left(\frac{L}{c}\right)^{\frac{1}{\beta}}, \tag{25a}$$

$$\mathrm{d}\mathcal{M} = \frac{\mathrm{d}L\, L^{\frac{1-\beta}{\beta}} c^{-\frac{1}{\beta}}}{\beta}. \tag{25b}$$

4.1. Frèchet $\mathcal{M}-L$ Distribution

To stress the astrophysical environment, we consider the change of variable $x = \mathcal{M}$, the mass, in Equation (2) for the Frèchet PDF

$$f\left(\mathcal{M}; b, \alpha\right) = \frac{\left(\frac{M}{b}\right)^{-\alpha} \alpha \mathrm{e}^{-\left(\frac{M}{b}\right)^{-\alpha}}}{M}. \tag{26}$$

To obtain a PDF in luminosity, L, we apply the transformation (24)

$$f_{ML}\left(L; b, \alpha, c, \beta\right) = \frac{L^{\frac{-\alpha-\beta}{\beta}} c^{\frac{\alpha}{\beta}} b^{\alpha} \alpha \mathrm{e}^{-L^{-\frac{\alpha}{\beta}} c^{\frac{\alpha}{\beta}} b^{\alpha}}}{\beta}, \tag{27}$$

where the suffix ML means mass-luminosity relationship. The DF is

$$F_{ML}\left(L; b, \alpha, c, \beta\right) = \mathrm{e}^{-L^{-\frac{\alpha}{\beta}} c^{\frac{\alpha}{\beta}} b^{\alpha}}, \tag{28}$$

the average value is defined for $\alpha > \beta$

$$\mu_{ML}\left(b, \alpha, c, \beta\right) = b^{\beta} c \Gamma\left(\frac{\alpha-\beta}{\alpha}\right), \tag{29}$$

the variance is defined for $\alpha > 2\beta$

$$\sigma^2_{ML}\left(b, \alpha, c, \beta\right) = b^{2\beta} c^2 \Gamma\left(\frac{-2\beta+\alpha}{\alpha}\right) - b^{2\beta} c^2 \Gamma\left(\frac{\alpha-\beta}{\alpha}\right)^2, \tag{30}$$

the rth moment about the origin is defined for $\alpha > r\beta$

$$\mu'_{ML}\left(b, \alpha, c, \beta, r\right) = b^{\beta r} c^r \Gamma\left(\frac{-\beta r+\alpha}{\alpha}\right), \tag{31}$$

the mode is at

$$\mathrm{mode}\left(b, \alpha, c, \beta\right)_{ML} = b^{\beta} c \left(\alpha+\beta\right)^{-\frac{\beta}{\alpha}} \alpha^{\frac{\beta}{\alpha}}, \tag{32}$$

the median is at

$$q_{ML}\left(b,\alpha,c,\beta\right)=b^{\beta}c\ln\left(2\right)^{-\frac{\beta}{\alpha}}, \tag{33}$$

and the random generation of the $\mathcal{M}-L$ Frècet variate X is given by

$$X:b,\alpha,c,\beta\approx b^{\beta}c\left(-\ln\left(R\right)\right)^{-\frac{\beta}{\alpha}}. \tag{34}$$

The astrophysical parameter β is constant and the three parameters b, α and c can be derived by the numerical solution of the following equations which arise from MLE

$$-\frac{\alpha\left(c^{\frac{\alpha}{\beta}}\left(\sum_{i=1}^{n}x_i^{-\frac{\alpha}{\beta}}\right)b^{\alpha}-n\right)}{b}=0, \tag{35a}$$

$$\frac{\left(\sum_{i=1}^{n}\left(b^{\alpha}c^{\frac{\alpha}{\beta}}\left(-\beta\ln\left(b\right)+\ln\left(x_i\right)-\ln\left(c\right)\right)x_i^{-\frac{\alpha}{\beta}}-\ln\left(x_i\right)\right)\right)\alpha+n\left(\ln\left(b\right)\alpha\beta+\alpha\ln\left(c\right)+\beta\right)}{\alpha\beta}=0, \tag{35b}$$

$$-\frac{\alpha\left(c^{\frac{\alpha}{\beta}}\left(\sum_{i=1}^{n}x_i^{-\frac{\alpha}{\beta}}\right)b^{\alpha}-n\right)}{\beta c}=0. \tag{35c}$$

4.2. The Truncated Frèchet $\mathcal{M}-L$ Distribution

The starting point is Equation (15) for the truncated Frèchet PDF with the variable x replaced by the mass. We apply the transformation (24) and the truncated Frèchet $\mathcal{M}-L$ PDF is

$$f_{MLT}\left(L;b,\alpha,c,\beta,L_l,L_u\right)=\frac{L^{\frac{-\alpha-\beta}{\beta}}c^{\frac{\alpha}{\beta}}b^{\alpha}\alpha\mathrm{e}^{-L^{-\frac{\alpha}{\beta}}c^{\frac{\alpha}{\beta}}b^{\alpha}}}{\beta\left(\mathrm{e}^{-L_u^{-\frac{\alpha}{\beta}}c^{\frac{\alpha}{\beta}}b^{\alpha}}-\mathrm{e}^{-L_l^{-\frac{\alpha}{\beta}}c^{\frac{\alpha}{\beta}}b^{\alpha}}\right)}, \tag{36}$$

