# Relativistic Motion with Viscosity. I Newton's Law of Resistance

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## Abstract

The equation of motion for a relativistic neutral particle that moves in a medium characterized by a friction proportional to the square of the velocity is analyzed. The relativistic trajectory is derived in a numerical way and in the form of a Taylor series. The astrophysical applications cover the trajectory of SN 1993J and the light curve of gamma ray bursts.

## Keywords

Supernovae, General, Supernovae, (Individual: SN1993j), Gamma-Ray Burst, (Individual: GRB 130427A), Gamma-Ray Burst, (Individual: GRB 060729)

## 1. Introduction

Relativistic viscosity has the following applications: change in the mean particle momentum and spreading around the mean for the cosmic rays (CR) [1]; acceleration of CR in shear flows, such as active galactic nuclei (AGN), gamma ray burst (GRB) and jets [2] [3]; interaction of a neutral particle with the microwave background radiation (CMB) [4] [5]; and generation of the CMB in the expanding universe [6]. The Lagrangian and a Hamiltonian for a relativistic particle moving in a dissipative medium characterized by a force that depends on the square of the velocity of the particle have been derived [7]. This paper is a highly idealized attempt to model SN light curves by assuming that the resistivity of the ambient interstellar medium is quadratic to the velocity of the SN envelope. Section 2 derives a relativistic equation of motion in the presence of viscosity proportional to the square of the velocity. Section 3 applies the relativistic results to the motion of SN 1993J, to the light curve of GRB 130427A and GRB 060729. The CMB is not related with the model that is presented here.

## 2. The Equation of Motion

We assume a one dimensional motion with a resistive force of Newtonian type, $F_{res} = -Bm_0 v(t)^2$, where $B$ is a constant, $m_0$ is the considered mass and $v(t)$ is the velocity. In the following, we will consider only positive and decreasing velocities. Newton's second law in special relativity is:

$$F = \frac{dp}{dt} = \frac{d}{dt}(mv(t)) = \frac{d}{dt}\left(\frac{m_0 v(t)}{\sqrt{1 - \frac{v(t)^2}{c^2}}}\right) = \frac{d}{dt}(m_0 v\gamma), \tag{1}$$

where $F$ is the force, $p$ is the relativistic momentum, $m$ is the relativistic mass, $m_0$ is the rest mass, $c$ is the velocity of light, $v(t)$ is the velocity and $\gamma = \frac{1}{\sqrt{1 - \frac{v(t)^2}{c^2}}}$ is the Lorentz factor; see equation (7.16) in [8]. The first order differential equation in the velocity that governs the motion is

$$\frac{\frac{d}{dt} v(t)}{\left(1 - \frac{(v(t))^2}{c^2}\right)^{\frac{3}{2}}} = -B(v(t))^2. \tag{2}$$

The solution to this first order differential equation for $v(t)$ in an implicit form is

$$\frac{c^2 - 2v^2}{cv}\frac{1}{\sqrt{c^2 - v^2}} - \frac{c^2 - 2v_0^2}{cv_0}\frac{1}{\sqrt{c^2 - v_0^2}} = B(t - t_0), \tag{3}$$

where $v_0$ is the velocity at $t = t_0$. An explicit solution for the velocity can be obtained by considering the physical solution of the previous algebraic equation of fourth degree in $v$

