

# Periodic Signature Change in Spacetimes of Embedding Class One

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**How to cite this paper:** Kuhfittig, P.K.F. (2022) Periodic Signature Change in Spacetimes of Embedding Class One. *International Journal of Astronomy and Astrophysics*, 12, 167-176.

<https://doi.org/10.4236/ijaa.2022.122010>

**Received:** March 4, 2022

**Accepted:** April 26, 2022

**Published:** April 29, 2022

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## Abstract

The idea of an oscillating Universe has remained a topic of interest even after the discovery of dark energy. This paper confirms this idea by means of another well-established theory in general relativity, the embedding of curved spacetimes in higher-dimensional flat spacetimes: an  $n$ -dimensional Riemannian space is said to be of embedding class  $m$  if  $m+n$  is the lowest dimension of the flat space in which the given space can be embedded; here  $m = \frac{1}{2}n(n+1)$ . So a four-dimensional Riemannian space is of class two and can therefore be embedded in a six-dimensional flat space. A line element of class two can be reduced to a line element of class one by a suitable coordinate transformation. The extra dimension can be either spacelike or timelike, leading to accelerating and decelerating expansions, respectively. Accordingly, it is proposed in this paper that the free parameter occurring in the transformation be a periodic function of time. The result is a mathematical model that can be interpreted as a periodic change in the signature of the embedding space. This signature change may be the best model for an oscillating Universe and complements various models proposed in the literature.

## Keywords

Signature Change, Oscillating Universe, Embedding Class One

## 1. Introduction

The idea of a signature change in general relativity is not new: the Schwarzschild line element

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (1)$$

implies that space and time are interchanged when crossing the event horizon of a black hole. Signature-changing events can also play an important role in quantum gravity in part because a quantum field defined on a manifold may contain regions of different signatures [1]. Modifications of Einstein's theory have led to similar considerations, an example of which is  $f(R, T)$  gravity discussed in Ref. [2]. Here  $R$  is the Ricci scalar and  $T$  is the trace of the energy-momentum tensor. Given the current accelerated expansion, a transition from a previous decelerated phase suggests a time-varying deceleration parameter. According to Ref. [2], the evolving deceleration parameter displays a signature-flipping behavior. The outcome is an oscillating Universe. A similar argument is presented in Ref. [3], based on  $f(Q, T)$  teleparallel gravity. Refs. [4] and [5] discuss the concept of an oscillatory Universe from a different perspective, a dynamic dark-energy equation of state. This process would continue indefinitely without violating any known conservation laws.

Totally different scenarios are considered in Refs. [6] and [7]. The discussion in Ref. [6] involves a brane-world model in a five-dimensional anti-de Sitter space. It is argued that the present accelerated expansion may simply be an indication that our brane world has undergone a signature change. Ref. [7] returns to quantum gravity via covariant models in loop quantum gravity. It is asserted that these models generically imply dynamical signature changes at high densities.

Other mathematical models of interest to us involve various embedding theorems. These have a long history in the general theory of relativity, aided in large part by Campbell's theorem [8]. According to Ref. [9], the five-dimensional field equations in terms of the Ricci tensor, *i.e.*,  $R_{AB} = 0$ ,  $A, B = 0, 1, 2, 3, 4$ , help explain the origin of matter. The reason is that the vacuum field equations in five dimensions yield the usual Einstein field equations *with matter*, called the induced-matter theory [10] [11]: what we perceive as matter can be viewed as the impingement of the extra dimension onto our spacetime. Moreover, the extra dimension can be either spacelike or timelike. According to Ref. [12], the particle-wave duality can in principle be solved because the five-dimensional dynamics has two modes, depending on whether the extra dimension is spacelike or timelike. So it is conceivable that the five-dimensional relativity theory could lead to a unification of general relativity and quantum field theory.

To become a useful mathematical model, we need to go beyond a single extra dimension. This requires the concept of embedding: an  $n$ -dimensional Riemannian space is said to be of embedding class  $m$  if  $m+n$  is the lowest dimension of the flat space in which the given space can be embedded; here  $m = \frac{1}{2}n(n+1)$ . So a four-dimensional Riemannian space is of class two and can therefore be embedded in a six-dimensional flat space. Moreover, a line element of class two can be reduced to a line element of class one by a suitable coordinate transformation [13] [14] [15] [16]. As noted above, the extra dimension can be spacelike or timelike. Not only is the result a powerful mathematical model, it is consistent

with observation since the extra dimension could not be directly observed.