where L, L_l and L_u are the luminosity, the lower luminosity and the upper luminosity; the suffix MLT denotes mass-luminosity relationship. The DF is

$$F_{MLT}\left(L;b,\alpha,c,\beta,L_l,L_u\right)=\frac{\mathrm{e}^{-L^{-\frac{\alpha}{\beta}}c^{\frac{\alpha}{\beta}}b^{\alpha}}}{\mathrm{e}^{-L_u^{-\frac{\alpha}{\beta}}c^{\frac{\alpha}{\beta}}b^{\alpha}}-\mathrm{e}^{-L_l^{-\frac{\alpha}{\beta}}c^{\frac{\alpha}{\beta}}b^{\alpha}}}. \tag{37}$$

The rth moment about the origin is

$$\mu_r'\left(b,\alpha,c,\beta,L_l,L_u\right)=\frac{b^{\beta r}c^{r}\left(\Gamma\left(\frac{-\beta r+\alpha}{\alpha},L_l^{-\frac{\alpha}{\beta}}c^{\frac{\alpha}{\beta}}b^{\alpha}\right)-\Gamma\left(\frac{-\beta r+\alpha}{\alpha},L_u^{-\frac{\alpha}{\beta}}c^{\frac{\alpha}{\beta}}b^{\alpha}\right)\right)}{-\mathrm{e}^{-L_u^{-\frac{\alpha}{\beta}}c^{\frac{\alpha}{\beta}}b^{\alpha}}+\mathrm{e}^{-L_l^{-\frac{\alpha}{\beta}}c^{\frac{\alpha}{\beta}}b^{\alpha}}}, \tag{38}$$

the median is at

$$q_{1/2}\left(b,\alpha,c,\beta,L_l,L_u\right)=b^{\beta}c\left(-\ln\left(\frac{\mathrm{e}^{-L_u^{-\frac{\alpha}{\beta}}c^{\frac{\alpha}{\beta}}b^{\alpha}}}{2}-\frac{\mathrm{e}^{-L_l^{-\frac{\alpha}{\beta}}c^{\frac{\alpha}{\beta}}b^{\alpha}}}{2}\right)\right)^{-\frac{\beta}{\alpha}}, \tag{39}$$

the mode is at

$$\mathrm{mode}\left(b,\alpha,c,\beta,L_l,L_u\right)=b^{\beta}\left(\alpha+\beta\right)^{-\frac{\beta}{\alpha}}\alpha^{\frac{\beta}{\alpha}}c, \tag{40}$$

and the random generation of the variate is given by

$$X:b,\alpha,c,\beta,L_l,L_u\approx b^{\beta}c\left(-\ln\left(-R\left(-\mathrm{e}^{-L_u^{-\frac{\alpha}{\beta}}c^{\frac{\alpha}{\beta}}b^{\alpha}}+\mathrm{e}^{-L_l^{-\frac{\alpha}{\beta}}c^{\frac{\alpha}{\beta}}b^{\alpha}}\right)\right)\right)^{-\frac{\beta}{\alpha}}. \tag{41}$$

The parameter β is fixed by the astrophysics and the three parameters b,α,c are obtained by solving the following equations that arise from MLE

$$\frac{n\alpha}{b}+\frac{n\left(\frac{L_u^{-\frac{\alpha}{\beta}}c^{\frac{\alpha}{\beta}}b^{\alpha}\alpha\mathrm{e}^{-L_u^{-\frac{\alpha}{\beta}}c^{\frac{\alpha}{\beta}}b^{\alpha}}}{b}-\frac{L_l^{-\frac{\alpha}{\beta}}c^{\frac{\alpha}{\beta}}b^{\alpha}\alpha\mathrm{e}^{-L_l^{-\frac{\alpha}{\beta}}c^{\frac{\alpha}{\beta}}b^{\alpha}}}{b}\right)\left(\mathrm{e}^{-L_u^{-\frac{\alpha}{\beta}}c^{\frac{\alpha}{\beta}}b^{\alpha}}-\mathrm{e}^{-L_l^{-\frac{\alpha}{\beta}}c^{\frac{\alpha}{\beta}}b^{\alpha}}\right)}{\left(-\mathrm{e}^{-L_u^{-\frac{\alpha}{\beta}}c^{\frac{\alpha}{\beta}}b^{\alpha}}+\mathrm{e}^{-L_l^{-\frac{\alpha}{\beta}}c^{\frac{\alpha}{\beta}}b^{\alpha}}\right)^{2}} \tag{42a}$$

$$+\sum_{i=1}^{n}\left(-\frac{x_i^{-\frac{\alpha}{\beta}}c^{\frac{\alpha}{\beta}}b^{\alpha}\alpha}{b}\right)=0,$$

