

# Transport in Astrophysics: VI. Ultra-High Energy Cosmic Rays

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# Abstract

Two new solutions of the homogeneous diffusion equation in 1D are derived in the presence of losses and a trigonometric profile for a profile of density. A simulation for the ankle in the energy distribution of cosmic rays (CRs) is provided in the framework of the fine tuning of the involved parameters. A theoretical image for the overall diffusion of CRs in galactic coordinates is provided.

# **Keywords**

Cosmic Rays, Particle Diffusion, Random Walks

# **1. Introduction**

In recent years, the Pierre Auger Observatory has been observing ultra-high energy cosmic rays (UHECRs), focusing on the flux at around  $5 \times 10^{18}$  eV (the so-called "ankle") [1] [2], and on their extra-galactic origin as determined from the arrival directions above 8 EeV [3]. We now explore the theoretical diffusion of CRs. The diffusion of the cosmic rays (CR) from clusters of galaxies once they are accelerated has been analyzed by many approaches, we select some of them. A demonstration that clusters of galaxies are able to keep CRs produce the diffuse flux of high-energy gamma and neutrino radiation due to the interaction with the intracluster gas [4]. The acceleration of electrons and CRs in the pair of large radio lobes in the Virgo cluster has been analyzed in [5], assuming that the CR streaming velocity is of the order of the sound velocity the observed bimodality of cluster radio halos appears to be a natural consequence of the interplay of CR transport processes [6], the study of diffuse radio sources allows explaining the acceleration and propagation of CRs [7]; the active nuclei of NGC 1275, the central dominant galaxy of the cluster, and IC 310, lying at about 0.6 degree

from the center, have been detected as point-like VHE gamma-ray emitters and therefore associated with the acceleration and diffusion of CRs [8]: low Mach shocks, such as the shocks of colliding clusters of galaxies, can accelerate CRs [9]. On assuming that losses are dominated by CR transport some involved parameters are found for the Coma cluster, [10]. A Monte Carlo program, CRPropa 3.2, has been built which simulates the propagation of high-energy CRs in the Universe [11]. The CRs provide a significant contribution to the pressure in the circumgalactic medium [12]. The diffuse radio emission from galaxy clusters is a manifestation of the same cosmic-ray ion population [13]. The present paper derives two new solutions for the diffusion in 1D, see Section 2. The astrophysical applications to CRs allow deriving an energy spectrum with the differential number of CRs  $\propto E^{-3}$  and explaining the ankle, see Section 3. The diffusion of CRs in the cosmic voids is modeled in the framework of Voronoi diagrams, see Section 4.

# 2. Diffusion

We briefly review the theory of a random walk and its connected diffusion coefficient (which is a function of the selected energy), the general equation of diffusion for Crs, the temporary 1D diffusion with an existing profile of matter in presence of losses and the temporary 1D diffusion with an existing oscillating profile.

#### 2.1. The Random Walk

The dependence on time of the mean square displacement from the starting point of the random walk,  $\overline{R^2(t)}$ , according to equation (8.38) in [14] is

$$R^{2}(t) \approx 2dDt \quad (t \to \infty), \tag{1}$$

where d is the number of spatial dimensions and D the diffusion coefficient. From Equation (1), the diffusion coefficient is derived in the continuum limit as t is very large

$$D \approx \left(t \to \infty\right) \frac{\overline{R}^2}{2dt}.$$
 (2)

Using discrete time steps, the average square radius after N steps, equation (12.5) in [14], is

$$\langle R^2(N) \rangle \sim 2dDN,$$
 (3)

from which the diffusion coefficient is derived:

$$D = \frac{\left\langle R^2\left(N\right)\right\rangle}{2dN}.$$
(4)

If  $\langle R^2(N) \rangle \sim N$ , the diffusion coefficient is

$$D = \frac{1}{2d} \lambda v_{tr}, \tag{5}$$

when the step length of the walker or mean free path between successive colli-

sions is  $\lambda$  and the transport velocity is  $v_{tr}$ .