$$v^4 + pv^2 + q = 0, \tag{4}$$

with

$$p = -c^2, \tag{5}$$

and

$$q = \frac{(c^2 - v_0^2)^{\frac{3}{2}} c^3 v_0^2}{NQ} \tag{6}$$

where

$$\begin{aligned}
NQ = {} & B^2\sqrt{c^2 - v_0^2}\, c^3 t^2 v_0^2 - 2B^2\sqrt{c^2 - v_0^2}\, c^3 t t_0 v_0^2 + B^2\sqrt{c^2 - v_0^2}\, c^3 t_0^2 v_0^2 \\
& - B^2\sqrt{c^2 - v_0^2}\, c t^2 v_0^4 + 2B^2\sqrt{c^2 - v_0^2}\, c t t_0 v_0^4 - B^2\sqrt{c^2 - v_0^2}\, c t_0^2 v_0^4 \\
& + 2Bc^4 t v_0 - 2Bc^4 t_0 v_0 - 6Bc^2 t v_0^3 + 6Bc^2 t_0 v_0^3 + 4Bt v_0^5 \\
& - 4Bt_0 v_0^5 + \sqrt{c^2 - v_0^2}\, c^3.
\end{aligned} \tag{7}$$

Before we continue, we will introduce the following simplification.

**Conjecture 1.** *In the presence of more than one solution for the temporal evolution of the velocity/space, we select the physical one that has a positive decreasing/increasing behavior.* According to this statement, the physical solution is

$$v(t) = \frac{1}{2}\sqrt{-2\sqrt{p^2 - 4q} - 2p}. \tag{8}$$

We already know that $v = v_0$ at $t = t_0$. Once we know that $v = v_1$ at $t = t_1$, it is possible to derive the unknown parameter $B$ from the previous formula

$$B = \frac{c^4 v_1 - 2c^2 v_0^2 v_1 - c^2 v_1^3 + 2v_0^2 v_1^3 - \sqrt{c^2 - v_0^2}\sqrt{v_0^2\left(c^2 - v_1^2\right)\left(c^2 - 2v_1^2\right)^2}}{\sqrt{c^2 - v_0^2}\, v_0 v_1 \left(t_0 - t_1\right)\left(c^2 - v_1^2\right)c}. \tag{9}$$

At this moment, we are unable to obtain an analytical solution for $r(t)$. Therefore, the trajectory is obtained by a numerical integration of Equation (8).

An approximation for the trajectory is represented by a Taylor series expansion about $r = r_0$ of order 2

$$r(t; t_0, r_0, v_0, B) = r_0 + v_0\left(t - t_0\right) - \frac{B v_0^2 \left(\frac{c^2 - v_0^2}{c^2}\right)^{\frac{3}{2}} \left(t - t_0\right)^2}{2} \\ + \frac{B^2 v_0^3 \left(c^2 - v_0^2\right)^2 \left(2c^2 - 5v_0^2\right)\left(t - t_0\right)^3}{6c^6}, \tag{10}$$

which means the following approximate velocity as function of time

$$v(t; t_0, r_0, v_0, B) = v_0 - B v_0^2 \left(\frac{c^2 - v_0^2}{c^2}\right)^{\frac{3}{2}} \left(t - t_0\right) \\ + \frac{B^2 v_0^3 \left(c^2 - v_0^2\right)^2 \left(2c^2 - 5v_0^2\right)\left(t - t_0\right)^2}{2c^6}. \tag{11}$$

We now analyze the mildly relativistic case for which the relativistic momentum in the case of a unit mass is

$$p(t) = v(t) + \frac{v(t)^3}{2c^2}, \tag{12}$$

which means the following first order differential equation for the velocity

$$\frac{d}{dt}v(t) + \frac{3v(t)^2\left(\frac{d}{dt}v(t)\right)}{2c^2} = -Bv(t)^2. \tag{13}$$

The velocity in the mildly relativistic case is

$$v(t) = \frac{2Bc^2 t_0 v_0 - 2Bc^2 v_0 t - 2c^2 + 3v_0^2 + \sqrt{VA}}{6v_0} \tag{14}$$

where

$$VA = 4B^2 c^4 t^2 v_0^2 - 8B^2 c^4 t t_0 v_0^2 + 4B^2 c^4 t_0^2 v_0^2 + 8Bc^4 t v_0 - 8Bc^4 t_0 v_0$$
$$- 12Bc^2 t v_0^3 + 12Bc^2 t_0 v_0^3 + 4c^4 + 12c^2 v_0^2 + 9v_0^4. \tag{15}$$