$$\frac{n\ln\left(c\right)}{\beta}+n\ln\left(b\right)+\frac{n}{\alpha}+C_2$$

$$+\sum_{i=1}^{n}\left(-\frac{\ln\left(x_i\right)}{\beta}+\frac{x_i^{-\frac{\alpha}{\beta}}\ln\left(x_i\right)c^{\frac{\alpha}{\beta}}b^{\alpha}}{\beta}-\frac{x_i^{-\frac{\alpha}{\beta}}c^{\frac{\alpha}{\beta}}\ln\left(c\right)b^{\alpha}}{\beta}-x_i^{-\frac{\alpha}{\beta}}c^{\frac{\alpha}{\beta}}b^{\alpha}\ln\left(b\right)\right)=0, \tag{42b}$$

$$\frac{n\alpha}{\beta c}+\frac{n\left(\frac{L_u^{-\frac{\alpha}{\beta}}c^{\frac{\alpha}{\beta}}\alpha b^{\alpha}\mathrm{e}^{-L_u^{-\frac{\alpha}{\beta}}c^{\frac{\alpha}{\beta}}b^{\alpha}}}{\beta c}-\frac{L_l^{-\frac{\alpha}{\beta}}c^{\frac{\alpha}{\beta}}\alpha b^{\alpha}\mathrm{e}^{-L_l^{-\frac{\alpha}{\beta}}c^{\frac{\alpha}{\beta}}b^{\alpha}}}{\beta c}\right)\left(\mathrm{e}^{-L_u^{-\frac{\alpha}{\beta}}c^{\frac{\alpha}{\beta}}b^{\alpha}}-\mathrm{e}^{-L_l^{-\frac{\alpha}{\beta}}c^{\frac{\alpha}{\beta}}b^{\alpha}}\right)}{\left(-\mathrm{e}^{-L_u^{-\frac{\alpha}{\beta}}c^{\frac{\alpha}{\beta}}b^{\alpha}}+\mathrm{e}^{-L_l^{-\frac{\alpha}{\beta}}c^{\frac{\alpha}{\beta}}b^{\alpha}}\right)^{2}} \tag{42c}$$

$$+\sum_{i=1}^{n}\left(-\frac{x_i^{-\frac{\alpha}{\beta}}c^{\frac{\alpha}{\beta}}\alpha b^{\alpha}}{\beta c}\right)=0,$$

where

$$
\begin{aligned}
C_2 = {} & \frac{1}{\left(\mathrm{e}^{-L_u^{-\frac{\alpha}{\beta}} c^{\frac{\alpha}{\beta}} b^{\alpha}} - \mathrm{e}^{-L_l^{-\frac{\alpha}{\beta}} c^{\frac{\alpha}{\beta}} b^{\alpha}}\right)^2 \beta} \times n c^{\frac{\alpha}{\beta}} b^{\alpha} \Bigg(\ln(b) \mathrm{e}^{-L_u^{-\frac{\alpha}{\beta}} c^{\frac{\alpha}{\beta}} b^{\alpha}} L_u^{-\frac{\alpha}{\beta}} \beta \\
& - \ln(b) \mathrm{e}^{-L_l^{-\frac{\alpha}{\beta}} c^{\frac{\alpha}{\beta}} b^{\alpha}} L_l^{-\frac{\alpha}{\beta}} \beta + \ln(c) \mathrm{e}^{-L_u^{-\frac{\alpha}{\beta}} c^{\frac{\alpha}{\beta}} b^{\alpha}} L_u^{-\frac{\alpha}{\beta}} - \ln(c) \mathrm{e}^{-L_l^{-\frac{\alpha}{\beta}} c^{\frac{\alpha}{\beta}} b^{\alpha}} L_l^{-\frac{\alpha}{\beta}} \\
& - \mathrm{e}^{-L_u^{-\frac{\alpha}{\beta}} c^{\frac{\alpha}{\beta}} b^{\alpha}} L_u^{-\frac{\alpha}{\beta}} \ln(L_u) + \mathrm{e}^{-L_l^{-\frac{\alpha}{\beta}} c^{\frac{\alpha}{\beta}} b^{\alpha}} L_l^{-\frac{\alpha}{\beta}} \ln(L_l) \Bigg) \left(\mathrm{e}^{-L_u^{-\frac{\alpha}{\beta}} c^{\frac{\alpha}{\beta}} b^{\alpha}} - \mathrm{e}^{-L_l^{-\frac{\alpha}{\beta}} c^{\frac{\alpha}{\beta}} b^{\alpha}} \right).
\end{aligned}
\tag{43}
$$

5. Astrophysical Applications

This section reviews some formulae that are useful in the conversion from the magnitude to the luminosity of a star, the adopted statistical tests, the application of the obtained results to the IMF for stars and the reliability of $\mathcal{M} - L$ relationship for NGC 2362.

5.1. Useful Formulae

The conversion from apparent magnitude, m, to absolute magnitude, M, is given by

$$
M = m + 5 - \frac{5 \ln(D)}{\ln(10)}, \tag{44}
$$

where D is the distance in pc and ln is the natural logarithm. The conversion from absolute magnitude to luminosity L is

$$
\frac{L}{L_{\odot}} = 10^{0.4 M_{\odot} - 0.4 M}, \tag{45}
$$

where $L_{\odot}$ and $M_{\odot}$ are the solar luminosity and absolute magnitude in the considered astronomical band, see Appendix A.4 in [13].