The relativistic gyroradius or Larmor radius of a single CR is

$$r_H = \frac{mc\gamma v_\perp}{qB},\tag{6}$$

where *m* is the mass of the particle, *c* is the speed of light,

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}},\tag{7}$$

is the Lorentz factor, v is the velocity of particle, q is the charge of particle, B is the magnetic field and  $v_{\perp}$  is its velocity perpendicular to the magnetic field, see formula (1.54) in [15] or formula (7.3) in [16]. We now assume  $v_{\perp} = c$  due to the fact that we are dealing with relativistic particles. The numerical value of the Larmor radius in the case of an accelerator for SI units is

$$r_L \approx \frac{3.335 E_{Gev}}{B_T} \,\mathrm{m},\tag{8}$$

where  $E_{Gev}$  is the energy expressed in GeV and  $B_T$  is the magnetic field expressed in Tesla. In the case of a CR, we express the Larmor radius in pc

$$r_L \approx \frac{1.081E_{PeV}}{ZB_{-6}} \,\mathrm{pc},\tag{9}$$

where Z is the atomic number,  $E_{PeV}$  is the energy expressed in  $10^{15}$  eV, and  $B_{-6}$  is the magnetic field expressed in  $10^{-6}$  gauss, see [17]. On assuming that the CRs diffuse with a mean free path equal to the relativistic gyroradius, the transport velocity is equal to the speed of light and d = 3, the diffusion coefficient according to Equation (5) is

$$D = \frac{0.055134E_{PeV}}{ZB_{-6}} \frac{\text{pc}^2}{\text{year}} = \frac{5.5134 \times 10^{-8} E_{GeV}}{ZB_{-6}} \frac{\text{pc}^2}{\text{year}}.$$
 (10)

## 2.2. The Diffusion-Loss Equation

The diffusion-loss equation as deduced by [18] (equation (20.1)) for light nuclei once the spallation phenomena are neglected has the form

$$\frac{\partial N_i}{\partial t} = D\nabla^2 N_i + \frac{\partial}{\partial E} \left[ b(E) N_i \right] + Q_i, \qquad (11)$$

here  $N_i$  is the number density of nuclei of species *i*,  $\nabla^2$  is the Laplacian operator, *D* is the scalar diffusion coefficient,  $\frac{\partial}{\partial E} [b(E)N_i]$  takes account of the energy balance, and  $Q_i$  represents the injection rate per unit volume. When only protons are considered, the energy dependence is neglected and the injection rate is included in the initial conditions, equation (11) becomes

$$\frac{\partial u(x, y, z, t)}{\partial t} = D\nabla^2 u(x, y, z, t), \qquad (12)$$

where u is the concentration of particles, see [14] (equation 12.34b).

## 2.3. Existing Profile and Losses

We analyze the case of 1D diffusion in a finite domain -L < x < L with an existing trigonometric profile with the maximum at the center of the box. The equation for the diffusion with losses is

$$\frac{\partial}{\partial t}u(x,t) = D\left(\frac{\partial^2}{\partial x^2}u(x,t)\right) - au(x,t), \tag{13}$$

where *D* is the diffusion coefficient, u(x,t) is the concentration of particles, and the parameter *a* represents the losses. The boundary conditions are periodic

$$u(-L,t) = u(L,t), \tag{14a}$$

$$\frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t).$$
(14b)

The initial condition is u(x,0) = f(x) with

$$f(x) = N_0 \cos\left(\frac{\pi x}{L}\right) + N_0, \qquad (15)$$

where  $N_0 \times 2$  is the maximum concentration. The solution is

$$u(x,t) = N_0 \left( e^{-\frac{t(D\pi^2 + L^2 a)}{L^2}} \cos\left(\frac{\pi x}{L}\right) + e^{-at} \right).$$
(16)

Figure 1 gives a 3D display of the solution as a function of time and Figure 2



**Figure 1.** Plot of the solution, u(x,t), as a function of x and t when L = 50 Mpc, Z = 1,  $E_{PeV} = 1$ ,  $a = 1.0 \times 10^{-8}$  and  $B_{-6} = 1$ .

gives the effect of increasing the losses on the solution.

Another example is to fix the variable x at L/2 and to see for which value the concentration go to nearly zero, in other words the CRs are confined, see **Figure 3**. A careful analysis of the above Figure reveals that for a critical value of a, the concentration drops drastically or the CRs are confined. The influence of the intensity of the magnetic field is reported in **Figure 4**; the decrease of the field



**Figure 2.** Plot of the solution, u(x), as a function of *x* when L = 50 Mpc,  $t = 1.0 \times 10^8$  yr, Z = 1,  $E_{PeV} = 1$  and  $B_{-6} = 1$  for different values of the parameter of the losses *a*.