The mathematical model proposed contains a free parameter  $K = K(t)$ , a periodic function of time that causes a periodic change in the signature of the embedding space. It is shown in this paper that a positive value of  $K$  can account for the accelerating expansion, while a negative value corresponds to a decelerating phase. The resulting oscillatory behavior can thereby be attributed to a signature change in the embedding space, complementing the above mathematical models.

This paper is organized as follows: Section 2 describes the embedding of a curved spacetime in a flat spacetime with an extra spacelike or timelike dimension. Section 3 discusses the free parameter  $K$ , now assumed to be a periodic function of time, to provide an effective model for a signature change. The result is the oscillatory behavior discussed in Section 4. In Section 5 we conclude.

## 2. The Embedding

Following Ref. [13], we begin with the spherically symmetric line element

$$ds^2 = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \tag{2}$$

using units in which  $c = G = 1$ . We also assume asymptotic flatness:  $e^{\nu(r)} \rightarrow 1$  and  $e^{\lambda(r)} \rightarrow 1$  as  $r \rightarrow \infty$ . As already noted, this metric of class two can be reduced to a metric of class one by a suitable transformation of coordinates. Since the extra dimension in the embedding space can be either spacelike or timelike, the two cases will be taken up separately.

### 2.1. An Extra Spacelike Dimension

In this subsection, we will consider the following five-dimensional embedding space:

$$ds^2 = -(dz^1)^2 + (dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2. \tag{3}$$

The transformation is given by  $z^1 = \sqrt{K} e^{\frac{\nu}{2}} \sinh \frac{t}{\sqrt{K}}$ ,  $z^2 = \sqrt{K} e^{\frac{\nu}{2}} \cosh \frac{t}{\sqrt{K}}$ ,  $z^3 = r \sin \theta \cos \phi$ ,  $z^4 = r \sin \theta \sin \phi$ , and  $z^5 = r \cos \theta$ , also discussed in Ref. [18]. (Some of the steps are repeated here for later reference.) The differentials of these components are

$$dz^1 = \sqrt{K} e^{\frac{\nu}{2}} \frac{\nu'}{2} \sinh \frac{t}{\sqrt{K}} dr + e^{\frac{\nu}{2}} \cosh \frac{t}{\sqrt{K}} dt, \tag{4}$$

$$dz^2 = \sqrt{K} e^{\frac{\nu}{2}} \frac{\nu'}{2} \cosh \frac{t}{\sqrt{K}} dr + e^{\frac{\nu}{2}} \sinh \frac{t}{\sqrt{K}} dt, \tag{5}$$

$$dz^3 = \sin \theta \cos \phi dr + r \cos \theta \cos \phi d\theta - r \sin \theta \sin \phi d\phi, \tag{6}$$

$$dz^4 = \sin \theta \sin \phi dr + r \cos \theta \sin \phi d\theta + r \sin \theta \cos \phi d\phi, \tag{7}$$

and

$$dz^5 = \cos \theta dr - r \sin \theta d\theta. \tag{8}$$

To facilitate the substitution into Equation (2), we first obtain the expressions for  $-(dz^1)^2 + (dz^2)^2$  and for  $(dz^3)^2 + (dz^4)^2 + (dz^5)^2$ :

$$-(dz^1)^2 + (dz^2)^2 = -e^\nu dt^2 + \frac{1}{4} K e^\nu (\nu')^2 dr^2$$

and

$$(dz^3)^2 + (dz^4)^2 + (dz^5)^2 = dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$

The substitution yields

$$ds^2 = -e^\nu dt^2 + \left(1 + \frac{1}{4} K e^\nu (\nu')^2\right) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \tag{9}$$

Metric (9) is therefore equivalent to metric (2) if

$$e^\lambda = 1 + \frac{1}{4} K e^\nu (\nu')^2, \tag{10}$$

where  $K > 0$  is a free parameter. The result is a metric of embedding class one. Equation (10) can also be obtained from the Karmarkar condition [17]:

$$R_{1414} = \frac{R_{1212}R_{3434} + R_{1224}R_{1334}}{R_{2323}}, \quad R_{2323} \neq 0. \tag{11}$$

In fact, Equation (10) is a solution of the differential equation

$$\frac{\nu'\lambda'}{1 - e^\lambda} = \nu'\lambda' - 2\nu'' - (\nu')^2, \tag{12}$$

which is readily solved by separation of variables. So  $K$  is actually an arbitrary constant of integration.