5.2. Statistics

The merit function χ^2 is computed according to the formula

$$
\chi^2 = \sum_{i=1}^{n} \frac{(T_i - O_i)^2}{T_i}, \tag{46}
$$

where n is the number of bins, T_i is the theoretical value, and O_i is the experimental value represented by the frequencies. The theoretical frequency distribution is given by

$$
T_i = N \Delta x_i p(x), \tag{47}
$$

where N is the number of elements of the sample, Δx_i is the magnitude of the size interval, and $p(x)$ is the PDF under examination. A reduced merit function χ^2_{red} is given by

$$
\chi^2_{red} = \chi^2 / NF, \tag{48}
$$

where $NF = n - k$ is the number of degrees of freedom, n is the number of bins, and k is the number of parameters. The goodness of the fit can be expressed by the probability Q, see equation 15.2.12 in [14], which involves the number of degrees of freedom and χ^2. According to [14] p. 658, the fit "may be acceptable" if $Q > 0.001$. The Akaike information criterion (AIC), see [15], is defined by

$$\mathrm{AIC} = 2k - 2\ln(L), \tag{49}$$

where L is the likelihood function and k the number of free parameters in the model. We assume a Gaussian distribution for the errors. The likelihood function can then be derived from the χ^2 statistic $L \propto \exp\left(-\frac{\chi^2}{2}\right)$ where χ^2 has been computed by equation (46), see [16] [17]. Now the AIC becomes

$$\mathrm{AIC} = 2k + \chi^2. \tag{50}$$

The Kolmogorov-Smirnov test (K-S), see [18] [19] [20], does not require the data to be binned. The K-S test, as implemented by the FORTRAN subroutine KSONE in [14], finds the maximum distance, D, between the theoretical and the astronomical DF, as well as the significance level P_{KS}; see formulas 14.3.5 and 14.3.9 in [14]. If $P_{KS} \ge 0.1$, then the goodness of the fit is believable.

5.3. The IMF for Stars

The *first* test is performed on NGC 2362 where the 271 stars have a range $1.47M_\odot \ge M \ge 0.11M_\odot$, see [21] and CDS catalog J/MNRAS/384/675/table1. According to [22], the distance of NGC 2362 is 1480 pc.

The *second* test is performed on the low-mass IMF in the young cluster NGC 6611, see [23] and CDS catalog J/MNRAS/392/1034. This massive cluster has an age of 2 - 3 Myr and contains masses from $1.5M_\odot \ge M \ge 0.02M_\odot$. Therefore, the brown dwarfs (BD) region, $\approx 0.2\mathcal{M}_\odot$ is covered. The *third* test is performed on the γ Velorum cluster where the 237 stars have a range $1.31M_\odot \ge M \ge 0.15M_\odot$, see [24] and CDS catalog J/A + A/589/A70/table5. The *fourth* test is performed on the young cluster Berkeley 59 where the 420 stars have a range $2.24M_\odot \ge M \ge 0.15M_\odot$, see [25] and CDS catalog J/AJ/155/44/table3. The results are presented in Table 1 for the Frèchet distribution with two parameters and in Table 2 for the truncated Frèchet distribution with four parameters, where the last column reports whether the results of the K-S test are better when

Table 1. Numerical values of χ^2_{red}, AIC, probability Q, D, the maximum distance between theoretical and observed DF, and P_{KS}, significance level, in the K-S test of the Frèchet distribution with two parameters for different astrophysical environments. The last column (F) indicates a P_{KS} higher (Y) or lower (N) than that for the lognormal distribution. The number of linear bins, n, is 10.

Cluster	parameters	AIC	χ^2_{red}	Q	D	P_{KS}	F
NGC 2362	$b = 0.44$, $a = 1.825$	81.58	9.69	1.49×10^{-13}	0.125	3.13×10^{-4}	N
NGC 6611	$b = 0.165$, $a = 0.912$	83.57	9.94	5.95×10^{-14}	0.15	1.43×10^{-4}	N
γ Velorum	$b = 0.267$, $a = 2.572$	23.7	2.46	0.015	0.046	0.68	Y
Berkeley 59	$b = 0.319$, $a = 2.68$	37.63	4.2	4.72×10^{-5}	5.1×10^{-2}	0.209	Y

Table 2. Numerical values of χ^2_{red}, AIC, probability Q, D, the maximum distance between theoretical and observed DF, and P_{KS}, significance level, in the K-S test of the truncated Frèchet distribution with four parameters for different astrophysical environments. The last column (F) indicates a P_{KS} higher (Y) or lower (N) than that for the lognormal distribution. The number of linear bins, n, is 10.