**Figure 3.** Plot of the solution, *u*, as a function of *a* when x = L/2, other parameters as in **Figure 2**.



**Figure 4.** Plot of the solution, u, as a function of the magnetic field,  $B_6$ , other parameters as in **Figure 2**.

produces a flattening in the concentration of CRs.

At position *x* the above solution has a maximum in time at

$$t_{\rm max} = -\frac{\ln\left(-\frac{L^2 a}{\cos\left(\frac{\pi x}{L}\right)\left(D\pi^2 + L^2 a\right)}\right)L^2}{D\pi^2} \,\rm{yr},\tag{17}$$

or

$$t_{\text{max}} = -\frac{0.6125 \ln \left(-\frac{L^2 a}{\cos\left(\frac{\pi x}{L}\right)\left(\frac{1.6324 E_{PeV}}{B_{-6}} + L^2 a\right)}\right) L^2 B_{-6}}{E_{PeV}} \text{ yr}, \qquad (18)$$

when Equation (10) is used. Figure 5 gives an example of  $t_{max}$  as a function of the energy.

# 2.4. Oscillating Profile

We analyze the case of 1D diffusion in a finite domain -L < x < L with an existing trigonometric profile with three maxima, the second at the center of the box. The equation for the diffusion is

$$\frac{\partial}{\partial t}w(x,t) = D\left(\frac{\partial^2}{\partial x^2}w(x,t)\right),\tag{19}$$

where *D* is the diffusion coefficient and w(x,t) is the concentration of particles. The boundary conditions are periodic



**Figure 5.** Plot of  $t_{\text{max}}$  as a function of the energy when L = 50 Mpc,  $x = 0.9 \times L$ , Z = 1 and  $B_{-6} = 1$ .

$$w(-L,t) = w(L,t), \tag{20a}$$

$$\frac{\partial w}{\partial x}(-L,t) = \frac{\partial w}{\partial x}(L,t).$$
(20b)

The initial condition is w(x,0) = f(x) with

$$f(x) = N_0 \cos\left(\frac{2\pi x}{L}\right) + N_0, \qquad (21)$$

where  $N_0 \times 2$  is the maximum concentration. The solution is

$$w(x,t) = N_0 \left( e^{\frac{4D\pi^2 t}{L^2}} \cos\left(\frac{2\pi x}{L}\right) + 1 \right).$$
(22)

We now insert in the above solution the astrophysical version of the diffusion coefficient as given by Equation (10)

$$w(x,t) = N_0 \left( e^{\frac{-6.5298E15t}{B_{-6}L^2}} \cos\left(\frac{2\pi x}{L}\right) + 1 \right).$$
(23)

Figure 6 gives the above astrophysical quantity as a function of time and space.



**Figure 6.** Contours of the solution, w(x,t), as a function of x and t when L = 50 Mpc, Z = 1,  $E_{PeV} = 5$  and  $B_{-6} = 1$ .

# **3. Astrophysical Application**

We introduce an explanation for the existing  $\propto E^{-3}$  dependence for the distribution in energy of the cosmic rays and we then model the ankle for the distribution in energy.

# 3.1. The Spectral Index

The *first* case to be analyzed is that of the existing profile with one maximum and in the presence of losses, see Equation (16). The solution in which we insert a subscript D is

$$u(x,t,D,D_0)_D = N_0 \left(\frac{D}{D_0}\right)^{\beta} \left(e^{-\frac{t(D\pi^2 + L^2 a)}{L^2}} \cos\left(\frac{\pi x}{L}\right) + e^{-at}\right)$$
(24)

where  $N_0$  is the number of injected particles when  $D = D_0$  and  $\beta$  is an exponent which characterizes the transient diffusion. In order to simplify the equation, we fix  $B_{-6} = 1$ ,  $D = 0.1654 E_{PeV} \frac{\text{pc}^2}{\text{year}}$  and  $D_0 = 0.1654 \frac{\text{pc}^2}{\text{year}}$  which

means that we analyze energies greater than 1 PeV. The solution as a function of the energy, in which we insert the subscript E, is therefore

$$u(x,t,E_{PeV})_{E} = N_{0}E_{PeV}^{\beta} \left(e^{-\frac{t(0.1654E_{PeV}\pi^{2}+L^{2}a)}{L^{2}}}\cos\left(\frac{\pi x}{L}\right) + e^{-at}\right).$$
 (25)

