Scientific Research Publishing

International Journal of Astronomy and Astrophysics, 2020, 10, 191-202
https://www.scirp.org/journal/ijaa
ISSN Online: 2161-4725
ISSN Print: 2161-4717

New Probability Distributions in Astrophysics: III. The Truncated Maxwell-Boltzmann Distribution

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How to cite this paper: Zaninetti, L. (2020) New Probability Distributions in Astrophysics: III. The Truncated Maxwell-Boltzmann Distribution. *International Journal of Astronomy and Astrophysics*, **10**, 191-202.
https://doi.org/10.4236/ijaa.2020.103010

Received: June 17, 2020
Accepted: August 16, 2020
Published: August 19, 2020

Abstract

The Maxwell-Boltzmann (MB) distribution for velocities in ideal gases is usually defined between zero and infinity. A double truncated MB distribution is here introduced and the probability density function, the distribution function, the average value, the rth moment about the origin, the root-mean-square speed and the variance are evaluated. Two applications are presented: 1) a numerical relationship between root-mean-square speed and temperature, and 2) a modification of the formula for the Jeans escape flux of molecules from an atmosphere.

Keywords

05.20.-y Classical Statistical Mechanics, 05.20.Dd Kinetic Theory

1. Introduction

The *Maxwell-Boltzmann* (MB) distribution, see [1] [2], is a powerful tool to explain the kinetic theory of gases. The range in velocity of this distribution spans the interval $[0,\infty]$, which produces several problems:

1) The maximum velocity of a gas cannot be greater than the velocity of light, c.

2) The kinetic theory is developed in a classical environment, which means that the involved velocities should be smaller than $\approx 1/10c$.

These items point toward the hypothesis of an upper bound in velocity for the MB. We will now report some approaches, including an upper bound in velocity: the ion velocities parallel to the magnetic field in a low density surface of a ionized plasma [3]; propagation of longitudinal electron waves in a collisionless, homogeneous, isotropic plasma, whose velocity distribution function is a truncated MB [4]; fast ion production in laser plasma [5]; the release of a dust particle from a plasma-facing wall [6]; an explanation of an anomaly in the Dark Matter (DAMA) experiment [7]; a distorted MB distribution of epithermal ions observed associated with the collapse of energetic ions [8]; and deviations to MB distribution that could have observable effects which can be measured trough the vapor spectroscopy at an interface [9]. However, these approaches do not clearly cover the effect of introducing a lower and an upper boundary in the MB distribution, which is the subject that will be analyzed in this paper.

This paper is structured as follows. Section 2 reviews the basic statistics of the MB distribution and it derives a new approximate expression for the median. Section 3 introduces the double truncated MB and it derives the connected statistics. Section 4 derives the relationship for root-mean-square speed versus temperature in the double truncated MB. Finally, Section 5.2 derives a new formula for Jeans flux in the exosphere.

2. The Maxwell-Boltzmann Distribution

Let V be a random variable defined in $[0,\infty]$; the MB probability density function (PDF), $f(v;a)$, is

$$f(v;a)=\frac{\sqrt{2}v^2\mathrm{e}^{-\frac{1}{2}\frac{v^2}{a^2}}}{\sqrt{\pi}a^3}, \tag{1}$$

where a is a parameter and v denotes the velocity, see [1] [2]. Conversion to the physics is done by introducing the variable a, which is defined as

$$a=\sqrt{\frac{kT}{m}}, \tag{2}$$

where m is the mass of the gas molecules, k is the Boltzmann constant and T is the thermodynamic temperature. With this change of variable, the MB PDF is

$$f_p(v;m,k,T)=\frac{\sqrt{2}v^2\mathrm{e}^{-\frac{1}{2}\frac{v^2 m}{kT}}}{\sqrt{\pi}\left(\frac{kT}{m}\right)^{\frac{3}{2}}}, \tag{3}$$

where the index p stands for physics. The distribution function (DF), $F(x;a)$, is

$$F(v;a)=\frac{\sqrt{2}a^2\left(a\sqrt{\pi}\sqrt{2}\,\mathrm{erf}\left(\frac{1}{2}\frac{\sqrt{2}v}{a}\right)-2v\mathrm{e}^{-\frac{1}{2}\frac{v^2}{a^2}}\right)}{2\sqrt{\pi}a^3} \tag{4}$$

$$F_p(v)=\frac{\sqrt{2}\left(\left(\frac{kT}{m}\right)^{\frac{3}{2}}\sqrt{\pi}\sqrt{2}\,\mathrm{erf}\left(\frac{1}{2}\sqrt{2}v\frac{1}{\sqrt{\frac{kT}{m}}}\right)m-2v\mathrm{e}^{-\frac{1}{2}\frac{v^2 m}{kT}}kT\right)}{2\sqrt{\pi}\left(\frac{kT}{m}\right)^{\frac{3}{2}}m}. \tag{5}$$