The indefinite integral of the velocity is

$$I(t) = \int v(t)\,\mathrm{d}t$$
$$= \frac{1}{24Bv_0^2 c^2} \left( -4B^2 c^4 t^2 v_0^2 + 8B^2 c^4 t t_0 v_0^2 - 8Bc^4 t v_0 + 12Bc^2 t v_0^3 \right.$$
$$+ 2B\sqrt{IA}\, c^2 t v_0 - 2B\sqrt{IA}\, c^2 t_0 v_0 + 24\ln\left(\sqrt{IA} - 3v_0^2\right)$$
$$\left. + \left(2t - 2t_0\right)Bc^2 v_0 + 2c^2\right)v_0^2 c^2 + 2\sqrt{IA}\, c^2 - 3\sqrt{IA}\, v_0^2 \Big), \tag{16}$$

where

$$IA = 4\left(1 + Bv_0\left(t - t_0\right)\right)^2 c^4 - 12v_0^2 \left(-1 + Bv_0\left(t - t_0\right)\right)c^2 + 9v_0^4, \tag{17}$$

which leads to the following trajectory in the mildly relativistic case

$$r(t) = I(t) - I(t_0). \tag{18}$$

The unknown parameter $B$ in the mildly relativistic case is

$$B = \frac{-v_0\left(2c^2 v_0 - 2c^2 v_1 + 3v_0^2 v_1 - 3v_0 v_1^2\right)}{2c^2 v_0^2 v_1 \left(t_0 - t_1\right)}. \tag{19}$$

## The Relativistic Luminosity

The relativistic transfer of energy through a surface, $A$, is

$$L_{m,r} = A\gamma^2 \left(\rho c^2 + p\right)v, \tag{20}$$

where $p$ is the pressure. For sake of simplicity, we take $p = 0$, and $\gamma$ is the Lorentz factor, see eqn. A31 in [9] or eqn. (43.44) in [10]. In the case of a spherical cold expansion

$$L_{m,r} = 4\pi r(t)^2 \frac{1}{1 - \beta(t)^2} \rho(t) c^3 \beta(t). \tag{21}$$

We now assume the following power-law behavior for the density in the advancing layer of radius $r$

$$\rho(r) = \rho_0 \left(\frac{r_0}{r}\right)^d, \tag{22}$$

which has the following temporal scaling

$$\rho(t) = \rho_0 \left(\frac{r_0}{r(t; t_0, r_0, v_0, B)}\right)^d, \tag{23}$$

where $r(t)$ is given by the Taylor series represented by Equation (10). The mechanical relativistic luminosity is

$$L_{m,r} = 4\pi r(t)^2 \frac{1}{1 - \beta(t)^2} \rho_0 \left(\frac{r_0}{r}\right)^d c^3 \beta(t). \tag{24}$$

Once we insert formulae (10) and (11) in this equation, we obtain an ap-

proximate expression for the mechanical relativistic energy, which has a complicated expression that is not here reported. We now parameterize the presence of the absorption introducing a the optical thickness $\tau_\nu$. The observed luminosity is assumed to be

$$L_{obs} = C_{obs} L_{m,r} \left(1 - e^{-\tau_\nu}\right), \tag{25}$$

where $C_{obs}$, which is here assumed to be constant in the interval of time considered here, allows the match between theory and observations. The optical thickness $\tau_\nu$ takes the value $\infty$ in the case of an optically thin medium or can be function of time to simulate the complex behavior of the observed luminosity; more details can be found in [11]. The observed absolute magnitude is

$$M_{obs} = -\log_{10}\left(L_{obs}\right) + k_{obs}, \tag{26}$$

where $k_{obs}$ is a constant of match between theory and observations.