Cluster	parameters	AIC	χ^2_{red}	Q	D	P_{KS}	F
NGC 2362	$b = 0.604$, $\alpha = 1.24$, $x_l = 0.2$, $x_u = 1.47$	37.294	4.88	5.34×10^{-5}	0.077	0.07	N
NGC 6611	$b = 0.66$, $\alpha = 0.44$, $x_l = 0.0189$, $x_u = 1.46$	25.7	2.95	7×10^{-3}	0.075	0.17	Y
γ Velorum	$b = 0.2$, $\alpha = 1.5$, $x_l = 0.15$, $x_u = 1.31$	14.85	1.14	0.33	0.06	0.33	Y
Berkeley 59	$b = 0.32$, $\alpha = 2.58$, $x_l = 0.16$, $x_u = 2.24$	35.13	4.52	1.36×10^{-4}	5.28×10^{-2}	0.185	Y

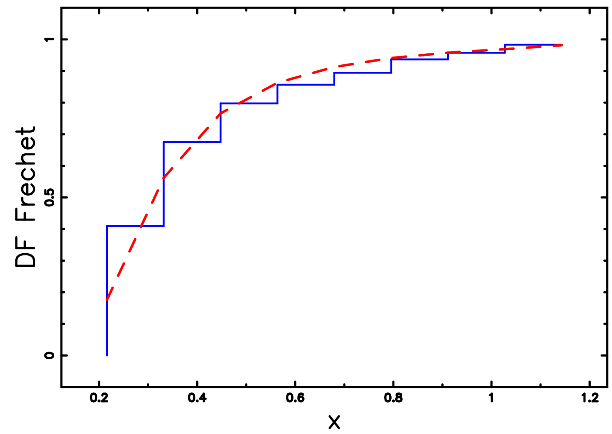

Figure 3. Empirical DF of the mass distribution for γ Velorum (bleu histogram) with a superposition of the Frèchet DF (red dashed line). Theoretical parameters as in **Table 1**.

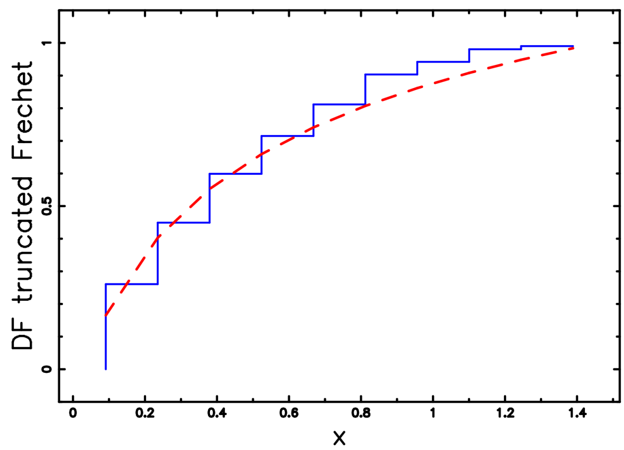

Figure 4. Empirical DF of the mass distribution for NGC 6611 (bleu histogram) with a superposition of the truncated Frèchet DF (red-dashed line). Theoretical parameters as in **Table 2**.

compared to the Weibull distribution (Y) or worse (N).

As an example, the empirical DF visualized through histograms and the theoretical Frèchet DF for γ Velorum are reported in **Figure 3**.

Figure 4 displays the theoretical truncated Frèchet DF and the empirical DF for NGC 6611.

5.4. $\mathcal{M}-L$ Relationship

We start with the sample in apparent magnitude for NGC 2362, see **Figure 5**. To have a sample in luminosity for NGC 2362, we convert the apparent magnitude in luminosity via formula (45); see **Figure 6**.

The data of **Figure 6** are now processed to obtain the parameters of the

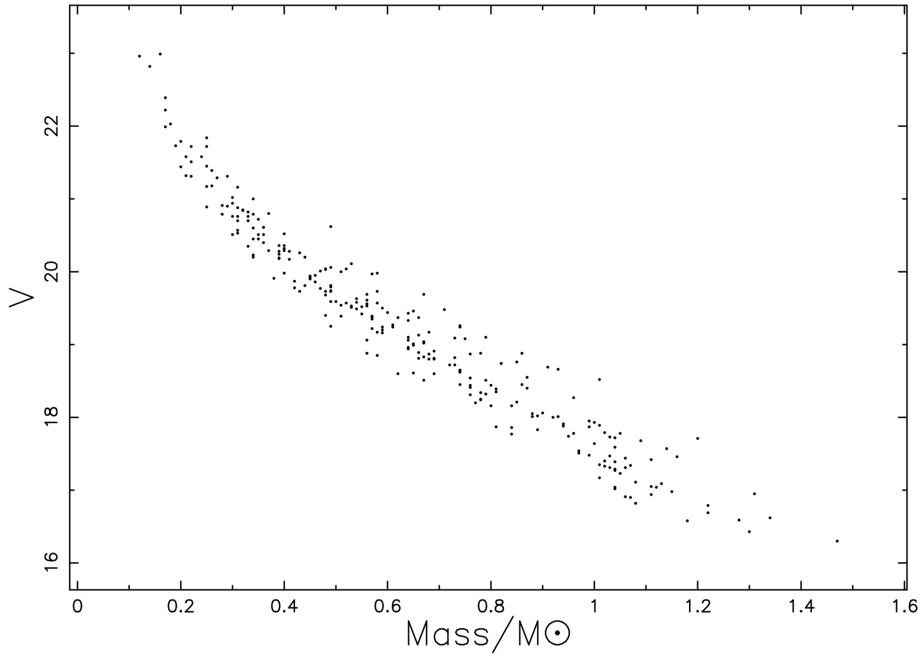

Figure 5. Apparent magnitude, V, versus mass for NGC 2362; data available from the Strasbourg Astronomical Data Centre (CDS), which are in the table with name J/MNRAS/384/675/table.