The *second* case to be analyzed is that where the existing profile has three maxima, see Equation (22). The solution, provided with the subscript D, is

$$w(x,t,D,D_0)_D = u(x,t) = N_0 \left(\frac{D}{D_0}\right)^{\beta} \left(e^{-\frac{4D\pi^2 t}{L^2}} \cos\left(\frac{2\pi x}{L}\right) + 1\right).$$
 (26)

The solution as a function of the energy, with subscript *E*, is

$$w(x,t,E_{PeV})_{E} = N_{0}E_{PeV}^{\beta} \left(e^{-\frac{6.529810648E_{PeV}t}{B_{-6}L^{2}}}\cos\left(\frac{2\pi x}{L}\right) + 1\right).$$
 (27)

# 3.2. Simulation of the Ankle

The ankle in the distribution of high-energy CRs is at  $\approx 10^7$  TeV and characterizes the transition from galactic to extra-galactic CRs. As an example, we report the energy spectrum of CRs from the Pierre Auger Observatory, see the green empty stars in **Figure 7**. It is important to say that in this figure, the spectrum of CRs is multiplied by  $E^3$  in order to obtain a better visualization of the ankle. We now present in **Figure 7** the theoretical solution for the case of an existing profile with one maximum and in the presence of losses, see Equation (25). **Figure 8** presents the comparison between the theoretical solution and an  $E^{-3}$  scaling of the flux. The value of  $N_0$  is chosen in order to match the experimental data.



**Figure 7.** Flux of *H* versus energy per nucleus in Gev multiplied by  $E^3$ : experimental data (empty green stars) according to [19] and theoretical distribution for an existing profile with one maximum and in the presence of losses, see Equation (16), (red full line) when d = 3, Z = 1,  $\beta = -3.1$ ,  $B_{-6} = 1$ ,  $a = 1.0 \times 10^{-20} \frac{1}{\text{yr}}$ , x = 14.58 Mpc and L = 18.23 Mpc.

In the case of the profile with three maxima, a comparison of the solution as represented by Equation (26) with the data, both multiplied by  $E^3$ , is given in **Figure 9. Figure 10** gives the theoretical solution and the  $E^{-3}$  scaling.



**Figure 8.** Flux of *H* versus energy per nucleus in Gev: theoretical distribution (green line), see Equation (16), and a flux of comparison which scales as  $E^{-3}$  (red line), parameters as in **Figure 7**.



**Figure 9.** Flux of *H* versus energy per nucleus in Gev multiplied by  $E^3$ : experimental data (empty green stars) according to [19] and theoretical distribution for an existing profile with three maxima, see Equation (22), (red full line) when d = 3, Z = 1,  $\beta = -3.1$ ,  $B_{-6} = 1$ , x = 8.28 Mpc and L = 18.23 Mpc.



**Figure 10.** Flux of *H* versus energy per nucleus in Gev: theoretical distribution (green line) and a flux of comparison which scales as  $E^{-3}$  (red line), parameters as in **Figure 9**.

## 3.3. Radio-Emission from Clusters

Another way to test the diffusion of the CRs is to assume that the intensity of synchrotron emission from relativistic electrons is proportional to the intensity of streaming CRs, see equation (6) in [6]. We now made a comparison between the observed intensity in a single cluster of galaxies, as an example Coma, with our theoretical concentration of CRs. The Coma cluster has redshift, z = 0.0231, and is observed both in the radio region [20], in the X-ray region [21] and in the gamma region [22]. We now focus on the intensity versus distance in Coma as given by **Figure 10** (bottom left) in [20] compared with one of the new solutions here derived, see **Figure 11**.

# 4. Network of Clusters

We have already modeled the diffusion of cosmic rays from a single cluster such as Coma on the distance of  $\approx 0.3$  Mpc, see Section 3.3. We now model a squared region of universe with side of  $\approx 200$  Mpc and a spherical shell characterized by a given redshift. Our model for the voids between the galaxies are the Voronoi diagrams here used as a useful tool. An accurate choice for the parameters of the model based on the statistics of the voids between galaxies will allow to derive an acceptable spatial displacement for the clusters.