The average value or mean, μ, is

$$\mu(a) = 2\frac{\sqrt{2}a}{\sqrt{\pi}}, \tag{6}$$

$$\mu(m,k,T)_p = 2\frac{\sqrt{2}}{\sqrt{\pi}}\sqrt{\frac{kT}{m}}, \tag{7}$$

the variance, σ^2, is

$$\sigma^2(a) = \frac{a^2(-8+3\pi)}{\pi} \tag{8}$$

$$\sigma^2(m,k,T)_p = \frac{kT(-8+3\pi)}{m\pi}. \tag{9}$$

The rth moment about the origin for the MB distribution is, μ'_r, is

$$\mu'_r(a) = \frac{2^{r/2+1}a^r\Gamma\left(r/2+\frac{3}{2}\right)}{\sqrt{\pi}} \tag{10}$$

$$\mu'_r(m,k,T)_p = \frac{2^{r/2+1}\left(\sqrt{\frac{kT}{m}}\right)^r\Gamma\left(r/2+\frac{3}{2}\right)}{\sqrt{\pi}}, \tag{11}$$

where

$$\Gamma(z) = \int_0^\infty \mathrm{e}^{-t}t^{z-1}\mathrm{d}t, \tag{12}$$

is the gamma function, see [10]. The root-mean-square speed, v_{rms}, can be obtained from this formula by inserting $r = 2$

$$v_{rms}(a) = \sqrt{3}a \tag{13}$$

$$v_{rms}(m,k,T)_p = \sqrt{3}\sqrt{\frac{kT}{m}}, \tag{14}$$

see Equations (7-10-16) in [11]. This equation allows us to derive the temperature once the root-mean-square speed is measured

$$T = \frac{1}{3}\frac{v_{rms}^2 m}{k}. \tag{15}$$

The coefficient of variation (CV) is

$$CV = \frac{\sigma(a)}{\mu(a)} = \sqrt{\frac{3}{8}\pi - 1}, \tag{16}$$

which is constant. The first three rth moments about the mean for the MB distribution, $\mu_r(a)$, are

$$\mu_2(a) = \frac{a^2(-8+3\pi)}{\pi} \tag{17}$$

$$\mu_3(a) = -2\frac{a^3\sqrt{2}(5\pi-16)}{\pi^{3/2}} \tag{18}$$

$$\mu_4(a) = \frac{a^4(15\pi^2+16\pi-192)}{\pi^2}. \tag{19}$$

The mode is at

$$v(a) = \sqrt{2}a \tag{20}$$

$$v(m,k,T)_p = \sqrt{2}\sqrt{\frac{kT}{m}}. \tag{21}$$

An approximate expression for the median can be obtained by a Taylor series of the DF around the mode. The approximation formula is

$$v(a) = -\frac{1}{4}a\left(-6 + \mathrm{e}\left(\mathrm{erf}\,(1) - \frac{1}{2}\right)\sqrt{\pi}\right)\sqrt{2}, \tag{22}$$

$$v(m,k,T)_p = -\frac{1}{4}\sqrt{\frac{kT}{m}}\left(-6 + \mathrm{e}\left(\mathrm{erf}\,(1) - \frac{1}{2}\right)\sqrt{\pi}\right)\sqrt{2}, \tag{23}$$

which has a percent error, δ, of $\delta \approx 0.04\%$ in respect to the numerical value. The entropy is

$$\ln\left(\sqrt{2}\sqrt{\pi}a\right) - \frac{1}{2} + \gamma, \tag{24}$$

$$\ln\left(\sqrt{2}\sqrt{\pi}\sqrt{\frac{kT}{m}}\right) - \frac{1}{2} + \gamma, \tag{25}$$

where γ is the Euler-Mascheroni constant, which is defined as

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right) = 0.57721\cdots, \tag{26}$$

see [10] for more details. The coefficient of skewness is

$$\frac{(-10\pi + 32)\sqrt{2}}{(-8 + 3\pi)^{\frac{3}{2}}} \approx 0.48569, \tag{27}$$