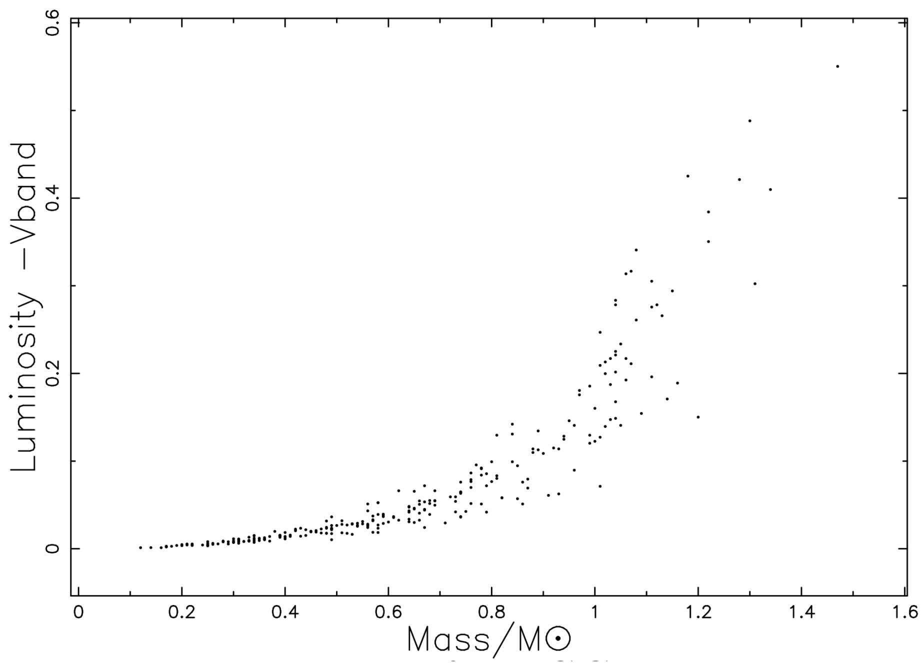

Figure 6. Luminosity, L, versus mass for NGC 2362 when $D = 1480\ \text{pc}$ and $M_{\odot} = 4.8$.

$\mathcal{M}-L$ Frècet distribution, see Table 3, and of the truncated $\mathcal{M}-L$ Frècet distribution, see Table 4.

Figure 7 displays the theoretical luminosity $\mathcal{M}-L$ Frèchet DF and the empirical DF, and Figure 8 displays the truncated $\mathcal{M}-L$ Frèchet DF.

Table 3. Numerical values of χ^2_{red}, AIC, probability Q, D, the maximum distance between theoretical and observed DF, and P_{KS}, significance level, in the K-S test of the $\mathcal{M}-L$ Frècet distribution in luminosity with four parameters. The number of linear bins, n, is 20.

Cluster	Parameters	AIC	χ^2_{red}	Q	D	P_{KS}
NGC 2362	$b = 0.092, a = 2.48$, c = 11.47, $\beta = 2.45$	23.17	0.948	0.511	0.154	3.93×10^{-6}

Table 4. Numerical values of χ^2_{red}, AIC, probability, Q, D, the maximum distance between theoretical and observed DF, and P_{KS}, significance level, in the K-S test of the truncated $\mathcal{M}-L$ Frècet distribution in luminosity with six parameters. The number of linear bins, n, is 20.

Cluster	Parameters	AIC	χ^2_{red}	Q	D	P_{KS}
NGC 2362	$b = 0.097, a = 1.47, c = 7.56, \beta = 2.45$ $L_l = 1.15 \times 10^{-3} L_\odot$, $L_u = 0.55 L_\odot$	25.24	0.946	0.507	0.0341	0.903

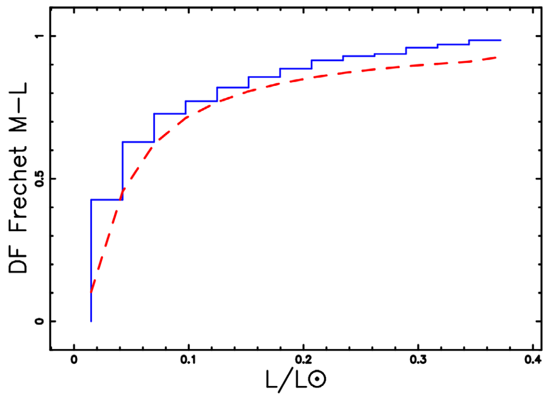

Figure 7. Empirical DF of the luminosity distribution for NGC 2362 (bleu histogram) with a superposition of the $\mathcal{M}-L$ Frèchet DF (red-dashed line). The theoretical parameters are as in Table 3.