#### 4.1. Astronomical Numbers

Here we model the distance, *d*, in the local universe with the pseudo-Euclidean cosmology:



**Figure 11.** Plot of the theoretical distribution for an existing profile with one maximum and losses, see Equation (16), as a function of x when L = 0.5 Mpc, Z = 1,  $E_{PeV} = 10^7$ ,  $a = 1.0 \times 10^{-8}$  and  $B_{-6} = 1$  (red full line) and observed Coma radio-data (green full stars).

$$d(z;c,H_0) = \frac{zc}{H_0},$$
(28)

where the Hubble constant,  $H_0$ , is expressed in km·s<sup>-1</sup>·Mpc<sup>-1</sup>, the velocity of light, *c*, is expressed in km·s<sup>-1</sup> and *z* is the redshift. Here we used

 $H_0 = 67.93 \text{ km} \cdot \text{s}^{-1} \cdot \text{Mpc}^{-1}$ , see [23]. The 2MASS Redshift Survey (2MRS) has 44,599 galaxies between 0 < z < 0.17 and covers 91% of the sky, see [24]. The redMaPPer catalog has 25,325 clusters of galaxies between 0.08 < z < 0.55 and 47,165 galaxies between 0.023 < z < 0.01 see [25]. A *first* catalog of cosmic voids can be found in [26], where the effective radius of the voids,  $R_{eff}$ , has been derived to be

$$R_{eff} = 18.23h^{-1}$$
 Mpc Pam et al. 2012 (29)

The *second* catalog is that with radii up to redshift  $0.12h^{-1}$  Mpc in (SDSS-DR7), see [27],

$$R_{eff} = 11.85 h^{-1} \,\mathrm{Mpc}$$
 Varela et al. 2012 (30)

The *third* catalog is that of the Baryon Oscillation Spectroscopic Survey, see [28],

$$R_{eff} = 57.53h^{-1} Mpc \quad Mao \ et \ al. \ 2017 \tag{31}$$

The *fourth* catalog is that of the VAST void catalog for SDSS DR7 which uses three algorithms, VoidFinder,  $V^2$ , and VoidRender, and two cosmologies: Planck2018 and WMAP5 see [29]. The first analysis uses the Planck2018 cos-

mology and VoidFinder algorithm and yields

$$R_{eff} = 14.39 h^{-1} \,\text{Mpc}$$
 Douglass et al. 2023 (32)

In the following, we will calibrate our code on this fourth evaluation.

#### 4.2. Voronoi Diagrams

We now review the existing knowledge about the Voronoi diagrams. The faces of the Voronoi Polyhedra share the same property, *i.e.*, they are equally distant from two nuclei or seeds. The intersection between a plane and the faces produces diagrams that are similar to the edges displacement in 2D Voronoi diagrams. From the point of view of the observations, it is very useful to study the intersection between a slice which crosses the center of the box and the faces of irregular polyhedron where the galaxies presumably reside. According to the nomenclature reported in [30], this cut is classified as  $V_p(2,3)$  and **Figure 12** gives a typical example. This figure also gives the spherical nodes that, in the absence of an official definition, can be defined as the locus of intersection between the lines of  $V_p(2,3)$ . The spherical nodes are equally distant from three or four nuclei. In the following, we will use Poissonian seeds; the parameters are the



**Figure 12.** Portion of the Poissonian Voronoi diagram  $V_p(2,3)$ ; cut on the X-Y plane, black points, the parameters are given in **Table 1**. The spherical nodes are marked by large green points.

number of nuclei, the side of the box in Mpc, and the number of pixels, for example 1400, that are used to build the diagrams, see [31]. The parameters adopted in the simulation of the pseudo-Euclidean cosmology are given in Table 1.

The cross-sectional area of the VP can also be visualized through a spherical cut that is characterized by a constant value of the distance to the center of the box, which in this case is expressed in terms of *z*. This intersection is not present in the Voronoi literature and therefore can be classified as a "new" topic. It may be called  $V_{P,s}(2,3)$ , where the indices *P*,*s* stand for Poissonian and sphere, respectively, see **Figure 13**. More details on the theory here presented for the Voronoi diagrams can be found in [32] [33].

#### 4.3. Astrophysical diffusion in a plane

We now map how the concentration of diffusion of the CR varies in the cosmic voids. The points of diffusion are the clusters of galaxies here identified with the 3D nodes of the Voronoi diagrams from which the diffusion starts. We now outline the adopted model for the 1D diffusion from many injection points (IPs) in a 2D space. The rules are:

1) The IPs are selected, as an example Figure 12 contains 574 nodes.