and the coefficient of kurtosis is

$$\frac{15\pi^2 + 16\pi - 192}{(-8 + 3\pi)^2} \approx 3.10816. \tag{28}$$

According to [12], a random number generation can be obtained via inverse transform sampling when the distribution function or cumulative distribution function, $F(x)$, is known: 1) a pseudo number generator gives a random number R between zero and one; 2) the inverse function $x = F^{-1}(R)$ is evaluated; and 3) the procedure is repeated for different values of R. In our case, the inverse function should be evaluated in a numerical way by solving for v the following nonlinear equation

$$F(v;a) - R = 0, \tag{29}$$

$$F(v;m,k,T)_p - R = 0, \tag{30}$$

where $F(v)$ and $F_p(v)$ are the two DF represented by Equations (4) and (5). As a practical example, by inserting in Equation (29) $a = 1$ and $R = 0.5$, we obtain in a numerical way $v = 1.538$.

3. The Double Truncated Maxwell-Boltzmann Distribution

Let V be a random variable that is defined in $[v_l, v_u]$; the *double truncated* version of the Maxwell-Boltzmann PDF, $f_t(v; a, v_l, v_u)$, is

$$f_t(v; a, v_l, v_u) = v^2 e^{-\frac{1}{2}\frac{v^2}{a^2}}, \tag{31}$$

where

$$C = \frac{-2}{CD}, \tag{32}$$

where

$$CD = a^2 \left(-a\sqrt{\pi}\sqrt{2}\,\mathrm{erf}\left(\frac{1}{2}\frac{\sqrt{2}v_u}{a}\right) + a\sqrt{\pi}\sqrt{2}\,\mathrm{erf}\left(\frac{1}{2}\frac{\sqrt{2}v_l}{a}\right) + 2v_u e^{-\frac{1}{2}\frac{v_u^2}{a^2}} - 2v_l e^{-\frac{1}{2}\frac{v_l^2}{a^2}} \right), \tag{33}$$

and $\mathrm{erf}(x)$ is the error function, which is defined as

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \mathrm{d}t, \tag{34}$$

see [10]. The physical meaning of a is still represented by Equation (2); however, due to the tendency to obtain complicated expressions, we will omit the double notation. The DF, $F_t(v; a, v_l, v_u)$, is

$$F_t(v; a, v_l, v_u) = \frac{Ca^2 \left(\sqrt{\pi}\sqrt{2}a\,\mathrm{erf}\left(\frac{1}{2}\frac{\sqrt{2}v}{a}\right) - 2ve^{-\frac{1}{2}\frac{v^2}{a^2}} \right)}{2}. \tag{35}$$

The average value $\mu_t(a, v_l, v_u)$, is

$$\mu_t(a, v_l, v_u) = Ca^2 \left(2e^{-\frac{1}{2}\frac{v_l^2}{a^2}} a^2 - 2e^{-\frac{1}{2}\frac{v_u^2}{a^2}} a^2 + e^{-\frac{1}{2}\frac{v_l^2}{a^2}} v_l^2 - e^{-\frac{1}{2}\frac{v_u^2}{a^2}} v_u^2 \right). \tag{36}$$

The rth moment about the origin for the double truncated MB distribution is, $\mu'_{r,t}(a, v_l, v_u)$,

$$\mu'_{r,t}(a, v_l, v_u) = \frac{MN}{r+3} \tag{37}$$

where

$$MN = C2^{\frac{r}{4}+\frac{5}{4}} a^2 \times \left(\left(\frac{v_u^2}{a^2}\right)^{-\frac{r}{4}-\frac{1}{4}} v_u^{r+1} e^{-\frac{1}{4}\frac{v_u^2}{a^2}} M_{\frac{r}{4}+\frac{1}{4}, \frac{r}{4}+\frac{3}{4}}\left(\frac{1}{2}\frac{v_u^2}{a^2}\right) - v_l^{r+1} e^{-\frac{1}{4}\frac{v_l^2}{a^2}} M_{\frac{r}{4}+\frac{1}{4}, \frac{r}{4}+\frac{3}{4}}\left(\frac{1}{2}\frac{v_l^2}{a^2}\right) \left(\frac{v_l^2}{a^2}\right)^{-\frac{r}{4}-\frac{1}{4}} \right) \tag{38}$$

where $M_{\mu,\nu}(z)$ is the Whittaker M function, see [10]. The root-mean-square speed, $v_{rms,t}(a, v_l, v_u)$, can be obtained from this formula by inserting $r = 2$, and is

