

# The Erdős-Faber-Lovász Conjecture for Gap-Restricted Hypergraphs

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## Abstract

An edge coloring of hypergraph  $H$  is a function  $\psi: E(H) \rightarrow \{1, 2, \dots, w\}$  such that  $\psi(e) \neq \psi(f)$  holds for any pair of intersecting edges  $e, f \in E(H)$ . The minimum number of colors in edge colorings of  $H$  is called the chromatic index of  $H$  and is denoted by  $\chi'(H)$ . Erdős, Faber and Lovász proposed a famous conjecture that  $\chi'(H) \leq n$  holds for any loopless linear hypergraph  $H$  with  $n$  vertices. In this paper, we show that  $\chi'(H) \leq n$  is true for gap-restricted hypergraphs. Our result extends a result of Alesandroni in 2021.

## Keywords

Linear Hypergraph, Chromatic Index, Erdős-Faber-Lovász Conjecture, Edge Cardinality

## 1. Introduction

Let  $H = (V(H), E(H))$  be a hypergraph with vertex set  $V(H)$  and edge set  $E(H)$ . For each edge  $e \in E(H)$ ,  $e$  is a subset of  $V(H)$ . Let  $|e|$  denote the number of vertices in edge  $e \in E(H)$ , and call it edge cardinality of  $e$ . An edge  $e \in E(H)$  with  $|e| = k$  is called a  $k$ -edge in hypergraph  $H$ . For an edge  $e \in E(H)$  with  $|e| \geq k$ , we call edge  $e$  a  $k^+$ -edge. If  $|e| \geq 2$  holds for each edge  $e \in E(H)$ , then  $H$  is called loopless hypergraph. We call a vertex  $v \in V(H)$  incident with an edge  $e \in E(H)$  if  $v \in e$ . A pair of distinct edges  $e, f \in E(H)$  is intersecting if  $e \cap f \neq \emptyset$ . An edge coloring of  $H$  is a function  $\psi: E(H) \rightarrow \{1, 2, \dots, w\}$  such that any pair of intersecting edges  $e, f \in E(H)$  satisfies that  $\psi(e) \neq \psi(f)$ . The chromatic index  $\chi'(H)$  of hypergraph  $H$  is the minimum number of colors of all edge colorings of  $H$ , i.e., the minimum value of  $w$ . We call a hyper-

graph  $H$  linear if  $|e \cap f| \leq 1$  holds for any pair of distinct edges  $e, f \in E(H)$ . For loopless linear hypergraphs, there is a famous conjecture proposed by Erdős, Faber and Lovász in 1972.

**Conjecture 1. [1] (The Erdős-Faber-Lovász Conjecture)** For any loopless linear hypergraph  $H$  with  $n$  vertices,  $\chi'(H) \leq n$ .

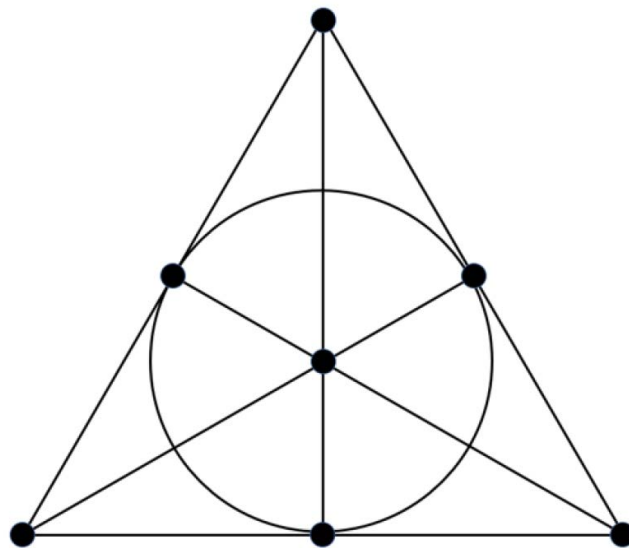
A hypergraph  $H$  is called intersecting hypergraph if  $H$  satisfies that, any pair of distinct edges are intersecting. We call a hypergraph  $H$   $r$ -uniform if  $|e| = r$  holds for each edge  $e \in E(H)$ . If Conjecture 1 is true, it is known that there are three classes of loopless linear hypergraphs, finite projective planes, degenerate planes and complete graphs with odd vertices, which are extremal hypergraphs of Conjecture 1.

