International Journal of Astronomy and Astrophysics, 2021, 11, 481-488
https://www.scirp.org/journal/ijaa
ISSN Online: 2161-4725
ISSN Print: 2161-4717

Relativistic Motion with Viscosity: II Stokes's Law of Resistance

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How to cite this paper: Zaninetti, L. (2021) Relativistic Motion with Viscosity: II Stokes's Law of Resistance. *International Journal of Astronomy and Astrophysics*, **11**, 481-488.
https://doi.org/10.4236/ijaa.2021.114025

Received: September 15, 2021
Accepted: November 12, 2021
Published: November 15, 2021

Abstract

The deduction of a relativistic and mildly relativistic equation of motion in the presence of a drag force proportional to the velocity is presented. The obtained results are used to model the trajectory of the supernova SN1993J and the light curves of gamma-ray bursts.

Keywords

Supernovae, General Supernovae, Individual (SN 1993J) ISM, Supernova Remnants GRB, Individual (GRB 130427A) GRB, Individual (GRB 120521C) GRB, Individual (GRB 130606A)

1. Introduction

A relativistic treatment of the equation of motion in the presence of a resistive force proportional to the velocity has been investigated in the following models: a model for the Newtonian scattering of photons [1], a motion through a uniform adiabatic medium on the steady-state accretion of matter onto a Schwarzschild black hole [2], an extreme mass-ratio inspirals around strongly accreting supermassive black holes [3], and ultra-relativistic detonations in the framework of the cosmological first-order phase transitions [4]. In Section 2, this paper explores the relativistic law of motion in the presence of viscosity proportional to the velocity. Section 3 is devoted to the astrophysical applications.

2. The Equation of Motion

2.1. The Classic Case

We assume a one-dimensional motion with a resistive force of Stokes type [5], $F_{res} = -Amv(t)$, where A is a constant, m is the considered mass and $v(t)$ is the velocity. The differential equation which governs the motion is

$$v(t)=\frac{v_0 e^{-At}}{e^{-t_0 A}}, \tag{1}$$

which has an analytical solution in an explicit form

$$v(t;A,v_0,t_0)=v(t)=\frac{v_0 e^{-At}}{e^{-t_0 A}}, \tag{2}$$

where v_0 is the velocity at $t=t_0$. The equation of motion in the explicit form is

$$r(t;A,v_0,t_0,r_0)=-\frac{v_0\left(e^{-At}-e^{-t_0 A}\right)e^{t_0 A}}{A}+t_0 \tag{3}$$

where r_0 is the distance at $t=t_0$. The numerical value of the constant A is

$$A=\frac{\ln\left(\frac{v_1}{v_0}\right)}{t_0-t_1}, \tag{4}$$

where v_1 is the velocity at $t=t_1$.

2.2. The Relativistic Case

We assume a one-dimensional motion with a resistive force of Stokes type, $F_{res}=-Am_0 v(t)$, where A is a constant, m_0 is the considered rest mass and $v(t)$ is the velocity. Newton's second law in special relativity is:

$$F=\frac{d}{dt}\left(\frac{m_0 v(t)}{\sqrt{1-\frac{v(t)^2}{c^2}}}\right), \tag{5}$$

where F is the force, m_0 is the rest mass, c is the velocity of light and $v(t)$ is the velocity; see Equation (7.16) in [6]. The first order differential equation in the velocity which governs the relativistic motion is

$$\frac{\frac{d}{dt}v(t)}{\left(1-\frac{(v(t))^2}{c^2}\right)^{\frac{3}{2}}}=-A(v(t)). \tag{6}$$