$$v_{rms,t}(a,v_l,v_u)=\sqrt{\frac{NV}{5\left(\frac{v_u^2}{a^2}\right)^{3/4}\left(\frac{v_l^2}{a^2}\right)^{3/4}}}, \tag{39}$$

where

$$\begin{aligned} NV = 2C2^{3/4}a^2\Bigg(& v_u^3\mathrm{e}^{-1/4\frac{v_u^2}{a^2}}M_{3/4,5/4}\left(1/2\frac{v_u^2}{a^2}\right)\left(\frac{v_l^2}{a^2}\right)^{3/4} \\ & -v_l^3\mathrm{e}^{-1/4\frac{v_l^2}{a^2}}M_{3/4,5/4}\left(1/2\frac{v_l^2}{a^2}\right)\left(\frac{v_u^2}{a^2}\right)^{3/4}\Bigg). \end{aligned} \tag{40}$$

The variance $\sigma_t^2(a,v_l,v_u)$ is defined as

$$\sigma_t^2(a,v_l,v_u)=\mu_{2,t}'(a,v_l,v_u)-\left(\mu_{1,t}'(a,v_l,v_u)\right)^2 \tag{41}$$

and has the following explicit form

$$\begin{aligned} & \sigma_t^2(a,v_l,v_u) \\ & =4\Bigg(\left((v_l+2v_u)a^2+v_lv_u\left(v_l+\frac{1}{2}v_u\right)\right)\left(a^2+1/2v_u^2\right)C^2a^4\mathrm{e}^{-\frac{1}{2}\frac{v_l^2+2v_u^2}{a^2}} \\ & -2\left(\left(v_l+\frac{1}{2}v_u\right)a^2+\frac{1}{4}v_lv_u(v_l+2v_u)\right)C^2a^4\left(a^2+\frac{1}{2}v_l^2\right)\mathrm{e}^{-\frac{1}{2}\frac{2v_l^2+v_u^2}{a^2}} \\ & +\left(a^2+\frac{1}{2}v_u^2\right)\Bigg(C\,\mathrm{erf}\left(\frac{1}{2}\frac{\sqrt{2}v_l}{a}\right)a^3\sqrt{2}\sqrt{\pi} \\ & -C\,\mathrm{erf}\left(\frac{1}{2}\frac{\sqrt{2}v_u}{a}\right)a^3\sqrt{2}\sqrt{\pi}+4\Bigg)Ca^2\left(a^2+\frac{1}{2}v_l^2\right)\mathrm{e}^{-\frac{1}{2}\frac{v_l^2+v_u^2}{a^2}} \\ & +C^2a^4\left(a^2+\frac{1}{2}v_l^2\right)^2v_l\mathrm{e}^{-\frac{3}{2}\frac{v_l^2}{a^2}}-\left(a^2+\frac{1}{2}v_u^2\right)^2C^2a^4v_u\mathrm{e}^{-\frac{3}{2}\frac{v_u^2}{a^2}} \\ & +\left(\frac{3}{4}a^2v_l+\frac{1}{4}v_l^3\right)\mathrm{e}^{-\frac{1}{2}\frac{v_l^2}{a^2}}+\left(-\frac{3}{4}a^2v_u-\frac{1}{4}v_u^3\right)\mathrm{e}^{-\frac{1}{2}\frac{v_u^2}{a^2}} \\ & -\frac{1}{2}\Bigg(\Bigg(C\,\mathrm{erf}\left(\frac{1}{2}\frac{\sqrt{2}v_l}{a}\right)a^3\sqrt{2}\sqrt{\pi} \\ & -C\,\mathrm{erf}\left(\frac{1}{2}\frac{\sqrt{2}v_u}{a}\right)a^3\sqrt{2}\sqrt{\pi}+4\Bigg)C\left(a^2+\frac{1}{2}v_l^2\right)^2\mathrm{e}^{-\frac{v_l^2}{a^2}} \\ & +\left(a^2+\frac{1}{2}v_u^2\right)^2\Bigg(C\,\mathrm{erf}\left(\frac{1}{2}\frac{\sqrt{2}v_l}{a}\right)a^3\sqrt{2}\sqrt{\pi} \\ & -C\,\mathrm{erf}\left(\frac{1}{2}\frac{\sqrt{2}v_u}{a}\right)a^3\sqrt{2}\sqrt{\pi}+4\Bigg)C\mathrm{e}^{-\frac{v_u^2}{a^2}} \\ & +\frac{3}{4}\sqrt{\pi}\sqrt{2}\left(-\mathrm{erf}\left(\frac{1}{2}\frac{\sqrt{2}v_u}{a}\right)+\mathrm{erf}\left(\frac{1}{2}\frac{\sqrt{2}v_l}{a}\right)\right)a\Bigg)a^2\Bigg)Ca^2. \end{aligned} \tag{42}$$

Although the coefficients of skewness and kurtosis for the truncated MB exist, they have a complicated expression.