An  $m$ -ordered finite projective plane  $H$  with  $n$  vertices and  $n$  edges, where  $n = m^2 + m + 1$ , is a hypergraph satisfying:

- 1)  $H$  is  $(m+1)$ -uniform and each vertex  $v \in V(H)$  incident with  $m+1$  edges.
- 2) For any two distinct vertices, there is exactly one edge containing both vertices.
- 3) For any two distinct edges, there is exactly one vertex incident with both edges.
- 4) There are four vertices so that no edge can contain more than two of these vertices.

It is known that for each prime power  $\hat{p}$ ,  $\hat{p}$ -ordered finite projective plane exist. A Fano plane in **Figure 1** is a finite projective plane with order 2.

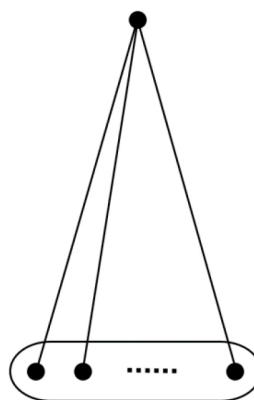
We briefly explain why it is an extremal hypergraph of Conjecture 1. For a finite projective plane  $H$  with order  $m$  and  $n = m^2 + m + 1$  vertices, since  $H$  is an intersecting hypergraph, each edge in  $H$  has a different color in any edge coloring of  $H$ . Also by  $|E(H)| = n$ , we have  $\chi'(H) = n$ .



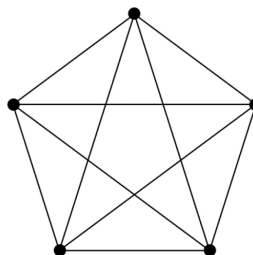
**Figure 1.** Finite projective plane.

The hypergraph in **Figure 2** is also intersecting linear hypergraph, which is called degenerate plane. A degenerate plane  $D$  with  $n$  vertices is formed by  $n-1$  2-edges which have a common vertex  $v$  and a  $(n-1)$ -edge which is formed by all vertices in  $V(D) \setminus \{v\}$ . Since degenerate plane  $D$  is an intersecting hypergraph with  $n$  edges, there is  $\chi'(D) = n$ . In **Figure 3**, it is a complete graph with odd vertices. A complete graph  $K_n$  with odd integer  $n$  is a graph formed by  $n$  vertices and any pair of vertices is connected by a 2-edge. It is well known that  $\chi'(K_n) = n$  for any odd  $n$ .

In 1988, Chang and Lawer [2] showed that  $\chi'(H) \leq \left\lceil \frac{3n}{2} - 2 \right\rceil$  is true for every loopless linear hypergraph  $H$  with  $n$  vertices. In 1992, based on Seymour's result [3] which related to the size of matchings in loopless linear hypergraphs, Kahn and Seymour [4] proved that the fractional chromatic index of any loopless linear hypergraph  $H$  is no more than the number of vertices in  $H$ . In the same year, Kahn [5] also gave an asymptotic bound of chromatic indices,  $n + o(n)$ , for loopless linear hypergraphs according to Conjecture 1. In 2008, Sánchez-Arroyo [6] verified that Conjecture 1 is true for any linear hypergraph  $H$  with  $n$  vertices which satisfy that  $|e| > \sqrt{n}$  for any edge  $e \in E(H)$ . Romero and Alonso-Pecina [7] showed that the conjecture is true for loopless linear hypergraphs with at most 12 vertices in 2014. In 2019, Faber and Harris [8] showed that  $\chi'(H) \leq n$  is true for any linear hypergraph  $H$  with  $n$  vertices and medium-size edges. In 2020, Bretto, Faisant and Hennecart [9] confirmed that, for weakly triangulated hypergraphs, Conjecture 1 is true.



**Figure 2.** Degenerate plane.



**Figure 3.**  $K_{2t+1}$ .

In 2021, Alesandroni [10] gave a result on EFL Conjecture in dual version. For any hypergraph  $H = (V(H), E(H))$ , the dual of  $H$  is a hypergraph  $H^*$  with  $V(H^*) = E(H)$ ,  $E(H^*) = V(H)$  such that  $v \in e \in E(H)$  if and only if  $e \in v \in E(H^*)$ . A vertex coloring of  $H$  is a function  $\psi_v$  that maps from  $V(H)$  to a color set  $C$  such that no edge contains homochromatic vertices. Clearly,  $H$  has an edge coloring with  $w$  colors if and only if  $H^*$  has a vertex coloring with  $w$  colors. Since we are only talking about edge coloring of hypergraphs in this paper, in order to show more clearly how the result of Alesandroni relates to Conjecture 1, here we present the form of the result in the edge coloring version.

**Theorem 2.** [10] Let  $H$  be a loopless linear hypergraph with  $n$  vertices. For every  $k \in [2, \sqrt{n}]$ , if there are at most  $k^2$   $k$ -edges in  $H$ , then  $\chi'(H) \leq n$ .