### 2.2. An Extra Timelike Dimension

Here we consider the following embedding space:

$$ds^2 = -(dz^1)^2 - (dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2. \tag{13}$$

It is shown in Ref. [18] that the coordinate transformation is

$$z^1 = \sqrt{K} e^{\frac{\nu}{2}} \sin \frac{t}{\sqrt{K}}, \quad z^2 = \sqrt{K} e^{\frac{\nu}{2}} \cos \frac{t}{\sqrt{K}}, \quad z^3 = r \sin \theta \cos \phi, \quad z^4 = r \sin \theta \sin \phi,$$

and  $z^5 = r \cos \theta$ .

The substitution yields

$$ds^2 = -e^\nu dt^2 + \left(1 - \frac{1}{4} K e^\nu (\nu')^2\right) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \tag{14}$$

So

$$e^\lambda = 1 - \frac{1}{4} K e^\nu (\nu')^2, \tag{15}$$

where  $K$  is a constant of integration, as before.

### 3. A Variable Parameter: $K = K(t)$

The similarity between Equations (10) and (15) suggests a direct connection that

can be attributed to a signature change. In Equation (2), both  $\nu$  and  $\lambda$  are functions of  $r$ . So the parameter  $K$  in Equation (10) could be a function of time, which is also true of the parameter  $K$  in Equation (15), *i.e.*,  $K = K(t)$ . To see the significance, suppose that  $K(t)$  is a periodic real-valued function that is symmetric with respect to the  $t$ -axis, similar to, for example, the function  $f(t) = A \sin \omega t$ ; so  $|K(t)| = |-K(t)|$ . To see the effect on the above transformation, let us replace  $K$  by  $-K$  in Equations (4) and (5):

$$dz^1 = \sqrt{-K} e^{\frac{\nu}{2}} \frac{\nu'}{2} \sinh \frac{t}{\sqrt{-K}} dr + e^{\frac{\nu}{2}} \cosh \frac{t}{\sqrt{-K}} dt \tag{16}$$

and

$$dz^2 = \sqrt{-K} e^{\frac{\nu}{2}} \frac{\nu'}{2} \cosh \frac{t}{\sqrt{-K}} dr + e^{\frac{\nu}{2}} \sinh \frac{t}{\sqrt{-K}} dt. \tag{17}$$

Making use of the identities

$$\sinh(ix) = i \sin x \quad \text{and} \quad \cosh(ix) = \cos x,$$

we obtain

$$dz^1 = \sqrt{K} e^{\frac{\nu}{2}} \frac{\nu'}{2} \sin \frac{t}{\sqrt{K}} dr + e^{\frac{\nu}{2}} \cos \frac{t}{\sqrt{K}} dt \tag{18}$$

and

$$dz^2 = i\sqrt{K} e^{\frac{\nu}{2}} \frac{\nu'}{2} \cos \frac{t}{\sqrt{K}} dr - ie^{\frac{\nu}{2}} \sin \frac{t}{\sqrt{K}} dt. \tag{19}$$

The result is

$$-(dz^1)^2 + (dz^2)^2 = -e^\nu dt^2 - \frac{1}{4} K e^\nu (\nu')^2 dr^2.$$

Since

$$(dz^3)^2 + (dz^4)^2 + (dz^5)^2$$

remains the same, we obtain the line element

$$ds^2 = -e^\nu dt^2 + \left(1 - \frac{1}{4} K e^\nu (\nu')^2\right) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \tag{20}$$