An analytical solution to the above first order differential does not exist; however, a solution exists for $v(t)$ in an implicit form for the time

$$t=\frac{N}{D}, \tag{7}$$

$$\begin{aligned}N=&\left(-2c^3+2cv_0^2\right)\sqrt{c^2-v^2}-\Big(\Big(-2t_0 A+\ln\left(\sqrt{c^2-v^2}-c\right)\\&-\ln\left(c+\sqrt{c^2-v^2}\right)-\ln\left(\sqrt{c^2-v_0^2}-c\right)+\ln\left(c+\sqrt{c^2-v_0^2}\right)\Big)c^2\\&-2c\sqrt{c^2-v_0^2}-v_0^2\Big(-2t_0 A+\ln\left(\sqrt{c^2-v^2}-c\right)-\ln\left(c+\sqrt{c^2-v^2}\right)\\&-\ln\left(\sqrt{c^2-v_0^2}-c\right)+\ln\left(c+\sqrt{c^2 v_0^2}\right)\Big)\Big)(c-v)(c+v).\end{aligned} \tag{8}$$

and

$$D = 2A\left(c^2 - v^2\right)\left(c^2 - v_0^2\right), \tag{9}$$

where v_0 is the velocity at $t = t_0$. The constant A can be derived from the following formula

$$A = \frac{NN}{DD}, \tag{10}$$

where

$$\begin{aligned} NN = {} & \left(-2c^3 + 2cv_1^2\right)\sqrt{c^2 - v_0^2} + \left(c + v_0\right)\left(c - v_0\right)\Bigg(\Bigg(\ln\left(\sqrt{c^2 - v_1^2} - c\right) \\ & - \ln\left(c + \sqrt{c^2 - v_1^2}\right) - \ln\left(\sqrt{c^2 - v_0^2} - c\right) + \ln\left(c + \sqrt{c^2 - v_0^2}\right)\Bigg)c^2 \\ & + 2\sqrt{c^2 - v_1^2}\,c - v_1^2\Bigg(\ln\left(\sqrt{c^2 - v_1^2} - c\right) - \ln\left(c + \sqrt{c^2 - v_1^2}\right) \\ & - \ln\left(\sqrt{c^2 - v_0^2} - c\right) + \ln\left(c + \sqrt{c^2 - v_0^2}\right)\Bigg)\Bigg), \end{aligned} \tag{11}$$

and

$$DD = 2\left(t_0 - t_1\right)\left(c^2 - v_1^2\right)\left(c^2 - v_0^2\right), \tag{12}$$

where v_1 is the velocity at $t = t_1$.

2.3. The Mildly-Relativistic Case

The first order differential equation for the mildly-relativistic motion is

$$\frac{\mathrm{d}}{\mathrm{d}t}v(t) + \frac{3v(t)^2\left(\frac{\mathrm{d}}{\mathrm{d}t}v(t)\right)}{2c^2} = -Av(t), \tag{13}$$

which has solution

$$v\left(t;t_0,v_0\right) = \mathrm{e}^{-At - \frac{W\left(\frac{3}{2c^2\left(\mathrm{e}^B\right)^2\left(\mathrm{e}^{At}\right)^2}\right)}{2} - B}, \tag{14}$$

where W is the Lambert W function [7] and

$$B = -\frac{4t_0Ac^2 + 4\ln\left(v_0\right)c^2 + 3v_0^2}{4c^2}, \tag{15}$$

with v_0 being the velocity at $t = t_0$. The trajectory in the mildly relativistic case is

$$r\left(t;t_0,r_0,v_0\right) = -\frac{\mathrm{e}^{t_0A}c^2v_0\left(W\left(D\right)+3\right)\left(\mathrm{e}^{\frac{v_0^2}{c^2}}\right)^{\frac{3}{4}} - 3\mathrm{e}^{At}\sqrt{\mathrm{e}^{W(D)}}\left(\left(Ar_0 + v_0\right)c^2 + \frac{v_0^3}{2}\right)}{3\sqrt{\mathrm{e}^{W(D)}}\,\mathrm{e}^{At}c^2A}, \tag{16}$$

where

$$D = \frac{3v_0^2 \mathrm{e}^{\frac{-4(t-t_0)c^2 A + 3v_0^2}{2c^2}}}{2c^2}, \tag{17}$$

with r_0 being r at $t = t_0$. The constant A can be derived in the mildly relativistic case by the following formula

$$A(t_0, t_1, v_0) = \frac{-v_0^2 \left(4 \ln\left(\frac{v_0}{v_1}\right) c^2 + 3v_0^2 - 3v_1^2\right)}{4v_0^2 c^2 (t_0 - t_1)}, \tag{18}$$

where v_1 is the velocity at $t = t_1$.