## 3. Astrophysical Applications

The chosen astrophysical units are pc for length and yr for time. With these units, the initial velocity is $v_0 \left(\text{km} \cdot \text{s}^{-1}\right) = 9.7968 \times 10^5 v_0 \left(\text{pc} \cdot \text{yr}^{-1}\right)$.

As a first example, we apply the above results to the deduction of the parameter $B$ to SN 1993J for which observational times and velocities are available [12] [13], see **Table 1**. A test for the quality of the fits is represented by the merit function $\chi^2$

$$\chi^2 = \sum_j \frac{\left(r_{th} - r_{obs}\right)^2}{\sigma_{obs}^2},$$

where $r_{th}$, $r_{obs}$ and $\sigma_{obs}$ are the theoretical radius, the observed radius and the observed uncertainty, respectively. Once $B$ is derived on an observational basis, **Figure 1** reports the analytical velocity and **Figure 2** reports the numerical trajectory, with data as given in **Table 1**.

A comparison between the numerical solution with the Runge Kutta method and a Taylor series is reported in **Figure 3**.

The second example is applied to the light curve (LC) of GRB 130427A, which was the most luminous gamma-ray burst in the last 30 years, see **Figure 1** in [14]. **Figure 4** reports the X-flux as a function of time and the relative theoretical

**Table 1.** Numerical values of the parameters for the fit and the theoretical model applied to SN 1993J.

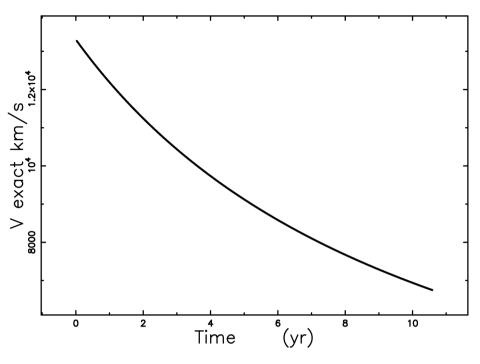| model | values | $\chi^2$ |
|---|---|---|
| Fit by a power law | $\alpha_{fit} = 0.828; C = 0.015$ | 43 |
| Newtonian viscosity | $r_0 = 3.0 \times 10^{-3}\,\text{pc}; v_0 = 13800\,\text{km/s}$ | 45.59 |
| B evaluation | $v_0 = 2.39 \times 10^{-2}\,\text{pc/yr}, v_1 = 8.6 \times 10^{-3}\,\text{pc/yr}$ | |
| B evaluation | $t_0 = 2.6 \times 10^{-2}\,\text{yr}, t_1 = 10.58 \times 10^{-2}\,\text{yr}$ | $B = 7.082\,\text{pc}^{-1}$ |

**Figure 1.** Analytical velocity versus time (full line) for SN 1993J as given by formula (8).
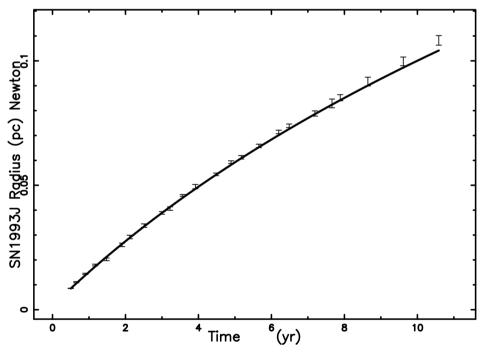


**Figure 2.** Numerical radius (full line) and astronomical data of SN 1993J with vertical error bars.

data with data as in **Table 2**.