It follows that

$$e^\lambda = 1 - \frac{1}{4} K e^\nu (\nu')^2, \tag{21}$$

in agreement with Equation (15). So changing the sign of the parameter  $K$  has the same effect as changing the extra spacelike dimension in the embedding space to a timelike dimension. The periodicity of  $K(t)$  therefore results in a periodic signature change in the embedding space and suggests an oscillating behavior. The periodicity of  $K$  produces a mathematical model that is consistent with the equation of state  $p = \omega\rho$  discussed in Ref. [5], where  $\omega(t) = \omega_0 + \omega_1 (t\dot{H}/H)$ . Here  $H = H(t)$  is the time varying Hubble parameter resulting in both expanding and contracting epochs of the Universe. Similarly,

Ref. [19] discusses a form of dark energy called quintom having an oscillating equation of state, which, in turn, leads to oscillations of the Hubble parameter and a recurring Universe.

### 4. The Oscillating Universe

Returning to line element (2), let us first list the Einstein field equations:

$$8\pi\rho = e^{-\lambda} \left( \frac{\lambda'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2}, \tag{22}$$

$$8\pi p = e^{-\lambda} \left( \frac{1}{r^2} + \frac{\nu'}{r} \right) - \frac{1}{r^2}, \tag{23}$$

and

$$8\pi p_t = \frac{1}{2} e^{-\lambda} \left[ \frac{1}{2} (\nu')^2 + \nu'' - \frac{1}{2} \lambda' \nu' + \frac{1}{r} (\nu' - \lambda') \right]. \tag{24}$$

Next, we state the Friedmann-Lemaître-Robertson-Walker (FLRW) model in the usual four dimensions [20]:

$$ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1-kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right], \tag{25}$$

where  $a^2(t)$  is a scale factor. In this paper we also make use of the Friedmann equation

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi}{3} (\rho + 3p), \tag{26}$$

again using units in which  $c = G = 1$ .

Before continuing, we need one more important observation: in the outer region of a galactic halo, we have [21]

$$ds^2 = -B_0 r^l dt^2 + e^{\lambda(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \tag{27}$$

where  $B_0$  is a constant and  $l = 0.000001$  is the tangential velocity. So  $\nu(r) = \ln B_0 + l \ln r$  and

$$\nu'(r) = \frac{l}{r} > 0. \tag{28}$$

On large scales,  $\nu'(r)$  becomes negligible, which is consistent with the FLRW model.

Returning once again to Equation (10), we obtain next

$$\lambda' = \frac{1}{1 + \frac{1}{4} K e^{\nu} (\nu')^2} \frac{1}{4} K e^{\nu} \left[ (\nu')^2 + 2\nu'\nu'' \right],$$

and from Equations (22) and (23), we have

$$8\pi(\rho + 3p) = e^{-\lambda} \left[ \frac{1/r}{1 + \frac{1}{4} K e^{\nu} (\nu')^2} \frac{1}{4} K e^{\nu} \left[ (\nu')^3 + 2\nu'\nu'' \right] - \frac{1}{r^2} \right] + \frac{1}{r^2} + 3e^{-\lambda} \left( \frac{1}{r^2} + \frac{\nu'}{r} \right) - \frac{3}{r^2}. \tag{29}$$

Finally, it follows from Equation (10) that Equation (29) can be written as

$$\begin{aligned}
 8\pi(\rho + 3p) &= -\frac{2}{r^2} + \frac{2e^{-\lambda}}{r^2} + e^{-\lambda} \left[ \frac{1/r}{1 + \frac{1}{4}Ke^{\nu}(\nu')^2} \cdot \frac{1}{4}Ke^{\nu}[(\nu')^3 + 2\nu'\nu''] + \frac{3\nu'}{r} \right] \\
 &= -\frac{2}{r^2} \frac{\frac{1}{4}Ke^{\nu}(\nu')^2}{1 + \frac{1}{4}Ke^{\nu}(\nu')^2} + \frac{1}{1 + \frac{1}{4}Ke^{\nu}(\nu')^2} \left[ \frac{1/r}{1 + \frac{1}{4}Ke^{\nu}(\nu')^2} \cdot \frac{1}{4}Ke^{\nu}[(\nu')^3 + 2\nu'\nu''] + \frac{3\nu'}{r} \right] \quad (30) \\
 &= \frac{1}{1 + \frac{1}{4}Ke^{\nu}(\nu')^2} \left[ -\frac{2}{r^2} \cdot \frac{1}{4}Ke^{\nu}(\nu')^2 + \frac{1/r}{1 + \frac{1}{4}Ke^{\nu}(\nu')^2} \cdot \frac{1}{4}Ke^{\nu}[(\nu')^3 + 2\nu'\nu''] + \frac{3\nu'}{r} \right].
 \end{aligned}$$