4. A Laboratory Application

The temperature as a function of root-mean-square speed for the MB is given by Equation (15). In the truncated MB distribution, the temperature can be found by solving the following nonlinear equation

$$v_{rms,t}\left(k, m, T, v_l, v_u\right) = v_{rms,m}, \tag{43}$$

where $v_{rms,m}$ is not a theoretical variable but is the root-mean-square speed measured in the laboratory and $v_{rms,t}$ is given by Equation (39). The laboratory measures of $v_{rms,m}$ started with [13], where a $v_{rms,m} = 388\ \text{m/s}$ at 400˚C was found for a metallic vapor. In the truncated MB distribution, there are three parameters that can be measured in the laboratory from a kinematical point of view, as follows: the lowest velocity, v_l; the highest velocity, v_u; and the root-mean-square speed, $v_{rms,m}$. Setting for simplicity $v_l = 0$, we will now explore the effect of the variation of v_u on the root-mean-square speed; see **Figure 1**. The *first* example of the influence of the upper limit in velocity on the temperature is given by potassium gas [14] [15], in which molecular mass is $6.492429890 \times 10^{-26}$ kg. In **Figure 2**, we evaluate in a numerical way the temperature when $v_l = 0$ and v_u is variable in the case of a measured value of $v_{rms,m}$.

The *second* example is given by diatomic nitrogen, N_2, in which molecular mass is $4.651737684 \times 10^{-26}$ kg. In **Figure 3**, we evaluate the temperature when $v_l = 0$ and v_u is a variable in the case of a measured value of $v_{rms,m}$.

5. The Jeans Escape

The standard formula for the escape of molecules from the exosphere is reviewed in the framework of the MB distribution. A new formula for the Jeans escape is derived in the framework of the truncated MB.

5.1. The Standard Case

In the exosphere, a molecule of mass m and velocity v_e is free to escape when

$$\frac{1}{2} m v_e^2 - G\frac{Mm}{R_{ex}} = 0, \tag{44}$$

where G is the Newtonian gravitational constant, M is the mass of the Earth, $R_{ex} = R + H$ is the radius of the exosphere, R is the radius of the Earth and H is the altitude of the exosphere. The flux of the molecules that are living in the exosphere Φ_j is

$$\Phi_j = \frac{1}{4} N_{ex} \mu_e, \tag{45}$$

where N_{ex} is the number of molecules per unit volume and μ_e is the average velocity of escape. In the presence of a given number of molecules per unit volume, the standard MB distribution in velocities in a unit volume, f_m, is

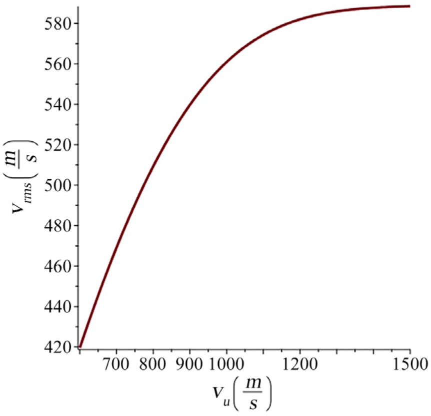

Figure 1. The theoretical root-mean-square speed as a function of the upper limit in velocity (continuous line) and standard value of the temperature (dotted line) when $a = 340$ and $v_l = 0$.

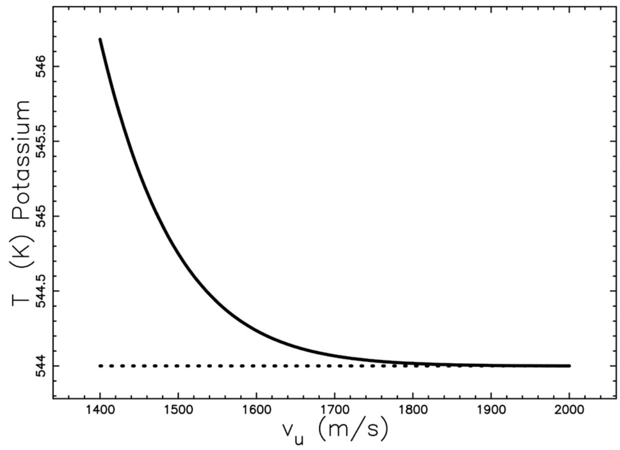

Figure 2. Temperature as a function of the upper limit in velocity for Potassium (continuous line) and standard value of the temperature (dotted line) when $v_l = 0$ and $v_{rms,m} = 589.111511\ \mathrm{m/s}$.