2.4. Astrophysical Luminosity

The mechanical relativistic luminosity is

$$L_{m,r} = 4\pi r(t)^2 \frac{1}{1-\beta(t)^2} \rho_0 \left(\frac{r_0}{r}\right)^d c^3 \beta(t), \tag{19}$$

where $r(t)$ is the temporary radius of the expansion, r_0 is the radius at $t = t_0$, ρ_0 is the density at $t = t_0$, d is a shape parameter and $\beta(t) = \frac{v(t)}{c}$. The observed luminosity, L_{obs}, is assumed to scale as

$$L_{obs} = C_{obs} L_{m,r} \left(1 - \mathrm{e}^{-\tau_\nu}\right), \tag{20}$$

where C_{obs} is a constant that allows the match between theory and observations, and $-\tau_\nu$ is the optical thickness.

3. Astrophysical Applications

The astrophysical units are chosen to be pc for the length and years for the time: the constant A is therefore expressed in $\frac{1}{\mathrm{yr}}$. A test for the quality of the fits is represented by the merit function χ^2

$$\chi^2 = \sum_j \frac{(r_{th} - r_{obs})^2}{\sigma_{obs}^2},$$

where r_{th}, r_{obs} and σ_{obs} are the theoretical radius, the observed radius and the observed uncertainty, respectively.

3.1. Application to SN 1993J

Figure 1 reports the numerical trajectory, of SN 1993J for which observational parameters are available [8] [9] with data as in **Table 1**.

3.2. Application to GRBs

A first example is applied to the light curve (LC) of GRB 130427A , which was the most luminous gamma-ray burst in the last 30 years; see **Figure 1** in [10]. **Figure 2** reports the X-flux as a function of the time and the relative theoretical data, with data as in **Table 2**.

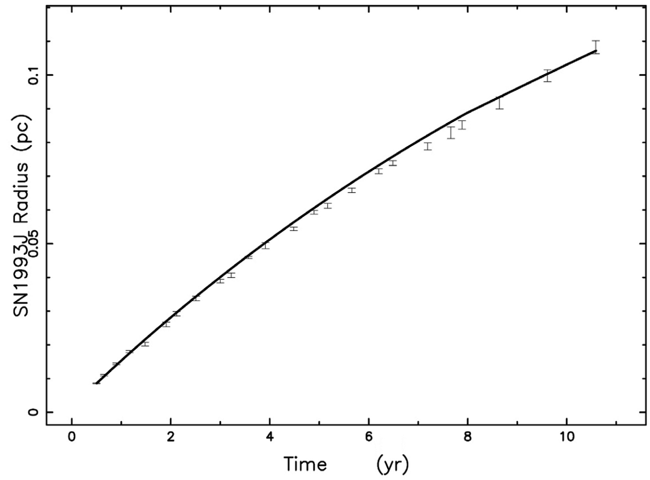

Figure 1. Numerical radius (full line) and astronomical data of SN 1993J with vertical error bars.

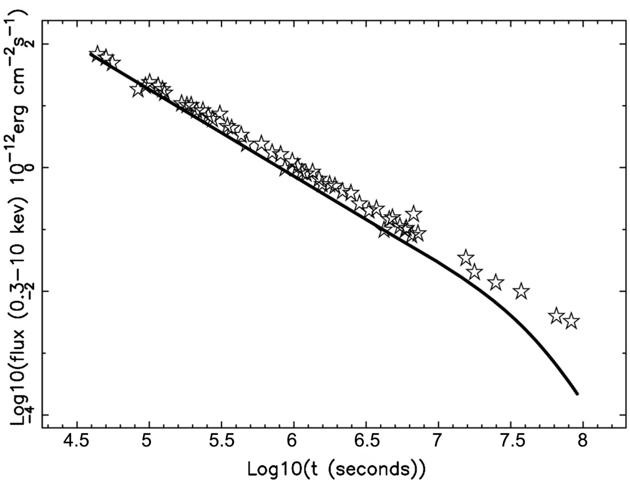

Figure 2. Flux in the X-ray as a function of time in seconds for GRB 130427A (empty stars) and theoretical curve as given by Equation (20) (full line) when $\tau_\nu = \infty$ with data are as in **Table 2**.