The third example is dedicated to the LC for GRB 060729 as observed by the Ultraviolet and Optical Telescope (UVOT) in the time interval $[10^{-2}\text{-}26]$ days, see **Figure 1** in [15]. **Figure 5** presents the LC of UVOT (U) apparent magnitude for GRB 060729 with data as in **Table 3**, and **Figure 6** presents the temporal
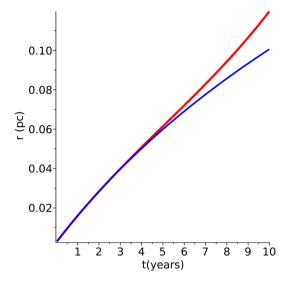
**Figure 3.** Numerical radius (blue line) and Taylor solution (red line) as represented by Equation (10).
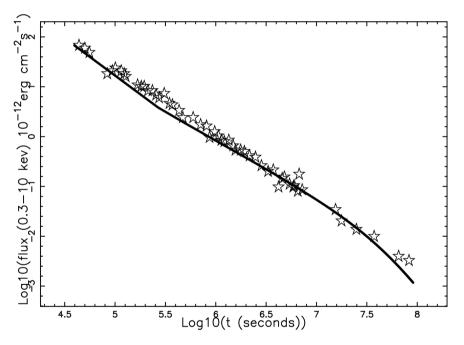


**Figure 4.** Flux in the X-ray as function of time in seconds for GRB 130427A (empty stars) and theoretical curve as given by Equation (25) (full line) when $\tau_{\nu} = \infty$ with data as in **Table 2**.

**Table 2.** Numerical values of the parameters for the theoretical model applied to GRB 130427A.

$$r_0 = 1.0 \times 10^{-4}\,\text{pc}; t_0 = 1.0 \times 10^{-4}\,\text{s}; \beta_0 = 0.99; B = 7.082\,\text{pc}^{-1}; d = 3.1$$

**Table 3.** Numerical values of the parameters for the theoretical model applied to GRB 060729.

$$r_0 = 1.0 \times 10^{-4}\,\text{pc}; t_0 = 2.0 \times 10^{-6}\,\text{s}; \beta_0 = 0.99; B = 7.082\,\text{pc}^{-1}; d = 3$$
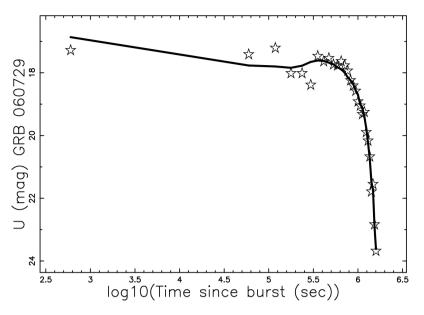
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**Figure 5.** The LC of UVOT (U) + HST (F330W) for GRB 060729 (empty stars) and theoretical curve with radius as given by the relativistic numerical model with data as in **Table 3**. The theoretical magnitude is given by Equation (26) (full line).
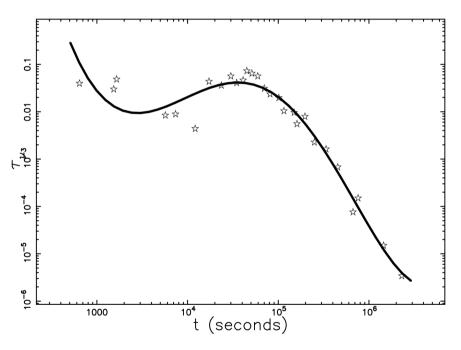


**Figure 6.** The time dependence of $\tau_\nu$ (empty stars) for GRB 060729 and a logarithmic polynomial approximation of degree 5 (full line). Parameters as in **Table 3**.

behavior of the optical depth.

## 4. Conclusions

**Physics.** We have derived velocity as a function of time for a relativistic neutral particle that moves in a dissipative medium in the presence of friction, which depends on the square velocity of the particle. The trajectory (*i.e.*, space as a

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function of time) can be deduced by numerical integration of the velocity or by a Taylor series of the differential equation of the second order.

**Astrophysics.** An application of the obtained results to SN 1993J allows us to derive the constant $B$ and draw a comparison between the observed and theoretical trajectory, see Table 1 and Figure 2. The LCs of GRB 130427A and GRB 060729 were simulated, see Figure 4 and Figure 5.

## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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