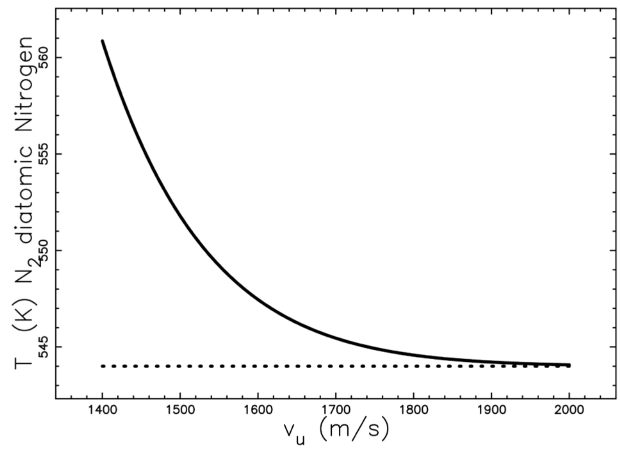

Figure 3. Temperature as a function of the upper limit in velocity for diatomic nitrogen, N_2, (continuous line) and standard value of the temperature (dotted line) when $v_l = 0$ and $v_{rms,m} = 695.9756308\ \mathrm{m/s}$.

$$f_m(v;m,k,T,N_{ex}) = N_{ex}\frac{\sqrt{2}v^2 e^{-\frac{1}{2}\frac{v^2 m}{kT}}}{\sqrt{\pi}\left(\frac{kT}{m}\right)^{\frac{3}{2}}}. \tag{46}$$

The average value of escape is defined as

$$\mu_e = \frac{\int_{v_e}^{\infty} v f_m(v;m,k,T,N_{ex})\,\mathrm{d}v}{\int_0^{\infty} f_m(v;m,k,T,N_{ex})\,\mathrm{d}v}. \tag{47}$$

In this integral, the following changes are made to the variables

$$\lambda = \frac{1}{2}\frac{mv^2}{kT}. \tag{48}$$

Therefore,

$$\mu_e = 2(\lambda_e+1)e^{-\lambda_e}\sqrt{2}\sqrt{\frac{kT}{\pi m}}, \tag{49}$$

with

$$\lambda_e = 2\frac{GM}{R_{ex}v_0^2}, \tag{50}$$

where v_0 is the mode as represented by Equation (21). The flux is now

$$\Phi_j = \frac{N_{ex}(\lambda_e+1)e^{-\lambda_e}v_0}{2\sqrt{\pi}}. \tag{51}$$

For more details see [16] [17] [18] [19]. On adopting the parameters of **Table 1** the Jeans escape flux for hydrogen is

$$\Phi_j = 3.98\times 10^{11}\,\text{molecules}\cdot\text{m}^{-2}\cdot\text{s}^{-1}, \tag{52}$$

and

$$\lambda_e = 7.78. \tag{53}$$

The Jeans escape flux for Earth at $T = 900\,\text{K}$ varies between $\Phi_j \approx 2.7\times 10^{11}\,\text{molecules}\cdot\text{m}^{-2}\cdot\text{s}^{-1}$; see [20] or **Figure 1** in [21]. and $\Phi_j \approx 4\times 10^{11}\,\text{molecules}\cdot\text{m}^{-2}\cdot\text{s}^{-1}$, see [22]. Therefore, our choice of parameters is compatible with the suggested interval in flux.

5.2. The Truncated Case

The average value of escape for a truncated MB distribution, $\mu_{e,t}$, is

$$\mu_{e,t} = \frac{\int_{v_e}^{\infty} v f_t(v;m,k,T,N_{ex},v_l,v_u)\,\mathrm{d}v}{\int_0^{\infty} f_m(v;m,k,T,N_{ex},v_l,v_u)\,\mathrm{d}v}. \tag{54}$$