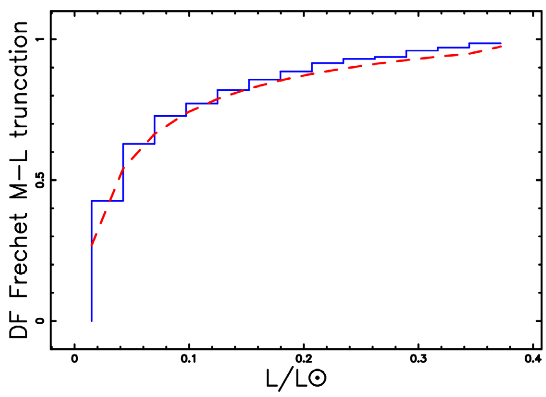

Figure 8. Empirical DF of the luminosity distribution for NGC 2362 (bleu histogram) with a superposition of the truncated $\mathcal{M}-L$ Frèchet DF (red-dashed line). The theoretical parameters are as in Table 4.

6. Conclusions

The truncated distributions

We derived the PDF, the DF, the average value, the rth moment, the median, the expression to generate the random variate and the MLE for the truncated Frècet distribution.

Astrophysical Applications

The application of this distribution to the IMF for stars gives better results than the lognormal distribution for two out of four samples, see Table 1. The truncated Frècet distribution gives better results than the Frècet distribution for two out of four samples, see Table 1 and Table 2.

The results for the mass distribution of γ Velorum cluster compared with other distributions are reported in Table 5, in which the Frècet distribution surprisingly produces the best results.

Table 5. Numerical values of D, the maximum distance between theoretical and observed DF, and P_{KS}, significance level, in the K-S test for different distributions in the case of γ Velorum cluster.

Distribution	Reference	D	P_{KS}
Frècet	here	0.046	0.68
Weibull	[26]	0.14	6.6×10^{-5}
Truncated Weibull	[26]	0.063	0.29
Truncated Sujatha	[27]	0.0614	0.322
Truncated Lindley	[28]	0.064	0.269
Generalized gamma	[29]	0.11	5.7×10^{-3}
Truncated generalized gamma	[29]	0.105	9.38×10^{-3}
Lognormal	[30]	0.091	0.034
Truncated lognormal	[30]	0.0529	0.509
Gamma	[31]	0.145	7.6×10^{-5}
Truncated gamma	[31]	0.0812	0.0828
Beta	[32]	0.1	0.015

The mass-luminosity relationship

We made a transformation that connects a pdf in mass into a pdf in luminosity, see Equation (24). The resulting distribution in luminosity has been applied to NGC 2362, see Table 4. Figure 7 and Figure 8 display the DF for the $\mathcal{M}-L$ Frèchet DF and the truncated $\mathcal{M}-L$ Frèchet DF. These results are compatible with $L \propto M^{2.45}$, which can be another way to confirm the $\mathcal{M}-L$ relationship.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Fréchet, M. (1927) Sur la loi de probabilité de l'écart maximum. *Annales de la Société Polonaise de Mathématique*, **6**, 93-116.

[2] Rosin, P. and Rammler, E. (2012) The Laws Governing the Fineness of Powdered Coal. *Journal of the Institute of Fuel*, **7**, 29-36.

[3] Gumbel, E.J. (1965) A Quick Estimation of the Parameters in Frechet's Distribution. *Review of the International Statistical Institute*, **33**, 349-363. https://doi.org/10.2307/1401700

[4] Abbas, K. and Yincai, T. (2012) Comparison of Estimation Methods for Frechet Distribution with Known Shape. *Caspian Journal of Applied Sciences Research*, **1**, 58-64.

[5] Ramos, P.L., Louzada, F., Ramos, E. and Dey, S. (2020) The Fréchet Distribution: Estimation and Application—An Overview. *Journal of Statistics and Management Systems*, **23**, 549-573. https://doi.org/10.1080/09720510.2019.1645400

[6] Harlow, D.G. (2002) Applications of the Fréchet Distribution Function. *International Journal of Materials and Product Technology*, **17**, 482-495. https://doi.org/10.1504/IJMPT.2002.005472

[7] Vivekanandan, N. (2015) Quantitative Assessment on Fitting of Gumbel and Frechet Distributions for Extreme Value Analysis of Rainfall. *International Journal of Scientific Research in Science and Technology*, **1**, 68.

[8] Abid, S.H. and Hassan, H.A. (2015) Some Additive Failure Rate Models Related with Moeu Distribution. *American Journal of Systems Science*, **4**, 1-10.

[9] Abid, S.H. (2016) Properties of Doubly-Truncated Fréchet Distribution. *American Journal of Applied Mathematics and Statistics*, **4**, 9-15.

[10] Olver, F.W.J., Lozier, D.W., Boisvert, R.F. and Clark, C.W. (2010) NIST Handbook of Mathematical Functions. Cambridge University Press, Cambridge.