When hypergraph  $H$  does not contain  $k$ -edge ( $k \in [2, \sqrt{n}]$ ), Theorem 2 implies a result as follows.

**Corollary 3.** For any loopless linear hypergraph  $H$  with  $n$  vertices, if  $|e| \geq \sqrt{n}$  holds for each  $e \in E(H)$ , then  $\chi'(H) \leq n$ .

In 2023, Kang *et al.* [11] verified Conjecture 1 on linear hypergraphs whose vertices number is sufficiently large. Recently, Wang and Zhang [12] generalized a result of Bretto, Faisant and Hennecart in [9].

In this paper, we first show that Conjecture 1 holds for a class of odd-edge hypergraphs and a class of even-edge hypergraphs (Theorem 4, Definition 2.1). Then we extend Theorem 4 to a more general class of loopless linear hypergraphs (Theorem 5). Based on Corollary 3, we further verify that Conjecture 1 holds for gap-restricted hypergraphs, and this result is better than Theorem 5. Finally, we give a corollary and some relevant discussion on some guesses of possible counterexamples of Conjecture 1.

## 2. Main Results

In this section, we give some results on Conjecture 1 according to the edge cardinality in loopless linear hypergraphs. First, we present a simple result for odd-edge (even-edge) hypergraphs.

**Definition 2.1. (Odd-edge (Even-edge) hypergraphs)** A loopless linear hypergraph  $H$  is called *odd-edge (even-edge)* if  $|e|$  is odd (even) for any edge  $e$  in  $H$ .

For any pair of distinct vertices  $u, v$  in hypergraph  $H$ , we call  $u$  and  $v$  adjacent if there exists an edge  $e \in E(H)$  satisfying that  $u, v \in e$ . Let  $d_k(e)$  be the number of  $k$ -edges which intersect with  $e$  and  $d_{k^+}(e)$  be the number of  $k^+$ -edges which intersect with  $e$  in  $H$ . Let  $r(H) = \max_{e \in E(H)} |e|$  be the rank of hypergraph  $H$  and  $ar(H) = \min_{e \in E(H)} |e|$  be the anti-rank of hypergraph  $H$ . We use  $d_k(v)$  to denote the number of  $k$ -edges which are incident with vertex  $v$  and  $d_{k^+}(v)$  to denote the number of  $k^+$ -edges which are incident with vertex  $v$  in  $H$ . Now, we give a result based on the restriction of  $d_k(e)$  for each integer  $k \in [ar(H), r(H)]$  and each  $k$ -edge  $e$ .

**Theorem 4.** Let  $H$  be an odd-edge (even-edge) hypergraph with  $n$  vertices. If

$$d_k(e) \leq \max \left\{ k^2 - 1, \frac{k^2 + n - 1}{2} \right\} \text{ holds for every integer } k \in [ar(H), r(H)]$$

and every  $k$ -edge  $e \in E(H)$ , then  $\chi'(H) \leq n$ .

*Proof.* We give an edge coloring of  $H$  with at most  $n$  colors to prove the theorem. Consider that coloring an edge with large cardinality is more likely to be affected by other colored edges than coloring an edge with small cardinality. We will color all edges in  $H$  according to a non-increasing order of edge cardinality. For any integer  $k \in [ar(H), r(H)]$ , let  $e \in E(H)$  be a  $k$ -edge. We use  $c_e$  to denote the number of colors which appear on intersecting edges of  $e$ .

When we are ready to color edge  $e$ , since the edge coloring is performed in a non-increasing order, the colored edges are either  $(k+1)^+$ -edges or  $k$ -edges.

Recall that  $d_k(e) \leq \max \left\{ k^2 - 1, \frac{k^2 + n - 1}{2} \right\}$ . We discuss two cases according to

the magnitude between  $k^2 - 1$  and  $\frac{k^2 + n - 1}{2}$ .

Case 1.  $k^2 - 1 \geq \frac{k^2 + n - 1}{2}$ .