2) The values of concentration from the multiple diffusion, the IPs, are recorded on a 2D grid  $\mathcal{M}(x, y)$  which covers the considered 2D space.

3) At each point of  $\mathcal{M}(x, y)$  we evaluate the distance of the nearest IP.

Table 1. Numerical values for the parameters of the Voronoi diagrams.

Cosmology	pixels	type of seeds	seeds	box-side [Mpc]	radius [ <i>z</i> ]
pseudo-Euclidean	1400	Poissonian	2231	353	0.039



**Figure 13.** The Voronoi diagram  $V_{P,s}(2,3)$  in the Hammer-Aitoff projection at 0.0169 < z < 0.0170 with the same parameters as in **Table 1**.

4) The value of  $\mathcal{M}(x, y)$  is computed with formula (25) where the variable x is the nearest distance.

**Figure 14** presents the results of such a diffusion from the theoretical clusters. **Figure 16** presents a cut of a given thickness,  $\Delta$ , of 2MRS and a number of  $V_P(2,3)$  chosen to scale as the number of galaxies. The intensity for the concentration of CR is reported along a line crossing the center of the box, see **Figure 15**, were some bridges can be observed.

#### 4.4. Astrophysical Diffusion in a Shell

We now outline the adopted model for the 1D diffusion in a shell. The rules are:

1) The IPs are selected, as an example, Figure 13 contains 193 clusters.

2) The values of concentration from the multiple diffusion, the IPs, are recorded on a 2D grid,  $\mathcal{M}(l,b)$ , where l and b identified with galactic latitude and galactic longitude.

3) At each point of  $\mathcal{M}(l,b)$  we evaluate the distance of the nearest IP.

4) The value of  $\mathcal{M}(l,b)$  is computed with formula (25) where the variable x is the nearest distance.

Figure 17 presents the results of such a diffusion from the theoretical clusters.

A comparison of the spatial distribution of galaxies with clusters of the red-MaPPer catalog is presented in Figure 18.



**Figure 14.** Contour for the concentration of diffusion or memory grid, see Equation (16), when d = 3, Z = 1,  $B_{-6} = 1$ ,  $a = 1.0 \times 10^{-10} \frac{1}{\text{yr}}$ ,  $t = 10^8 \text{ yr}$ ,  $E_{PeV} = 10^7$  and L = 50 Mpc.



Figure 15. Cut along a line for the concentration of diffusion, parameters as in Figure 14.



**Figure 16.** A cut of the 3D spatial distribution of 2MRS in the Z = 0 plane when  $\Delta = 4$  Mpc, the squared box has a side of 614 Mpc: we have 1384 galaxies (green filled circles). The same number of  $V_p(2,3)$  (red squares) with radial scaling as the number of real galaxies with the same parameters as in **Table 1**.



Galactic Longitude (1) [degrees]

Figure 17. Concentration of CR in a shell in Mollweide projection, parameters as in Figure 14.



**Figure 18.** The 2555 galaxies of 2MRS at 0.0169 < z < 0.0170 (green points) and the same number of  $V_{p,s}(2,3)$  in the Hammer-Aitoff projection (red points). The parameters are given in **Table 1**.

# **5.** Conclusions

#### PDE & Boundary Conditions

Two new solutions for the diffusion equation have been derived: the first one was derived for the case of a 1D diffusion and a trigonometric profile for the existing number of particles, see Equation (16); the second one gives the 1D diffusion for the case of a trigonometric profile for the number of particles, see Equation (22).

#### The Ankle

The ankle has been simulated trough a careful choice of the involved parameters both in the case of an existing profile with one maximum and in the presence of losses, see **Figure 7**, and in the case of an existing profile with three

#### maxima, see Figure 9.

#### Diffusion from clusters

We modeled the diffusion from the 3D nodes of the Voronoi diagrams in a plane for CRs with  $E_{PeV} = 10^7$ , see Figure 14, and in a shell, see Figure 17. A line cut of concentration in the plane of diffusion such as in Figure 15 can be further particularized for the Coma cluster which is not exactly spherical but presents a halo, a bridge and a relic [34]. In other words the complex morphology of the cut of radio-intensity in clusters can be modeled once the line-cut in intensity versus distance are provided by the radio-astronomers.

#### **Conflicts of Interest**

The author declares no conflicts of interest regarding the publication of this paper.

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