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Application of Sphere Packing Theory in Government Budget

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Abstract

The sphere packing problem involves packing non-overlapping spheres with maximum volume into a convex set. The purpose of this work is to propose a novel application of the sphere packing problem to the government budget issue and to formulate a mathematical model for reducing the government budget deficit. In the numerical experiment, we used statistical data on the Mongolian economy from 2013 to 2022. We considered 19 types of revenues and 28 types of expenditures. The numerical experiment shows that with specific adjustments to current government revenue and expenditure, the government can decrease the budget deficit.

Keywords

Sphere Packing, Optimization, Linear Programming, Fiscal Deficit, Government Budget

1. Introduction

The government budget represents a forecast of the government's anticipated revenues and planned expenditures over a specified period, typically spanning one year. Government revenues primarily encompass taxes, while expenditures comprise government spending. The governments and central banks of all nations strive to uphold a specific standard of development, economic performance, and monetary stability through the implementation of diverse policies. The economic impact is notably influenced by government budget policies, which encompass alterations in fiscal revenues, taxes, and expenditures. This research will assist in guiding the government through the process of implementing this alteration.

Due to socio-economic development and economic globalization, there is an urgent need to rely on scientific calculations in policy formulation, which is the basis for this research[1]. This research work is also useful for people to monitor how the government spends the taxes they pay.

In the last 10 years, the Mongolian government budget has been in deficit in 7 years and in surplus in 3 years. To clarify, the budget was in deficit between 2013 and 2017 and started to turn into a surplus in 2018-2019, but from 2020 to 2021 it turned into a deficit due to the pandemic and returned to a surplus in 2022. This reflects the lack of sound fiscal planning in the past. It also indicates that little attention has been paid to determining the conditions for optimal fiscal revenue and expenditure. Therefore, we propose a mathematical methodology called the sphere packing approach. This problem has important applications in science and technology. Recently, new applications of sphere packing theory have appeared in finance (Badam et al., 2021), business (Enkhbat & Tungalag, 2023) and the mining industry (Enkhbat, 2022). There are other studies that focus on optimizing government revenue and expenditure independently. Consequently, the government can integrate the outcomes of this study alongside findings from those existing research efforts.

In this paper, we define break-even range and outperformance range of each revenue and expenditure of the Mongolian government budget by calculating optimal parameters.

This paper is devoted to application of the sphere packing theory to the government financial policy which must run a balanced budget and maintain an optimal taxation level.

2. Research Method

Our research employs the sphere packing theory as its main methodology. This theory focuses on the arrangement of non-overlapping spheres to achieve maximum volume within a specified set. We used sphere packing problem with one sphere and linear programming to find out the set of the budget adequacy conditions with respect to optimal parameters.

Let us mention the latest result on the sphere packing theory (Enkhbat, 2022).

Let $B\left(x^0, r\right)$ be a ball with center $x^0 \in R^n$ and radius $r \in R$.

$$B\left(x^0, r\right) = \left\{x \in R^n \mid \left\|x - x^0\right\| \le r\right\}, \tag{1}$$

here $\|\cdot\|$ denotes the Euclidean norm. The n-dimensional volume of the Euclidean ball $B\left(x^0, r\right)$ is (Richard & David, 2001):

$$V(B) = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} r^n \tag{2}$$

where, Γ is Leonhard Euler's gamma function.

Let D be a polytope given by the following linear inequalities.

[1]Tax system renovation and tax optimization in Mongolia, business school, Mongolian national university, Ulaanbaatar 2017, MUIS press publisher, pp. 2.

$$D = \left\{ x \in R^n \mid \left\langle a^i, x \right\rangle \le b_i, i = \overline{1,m} \right\}, a^i \in R^n, b_i \in R.$$

The classical packing problem of inscribing k equal spheres into D with maximum volume is

$$\max_{(u,r)} V = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} kr^n \tag{3}$$

subject to

$$\left\langle a^i, u^j \right\rangle + r\left\| a^i \right\| \le b_i,\ i = \overline{1,m},\ j = \overline{1,k}. \tag{4}$$

$$\left\| u^i - u^j \right\|^2 \ge 4r^2,\ i, j = \overline{1,k}, i < j, \tag{5}$$

$$r \ge 0 \tag{6}$$

where, $u^1, u^2, \cdots, u^k$ centers of the spheres inscribed in D and r is their radius. For our further purpose, we use one sphere packing problem which is:

$$\max_{(x,r)} V = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} r^n \tag{7}$$

subject to

$$\left\langle a^i, x \right\rangle + r\left\| a^i \right\| \le b_i,\ i = \overline{1,m}. \tag{8}$$

$$r \ge 0. \tag{9}$$

Problem (10)-(12) is equivalent to the following linear programming.

$$\max_{(x,r)} \bar{V} = r \tag{10}$$

subject to

$$\left\langle a^i, x \right\rangle + r\left\| a^i \right\| \le b_i,\ i = \overline{1,m}. \tag{11}$$

$$r \ge 0. \tag{12}$$

Optimization of budget deficit

The total surplus of a government budget can be written as

$$f = \sum_{i=1}^{n} R_i - \sum_{j=n+1}^{n+m} C_i \tag{13}$$

where, f-total surplus or deficit, R_i- government i-th revenue, C_i- government i-th expenditure.