The question now arises: what happens when  $K(t)$  passes zero to become negative? We already know that if  $K$  is replaced by  $-K$ , Equation (10) becomes Equation (15), while Equation (30) becomes

$$\begin{aligned}
 8\pi(\rho + 3p) &= \frac{1}{1 - \frac{1}{4}Ke^{\nu}(\nu')^2} \left[ \frac{2}{r^2} \cdot \frac{1}{4}Ke^{\nu}(\nu')^2 - \frac{1/r}{1 - \frac{1}{4}Ke^{\nu}(\nu')^2} \right. \\
 &\quad \left. \cdot \frac{1}{4}Ke^{\nu}[(\nu')^3 + 2\nu'\nu''] + \frac{3\nu'}{r} \right], K > 0. \quad (31)
 \end{aligned}$$

This result could also be obtained directly from Equation (15).

In view of Equation (28), both  $(\nu')^3 + 2\nu'\nu''$  and  $3\nu'/r$  are negligibly small on large scales. So for  $K(t) > 0$ , Equation (30) implies that  $8\pi(\rho + 3p) < 0$ . This shows that  $\ddot{a}(t)$  in the Friedmann equation is positive, indicating an accelerated expansion. For  $K(t) < 0$ , we obtain from Equation (31) that  $8\pi(\rho + 3p) > 0$  and  $\ddot{a}(t) < 0$ , indicating a decelerating expansion.

The deceleration will continue until  $K(t) = 0$  to start a new cycle with  $\ddot{a}(t) > 0$ . At this point, the Universe would experience a big bounce or a big-bang singularity. For a discussion of the big bounce, see Poplawski [22] and references therein. For the case of a singularity, it needs to be stressed that the embedding theory goes well beyond the usual models in the following sense: during the decelerating phase, we have an extra timelike dimension. This is similar to Hawking’s imaginary time discussed in Ref. [23]: while ordinary time would still have a big-bang singularity, imaginary time avoids the singularity, also called the no-boundary proposal. Our extra timelike dimension would have the same effect, thereby allowing the oscillations to continue unhindered.

Having a plausible mathematical model does not automatically yield the best physical interpretation, especially if the model contains a free parameter. So let us recall from Section 2 that  $K$  started off as an arbitrary constant that was subsequently replaced by a periodic function of  $t$  without affecting the solution. The best physical interpretation of the resulting model appears to be the signature change discussed in Section 3 simply because the outcome complements the

various models in Refs. [1]-[7].

## 5. Conclusions

This paper begins with a brief review of the literature dealing with signature-changing events in various gravitational theories. This paper discusses these issues in the context of the well-established embedding theory in an  $n$ -dimensional Riemannian space. Such a space is said to be of embedding class  $m$  if  $m+n$  is the lowest dimension of the flat space in which the given space can be embedded; here  $m = \frac{1}{2}n(n+1)$ . So a four-dimensional Riemannian space is of class two and can therefore be embedded in a six-dimensional flat space. We also made use of the fact that a metric of class two can be reduced to a metric of class one by a suitable coordinate transformation. Moreover, Einstein's theory allows the extra dimension to be either spacelike or timelike.

It is first shown that a spacelike extra dimension leads to an accelerated expansion, thereby accounting for the mysterious dark energy. An extra timelike dimension, on the other hand, leads to a decelerating expansion. Our mathematical model includes a free parameter  $K$  which we can take to be a periodic function of time, resulting in an oscillating solution. This assumption can be justified for physical reasons: we have seen that the best physical interpretation comes from the well-established embedding theory that clearly points to a signature-changing event. This outcome is very much in line with the various models in Refs. [1]-[7]. An important additional feature of the embedding space is the extra timelike dimension during the decelerating phase. This allows the oscillations to continue whether the transition involves a big-bang singularity or just a big bounce.

## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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