Table 1. Numerical values for the parameters of Stokes's theoretical model applied to SN 1993J.

model	values	χ^2
Stokes's	$r_0 = 3.0\times10^{-3}$ pc; $v_0 = 13800$ km/s; $A = 0.07\frac{1}{\text{years}}$	85.7

A second example is applied to the LC in X-ray of GRB 120521C 2, see **Figure 2** in [11], which is reported in **Figure 3**, with temporal behavior of the optical depth as in **Figure 4**.

A third example is given by the LC in X-ray of GRB 130606A, see **Figure 2** in [11], which is reported in **Figure 5**, with the temporal behavior of the optical depth as in **Figure 6**.

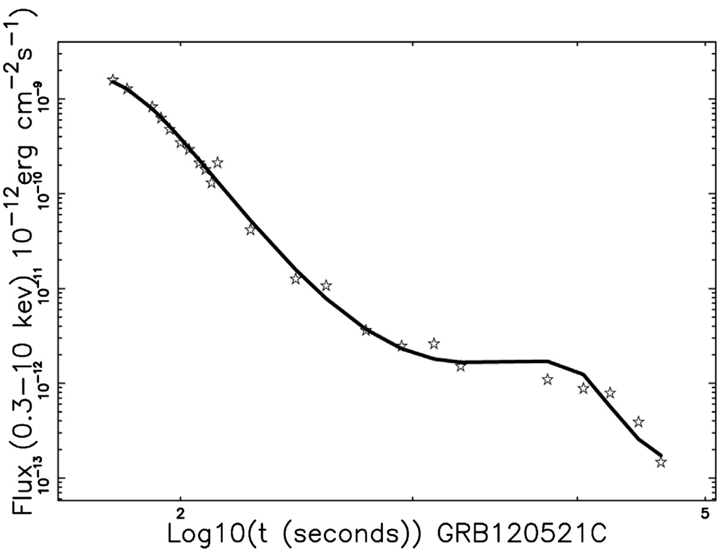

Figure 3. Flux in the X-ray as function of time in seconds for GRB 120521C (empty stars) and theoretical curve as given by Equation (20) (full line), with τ_ν as in **Figure 4** and with data as in **Table 2**.

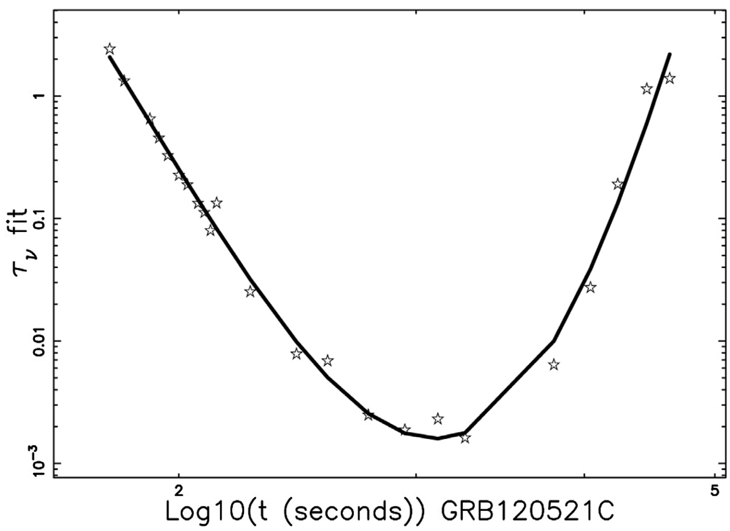

Figure 4. The time dependence of τ_ν (empty stars) for GRB 120521C and a logarithmic polynomial approximation of degree 5 (full line). Parameters as in **Table 2**.