[11] Lang, K.R. (1999) Astrophysical Formulae. Third Edition, Springer, New York. https://doi.org/10.1007/978-3-662-21639-2

[12] Zaninetti, L. (2005) The Initial Mass Function as Given by the Fragmentation. *Astronomische Nachrichten*, **326**, 754. https://doi.org/10.1002/asna.200510408

[13] Longair, M.S. (2011) High Energy Astrophysics. 3rd Edition, Cambridge University Press, Cambridge. https://doi.org/10.1017/CBO9780511778346

[14] Press, W.H., Teukolsky, S.A., Vetterling, W.T. and Flannery, B.P. (1992) Numerical Recipes in FORTRAN. The Art of Scientific Computing. Cambridge University Press, Cambridge.

[15] Akaike, H. (1974) A New Look at the Statistical Model Identification. *IEEE Transactions on Automatic Control*, **19**, 716-723. https://doi.org/10.1109/TAC.1974.1100705

[16] Liddle, A.R. (2004) How Many Cosmological Parameters? *MNRAS*, **351**, L49. https://doi.org/10.1111/j.1365-2966.2004.08033.x

[17] Godlowski, W. and Szydowski, M. (2005) Constraints on Dark Energy Models from Supernovae. In: Turatto, M., Benetti, S., Zampieri, L. and Shea, W., Eds., 1604-2004: *Supernovae as Cosmological Lighthouses*, Vol. 342 of Astronomical Society of the Pacific Conference Series, Astronomical Society of the Pacific, San Francisco, 508-516.

[18] Kolmogoroff, A. (1941) Confidence Limits for an Unknown Distribution Function. *The Annals of Mathematical Statistics*, **12**, 461-463.

https://doi.org/10.1214/aoms/1177731684

[19] Smirnov, N. (1948) Table for Estimating the Goodness of Fit of Empirical Distributions. *The Annals of Mathematical Statistics*, **19**, 279-281. https://doi.org/10.1214/aoms/1177730256

[20] Massey, F.J.J. (1951) The Kolmogorov-Smirnov Test for Goodness of Fit. *Journal of the American Statistical Association*, **46**, 68-78. https://doi.org/10.1080/01621459.1951.10500769

[21] Irwin, J., Hodgkin, S., Aigrain, S., Bouvier, J., Hebb, L., Irwin, M. and Moraux, E. (2008) The Monitor Project: Rotation of Low-Mass Stars in NGC 2362—Testing the Disc Regulation Paradigm at 5 Myr. *MNRAS*, **384**, 675-686. https://doi.org/10.1111/j.1365-2966.2007.12725.x

[22] Moitinho, A., Alves, J., Huélamo, N. and Lada, C.J. (2001) NGC 2362: A Template for Early Stellar Evolution. *ApJ Letters*, **563**, L73. https://doi.org/10.1086/338503

[23] Oliveira, J.M., Jeffries, R.D. and van Loon, J.T. (2009) The Low-Mass Initial Mass Function in the Young Cluster NGC 6611. *MNRAS*, **392**, 1034-1050. https://doi.org/10.1111/j.1365-2966.2008.14140.x

[24] Prisinzano, L., Damiani, F., *et al.* (2016) The Gaia-ESO Survey: Membership and Initial Mass Function of the γ Velorum Cluster. *A&A*, **589**, A70. https://doi.org/10.1051/0004-6361/201527875

[25] Panwar, N., Pandey, A.K., Samal, M.R., *et al.* (2018) Young Cluster Berkeley 59: Properties, Evolution, and Star Formation. *The Astronomical Journal*, **155**, 44. https://doi.org/10.3847/1538-3881/aa9f1b

[26] Zaninetti, L. (2021) New Probability Distributions in Astrophysics: V. The Truncated Weibull Distribution. *International Journal of Astronomy and Astrophysics*, **11**, 133-149. https://doi.org/10.4236/ijaa.2021.111008

[27] Zaninetti, L. (2021) New Probability Distributions in Astrophysics: VI. The Truncated Sujatha Distribution. *International Journal of Astronomy and Astrophysics*, **11**, 517-529. https://doi.org/10.4236/ijaa.2021.114028

[28] Zaninetti, L. (2020) New Probability Distributions in Astrophysics: II. The Generalized and Double Truncated Lindley. *International Journal of Astronomy and Astrophysics*, **10**, 39-55. https://doi.org/10.4236/ijaa.2020.101004

[29] Zaninetti, L. (2019) New Probability Distributions in Astrophysics: I. The Truncated Generalized Gamma. *International Journal of Astronomy and Astrophysics*, **9**, 393-410. https://doi.org/10.4236/ijaa.2019.94027

[30] Zaninetti, L. (2017) A Left and Right Truncated Lognormal Distribution for the Stars. *Advances in Astrophysics*, **2**, 197-213. https://doi.org/10.22606/adap.2017.23005

[31] Zaninetti, L. (2013) A Right and Left Truncated Gamma Distribution with Application to the Stars. *Advanced Studies in Theoretical Physics*, **23**, 1139-1147. https://doi.org/10.12988/astp.2013.310125

[32] Zaninetti, L. (2013) The Initial Mass Function Modeled by a Left Truncated Beta Distribution. *The Astrophysical Journal*, **765**, 128-135. https://doi.org/10.1088/0004-637X/765/2/128