The condition that the total surplus exceeds a given level of surplus is

$$f \ge \gamma \tag{14}$$

where, γ–minimum surplus criteria.

We introduce the following set:

$$D=\left\{x\in R^{m+n} \mid \sum_{i=1}^{n}R_i-\sum_{j=n+1}^{n+m}C_i\geq\gamma, R_i^{\min}\leq R_i\leq R_i^{\max},\right.$$
$$\left.C_j^{\min}\leq C_j\leq C_j^{\max}, i=1,\cdots,n; j=n+1,\cdots,n+m\right\} \tag{15}$$
$$R_i=x_iR_i^{\min}+(1-x_i)R_i^{\max}, 0\leq x_i\leq 1, i=1,\cdots,n$$
$$C_j=x_jC_j^{\min}+(1-x_j)C_j^{\max}, 0\leq x_j\leq 1,\ j=n+1,\cdots,n+m$$

The total surplus function can be written as:

$$f=\sum_{i=1}^{n}\left[x_i\left(R_i^{\min}-R_i^{\max}\right)+R_i^{\max}\right]-\sum_{j=n+1}^{n+m}\left[x_j\left(C_j^{\min}-C_j^{\max}\right)+C_j^{\max}\right]$$

Condition (15) reduces to the following inequalities:

$$\sum_{j=n+1}^{n+m}x_j\left(C_j^{\min}-C_j^{\max}\right)+\sum_{i=1}^{n}x_i\left(R_i^{\min}-R_i^{\max}\right)\leq\sum_{i=1}^{n}R_i^{\max}-\sum_{j=n+1}^{n+m}C_j^{\max}-\gamma$$

Denote by $B\left(x^0,r^0\right)$ a sphere with a center $x^0\in R^{n+m}$ and radius $r^0\in R$, $r^0>0$:

$$B\left(x^0,r^0\right)=\left\{x\in R^{n+m} \mid \left\|x-x^0\right\|\leq r^0\right\}$$

It is easy to see that any point $y\in B$ can be presented as

$$y=x^0+\alpha\frac{h}{\|h\|}$$

for any $h\in R^{n+m}$ and $0\leq\alpha\leq r^0$. If $B\left(x^0,r^0\right)\subset D$ then any point satisfies Condition (15). According to sphere packing theory, we must find a sphere $B\left(x^0,r^0\right)$ with the maximum radius r^0 such that $B\left(x^0,r^0\right)\subset D$. For this purpose, taking into account the constraints of *D*, we write following linear programming problem

$$\begin{cases} r\to\max \\ \sum_{i=1}^{n}x_i\left(R_i^{\max}-R_i^{\min}\right)-\sum_{j=n+1}^{n+m}x_j\left(C_j^{\max}-C_j^{\min}\right) \\ +r\sqrt{\sum_{i=1}^{n}\left(R_i^{\max}-R_i^{\min}\right)^2+\sum_{j=n+1}^{n+m}\left(C_j^{\max}-C_j^{\min}\right)^2}\leq\sum_{i=1}^{n}R_i^{\max}-\sum_{j=n+1}^{n+m}C_j^{\max}-\gamma \\ -x_j+r\leq 0,\ j=1,\cdots,n+m \\ x_j+r\leq 1,\ j=1,\cdots,n+m \end{cases} \tag{16}$$

Let $\left(x^0,r^0\right)$ be a solution to the above problem and if we take any

$$h=\left(h_1,h_2,\cdots,h_{n+m}\right)\in R^{n+m}$$

then

$$x^h=\left(x_1,x_2,\cdots,x_{n+m}\right)=\left(x_1^0+r^0\frac{h}{\|h\|},\cdots,x_i^0+r^0\frac{h_i}{\|h\|},\cdots,x_{n+m}^0+r^0\frac{h_{n+m}}{\|h\|}\right)$$

$$\overline{x}^h=\left(\overline{x}_1,\cdots,\overline{x}_{n+m}\right)=\left(x_1^0-r^0\frac{h}{\|h\|},\cdots,x_i^0-r^0\frac{h_i}{\|h\|},\cdots,x_{n+m}^0-r^0\frac{h_{n+m}}{\|h\|}\right)$$

$$\|h\| = \sqrt{h_1^2 \cdots h_{n+m}^2}$$

and $x^h, \overline{x}^h \in B\left(x^0, r^0\right)$, consequently, $x^h, \overline{x}^h \in D$ satisfy Conditions (15). It means that in order to ensure breakeven or surplus of a government budget, the government must keep its revenues and expenditures in the following intervals:

$$x_j^0 - r^0 \frac{h_j}{\|h\|} \le x_j \le x_j^0 - r^0 \frac{h_j}{\|h\|},\ j = 1, \cdots, n+m \tag{17}$$

3. Numerical Experiments

For the numerical experiment, we grouped all government revenues and expenditures from 2013 to 2022 into the following types in Table 1 and Table 2. (Mongolian Statistical Year Book, 2019; 2022) From these last 10 years of data, the maximum and minimum values of income and expenditure were considered.