Table 2. Numerical values of the parameters for the theoretical model.

GRB name	theoretical parameters
GRB 130427A	$r_0 = 9.9\times10^{-5}$ pc; $t_0 = 1.0\times10^{-3}$ year; $\beta_0 = 0.9$; $A = 1\frac{1}{\text{pc}}$; $d = 3.1$
GRB 120521C	$r_0 = 1.0\times10^{-4}$ pc; $t_0 = 1.0\times10^{-6}$ year; $\beta_0 = 0.9$; $A = 10000\frac{1}{\text{pc}}$; $d = 3$
GRB 130606A	$r_0 = 1.0\times10^{-4}$ pc; $t_0 = 1.0\times10^{-6}$ year; $\beta_0 = 0.9$; $A = 1000\frac{1}{\text{pc}}$; $d = 2$

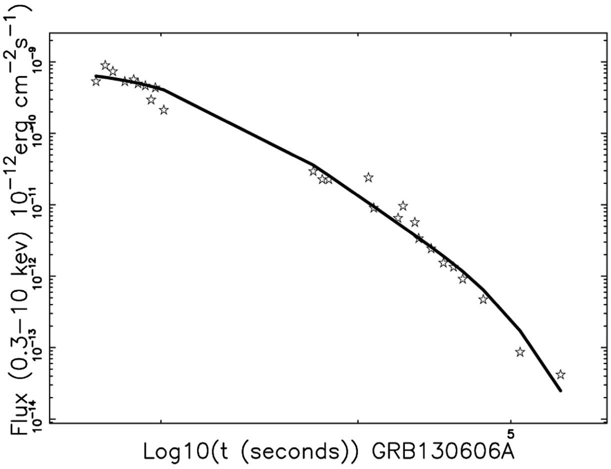

Figure 5. Flux in the X-ray as a function of time in seconds for GRB 130606A (empty stars) and theoretical curve as given by Equation (20) (full line), with τ_ν as in **Figure 6** and with data as in **Table 2**.

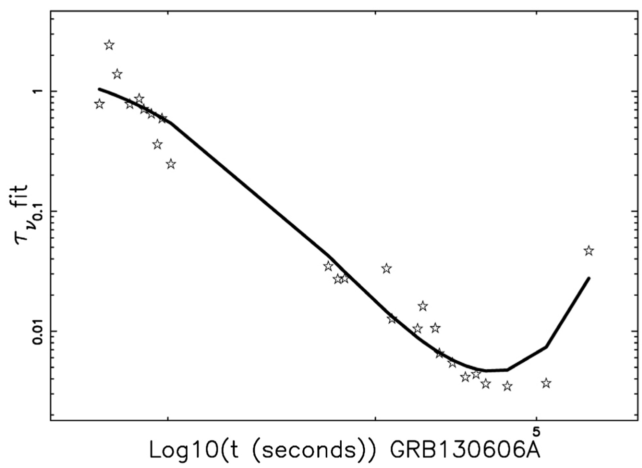

Figure 6. The time dependence of τ_ν (empty stars) for GRB 130606A and a logarithmic polynomial approximation of degree 5 (full line). Parameters as in **Table 2**.

4. Conclusions

We analyzed the one-dimensional relativistic motion in the presence of a resistive force proportional to the velocity. An analytical solution for the velocity was derived in an implicit form, see Equation (7). In the mildly relativistic case, we derived an analytical solution for both the velocity, see Equation (14), and the distance, see Equation (16), in terms of the Lambert W function.

A first test to evaluate the constant A in an astrophysical environment is on SN 1993J. A full relativistic treatment of the LC for GRBs was done for GRB 130427A, GRB 120521C and GRB 130606A in the framework of the optical thickness with a time dependence.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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