This integral can be solved by introducing the change of variable as given by Equation (48)

$$\mu_{e,t} = -2\frac{\left((\lambda_u+1)e^{-\lambda_u} - e^{-\lambda_e}(\lambda_e+1)\right)\sqrt{2}}{2\sqrt{\lambda_l}e^{-\lambda_l} - 2\sqrt{\lambda_u}e^{-\lambda_u} - \sqrt{\pi}\,\mathrm{erf}\left(\sqrt{\lambda_l}\right) + \sqrt{\pi}\,\mathrm{erf}\left(\sqrt{\lambda_u}\right)}\sqrt{\frac{kT}{m}}, \tag{55}$$

where λ_l is the lower value of λ and λ_u is the upper value of λ. The flux of the molecules that are living the exosphere in the truncated MB, $\Phi_{j,t}$, is

$$\Phi_{j,t} = \frac{N_{ex}\left(\left(\lambda_u+1\right)e^{-\lambda_u}-e^{-\lambda_e}\left(\lambda_e+1\right)\right)\sqrt{2}}{4\sqrt{\lambda_u}e^{-\lambda_u}+2\sqrt{\pi}\,\mathrm{erf}\left(\sqrt{\lambda_l}\right)-2\sqrt{\pi}\,\mathrm{erf}\left(\sqrt{\lambda_u}\right)-4\sqrt{\lambda_l}e^{-\lambda_l}}\sqrt{\frac{kT}{m}}. \quad (56)$$

The increasing flux of molecules is outlined when one parameter, λ_l, is variable; see **Figure 4**. In other words, an increase in λ_l produces an increase in the flux of the molecules. The dependence of the flux when two parameters are variable, λ_l and λ_u, is reported in **Figure 5**.

Table 1. Adopted physical parameters for the exosphere.

Parameter	Value
R_{ex}	6900 km
T	900 K
N_{ex}	10^{11} m^{-3}

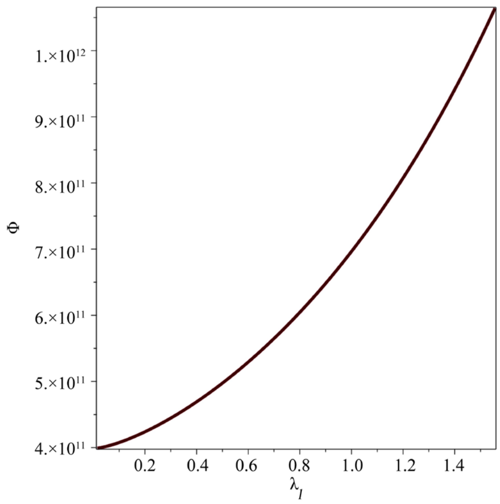

Figure 4. The flux of molecules as a function of λ_l with parameters as in **Table 1**, $\lambda_e = 7.78$ and $\lambda_u = 1000\lambda_e$.

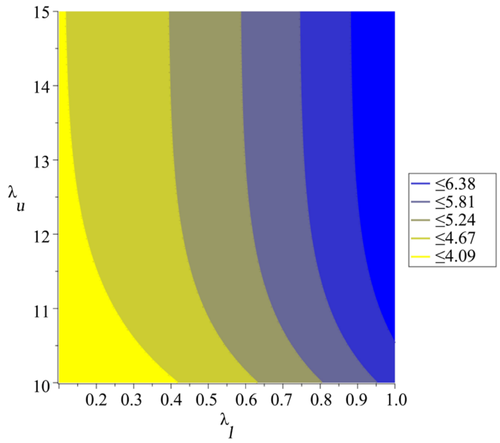

Figure 5. The flux of molecules as a function of λ_l and λ_u with parameters as in **Table 1**.

These Jeans escape fluxes for Earth are compatible with the observed values that were reported in Section 5.1.

6. Conclusion

This paper derived analytical formulae for the following quantities for a double truncated MB distribution: the PDF, the DF, the average value, the rth moment about the origin, the root-mean-square speed and the variance. The traditional correspondence between root-mean-square speed and temperature is replaced by the nonlinear Equation (43). The new formula (56) for the Jeans escape flux of molecules from an atmosphere is now a function of the lower and upper boundary in velocity.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Maxwell, J.C. (1860) V. Illustrations of the Dynamical Theory of Gases—Part I. On the Motions and Collisions of Perfectly Elastic Spheres. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, **19**, 19-32. https://doi.org/10.1080/14786446008642818

[2] Boltzmann, L. (1872) Weitere Studien über das Wärmegleichgewicht unter Gasmolekülen. *Sitzungsberichte Akademie der Wissenschaften*, **66**, 275-370.