Table 1. Optimal government revenue range in million tugrugs.

Revenue	Breakeven Range		Outperformance range compared to 2022	
Future Heritage fund	651,988	647,858	661,875	657,850
Stabilization fund	876,888	865,813	890,073	879,281
Corporate income tax	1,858,762	1,833,479	1,878,722	1,854,083
Personal income taxes	988,080	974,407	996,133	982,808
Unidentified business revenues	4883	4730	4954	4805
Social security revenue	2,244,885	2,193,600	2,264,802	2,214,823
Property taxes	175,160	170,573	176,679	172,209
VAT on Domestic goods and services	1,059,910	1,029,178	1,068,764	1,038,815
VAT on Imported goods	1,831,540	1,763,592	1,848,844	1,782,627
Excise taxes	716,291	699,854	720,038	704,019
Revenue of special purposes	16,320	15,945	16,397	16,032
Revenue of foreign activities	950,790	904,497	959,484	914,371
Other popular fees	1,451,425	1,371,939	1,465,128	1,387,666
Land payment	140,419	133,023	141,596	134,388
Fee on usage of natural resource	74,915	71,640	75,399	72,207
Other taxes	7756	7005	7859	7128
Popular non-tax revenue	1,131,317	1,092,592	1,136,305	1,098,567
Capital revenue	36,700	32,818	37,169	33,386
Grants and transfers	198,450	175,853	201,024	179,002

Table 2. Optimal government expenditure range in million tugrugs.

Expenditure	Breakeven Range		Outperformance range compared to 2022	
Wages and salaries	1,919,076	1,820,023	1,905,896	1,809,366
Accommodation cost	226,142	212,969	224,465	211,627
Supply and goods expenses	111,520	100,086	110,123	98,981
Normative expenses	261,246	246,576	259,524	245,228
Purchase of stationery, effects and repair	73,369	64,972	72,420	64,237
Diem and allowances	13,698	10,497	13,349	10,230
Fees of services done by others	343,849	260,684	335,092	254,046
Purchase of goods and services	333,733	298,694	330,163	296,017
Foreign interest payments	352,515	269,851	344,358	263,799
Domestic interest payments	357,813	299,569	352,238	295,477
Subsidy to Government organization	61,800	50,501	60,750	49,738
Subsidy to Private company	235,963	175,444	230,494	171,517
Government current transfers	1,831,893	1,179,610	1,774,531	1,138,860
Social security contributions	2,005,928	1,683,540	1,978,312	1,664,134
Social welfare contributions	941,634	719,865	923,113	706,992
Other contributions from employer	146,652	100,869	142,921	98,304
Contributions from Government to individual	23,840	16,284	23,239	15,875
One-time contributions for retirement	86,581	68,661	85,187	67,724
Contributions for employees who worked in the rural areas on sustainable basis	44,015	35,020	43,331	34,565
One-time contributions	465,074	308,253	453,401	300,572
Other transfers	201,298	130,975	196,172	127,639
Construction	1,578,324	1,173,288	1,549,392	1,154,671
Capital Repairs	110,399	94,615	109,294	93,911
Equipment, Repayable investment of Gov's budget	209,116	131,598	203,791	128,247
Other capital	234,794	150,885	229,136	147,364
Strategy reserve fund capital	19,279	14,675	18,974	14,488
Foreign financed capital	118,556	72,142	115,536	70,305
Net lending	161,452	35,099	153,373	30,238

The budget revenue mainly consists of tax and fee income. Other types of income include funds, social security and government operating income. Whereas, most of the budget expenses are government operating expenses, pensions, welfare and development expenses. For solving the linear programming problem (16), we used Gekko package in Python. The minimum and maximum optimal parameters are found by (17), and the optimal ranges of the revenues and expenditures are defined by the problem (15). For example, from **Table 1**, if "Future Heritage Fund" is budgeted between 648 billion and 652 billion tugrugs, the budget will have neither profit nor loss, and if the fund is budgeted between 658 billion and 662 billion tugrugs, the budget will perform better than the previous year's performance. In the same way, if the government budget other incomes and expenses in the intervals found in **Table 1** and **Table 2**, it will achieve the two types of belove intended results.

4. Conclusions

In this paper, break-even and budget adequacy conditions have been examined by the sphere packing theory. The sphere packing approach analytically represents optimal ranges of parameters via the center and radius of a sphere inscribed in a set of budget adequacy conditions. This helps government officials to make rational decisions in the budget planning. Numerical results show that under certain ranges of revenues and expenditures, the government has the budget surplus or break-even state. The proposed methodology can be applied to other fields of economics. The amount of income and expenditures can also be found in this way in order to increase the budget performance by a certain percentage. The deficiency of the proposed approach is that we cannot find all feasible points of the set of optimal parameters. Or else, additional constraints can be imposed in the optimization problem, and we can find more specific optimal solutions.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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