[3] Buzzi, J.M., Doucet, H.J. and Gresillon, D. (1970) Ion Distribution Functions in Collisionless Surface Ionized Plasmas. *The Physics of Fluids*, **13**, 3041. https://doi.org/10.1063/1.1692899

[4] Treguier, J. and Henry, D. (1976) Propagation of Electronic Longitudinal Modes in a Truncated Maxwellian Plasma. *Journal of Plasma Physics*, **15**, 447-461. https://doi.org/10.1017/S0022377800019954

[5] Kishimoto, Y., Mima, K., Watanabe, T. and Nishikawa, K. (1983) Analysis of Fast-Ion velocity Distributions in Laser Plasmas with a Truncated Maxwellian Velocity Distribution of Hot Electrons. *The Physics of Fluids*, **26**, 2308. https://doi.org/10.1063/1.864389

[6] Tomita, Y., Smirnov, R., Nakamura, H., Zhu, S., Takizuka, T. and Tskhakaya, D. (2007) Effect of Truncation of Electron Velocity Distribution on Release of Dust Particle from Plasma-Facing Wall. *Journal of Nuclear Materials*, **363**, 264-269. https://doi.org/10.1016/j.jnucmat.2007.01.025

[7] Fowlie, A. (2017) Halo-Independence with Quantified Maximum Entropy at DAMA/LIBRA. *Journal of Cosmology and Astroparticle Physics*, **2017**, 2-4. https://doi.org/10.1088/1475-7516/2017/10/002

[8] Ida, K., Kobayashi, T., Yoshinuma, M., Akiyama, T., Tokuzawa, T., Tsuchiya, H., Itoh, K. and LHD Experiment Group (2017) Observation of Distorted Maxwell-Boltzmann Distribution of Epithermal Ions in LHD. *Physics of Plasmas*, **24**, Article ID: 122502. https://doi.org/10.1063/1.4999644

[9] Todorov, P., De Aquino Carvalho, J.C., Maurin, I., Laliotis, A. and Bloch, D. (2019) Search for Deviations from the Ideal Maxwell-Boltzmann Distribution for a Gas at

an Interface. *Proceedings of the SPIE*, **11047**, Article ID: 110470.

[10] Olver, F.W.J., Lozier, D.W., Boisvert, R.F. and Clark, C.W. (2010) NIST Handbook of Mathematical Functions. Cambridge University Press, Cambridge.

[11] Reif, F. (2009) Fundamentals of Statistical and Thermal Physics. Waveland Press, Long Grove, Illinois.

[12] Devroye, L. (1986) General Principles in Random Variate Generation. Springer, New York. https://doi.org/10.1007/978-1-4613-8643-8

[13] Eldridge, J. (1927) Experimental Test of Maxwell's Distribution Law. *Physical Review*, **30**, 931-935. https://doi.org/10.1103/PhysRev.30.931

[14] Miller, R.C. and Kusch, P. (1955) Velocity Distributions in Potassium and Thallium Atomic Beams. *Physical Review*, **99**, 1314-1321. https://doi.org/10.1103/PhysRev.99.1314

[15] Hernandez, H. (2017) Standard Maxwell-Boltzmann Distribution: Definition and Properties. ForsChem Research Reports, Medellin.

[16] Jeans, J.H. (1955) The Dynamical Theory of Gases. Dover, New York.

[17] Shu, F.H. (1982) The Physical Universe. University Science Books, Mill Valley.

[18] Catling, D.C. and Kasting, J.F. (2017) Atmospheric Evolution on Inhabited and Lifeless Worlds. Cambridge University Press, Cambridge, 129. https://doi.org/10.1017/9781139020558

[19] Owen, J.E. (2019) Atmospheric Escape and the Evolution of Close-in Exoplanets. *Annual Review of Earth and Planetary Sciences*, **47**, 67-90. https://doi.org/10.1146/annurev-earth-053018-060246

[20] Vidal-Madjar, A., Blamont, J.E. and Phissamay, B. (1974) Evolution with Solar Activity of the Atomic Hydrogen Density at 100 Kilometers of Altitude. *Journal of Geophysical Research*, **79**, 233-241. https://doi.org/10.1029/JA079i001p00233

[21] Liu, W., Chiao, M., Collier, M.R., *et al*. (2017) The Structure of the Local Hot Bubble. *The Astrophysical Journal*, **834**, 33-38. https://doi.org/10.3847/1538-4357/834/1/33

[22] Bertaux, J. (1974) Lhydrogène atomique dans lexosphère terrestre: Mesures dintensité et de largeur de raie de lémission lyman alpha à bord du satellite ogo 5 et interprétation. These détat, Université Paris.