For each vertex  $v \in e$ , since hypergraph  $H$  is linear and  $|V(H)| = n$ , the number of vertices which are adjacent to  $v$  is no more than  $n - 1$ . Note that, for each vertex  $u \in V(H)$  which is adjacent to  $v$ ,  $u$  is either contained by edge  $e$  or by an intersecting edge of  $e$ . By  $|e| = k$ , there are  $k - 1$  vertices adjacent to  $v$  in edge  $e$ . Then the number of adjacent vertices of  $v$  which are contained by intersecting edges of  $e$  is no more than  $n - k$ . For each  $k$ -edge which is incident with  $v$ , there are  $k - 1$  vertices adjacent to  $v$  in the  $k$ -edge. By the definition of  $d_k(v)$ , there are  $(k - 1)(d_k(v) - 1)$  vertices adjacent to  $v$  in all  $k$ -edges which are incident with  $v$  except edge  $e$ . Recall that, for each  $(k + 1)^+$ -edge incident with vertex  $v$ , the  $(k + 1)^+$ -edge has at least  $k$  vertices which are adjacent to  $v$ . So, for each vertex  $v \in e$ , there is

$$d_{(k+1)^+}(v) \leq \left\lfloor \frac{n - k - (k - 1)(d_k(v) - 1)}{k} \right\rfloor. \tag{1}$$

Note that  $d_{(k+1)^+}(e) = \sum_{v \in e} d_{(k+1)^+}(v)$ . By (1), we have

$$\begin{aligned} d_{(k+1)^+}(e) &= \sum_{v \in e} d_{(k+1)^+}(v) \\ &\leq \sum_{v \in e} \left\lfloor \frac{n - k - (k - 1)(d_k(v) - 1)}{k} \right\rfloor. \end{aligned}$$

Also by  $|e| = k$ , there is

$$\begin{aligned} d_{(k+1)^+}(e) &\leq \sum_{v \in e} \left\lfloor \frac{n - k - (k - 1)(d_k(v) - 1)}{k} \right\rfloor \leq \left\lfloor \frac{\sum_{v \in e} (n - k - (k - 1)(d_k(v) - 1))}{k} \right\rfloor \\ &= \left\lfloor \frac{k(n - k) - (k - 1) \sum_{v \in e} (d_k(v) - 1)}{k} \right\rfloor = n - k + \left\lfloor \frac{-(k - 1)d_k(e)}{k} \right\rfloor. \end{aligned} \tag{2}$$

Recalling that the colored edges are either  $(k+1)^+$ -edges or  $k$ -edges, by (2), we have

$$\begin{aligned} c_e &\leq d_{(k+1)^+}(e) + d_k(e) \\ &\leq n - k + \left\lfloor \frac{-(k-1)d_k(e)}{k} \right\rfloor + d_k(e) \\ &= n - k + \left\lfloor \frac{d_k(e)}{k} \right\rfloor. \end{aligned} \tag{3}$$

By  $k^2 - 1 \geq \frac{k^2 + n - 1}{2}$  and  $d_k(e) \leq \max\left\{k^2 - 1, \frac{k^2 + n - 1}{2}\right\}$ , there is

$d_k(e) \leq k^2 - 1$ . Also by (3), we have

$$\begin{aligned} c_e &\leq n - k + \left\lfloor \frac{d_k(e)}{k} \right\rfloor \\ &\leq n - k + \left\lfloor \frac{k^2 - 1}{k} \right\rfloor \\ &= n - k + k - 1 = n - 1. \end{aligned}$$

Case 2.  $k^2 - 1 < \frac{k^2 + n - 1}{2}$ .

By  $d_k(e) \leq \max\left\{k^2 - 1, \frac{k^2 + n - 1}{2}\right\}$ , there is  $d_k(e) \leq \frac{k^2 + n - 1}{2}$ . Recall that  $H$

is an odd-edge (even-edge) hypergraph. For each integer  $k \in [ar(H), r(H)]$ , if there exists a  $k$ -edge in  $E(H)$ , then  $H$  must have no  $(k+1)$ -edge. It means that, before we color  $k$ -edge  $e$ , the colored edges are either  $(k+2)^+$ -edges or  $k$ -edges. Then we count the number of  $(k+2)^+$ -edges which intersect with  $e$  in  $H$  to estimate of  $c_e$ . As discussed above, for each vertex  $v \in e$ , the number of its adjacent vertices contained in intersecting edges of  $e$  is at most  $n - k$ . And similarly, the number of adjacent vertices of  $v$  which are contributed by intersecting  $k$ -edges of  $e$  is  $(k-1)(d_k(v) - 1)$ . Recall that, for each  $(k+2)^+$ -edge incident with vertex  $v$ , the  $(k+2)^+$ -edge contains at least  $k+1$  vertices adjacent to  $v$ . Then  $d_{(k+2)^+}(v) \leq \left\lfloor \frac{n - k - (k-1)(d_k(v) - 1)}{k+1} \right\rfloor$  is true for each vertex  $v \in e$ .

By  $|e| = k$  and  $d_{(k+2)^+}(e) = \sum_{v \in e} d_{(k+2)^+}(v)$ , the number of  $(k+2)^+$ -edges which intersect with  $e$  satisfies that

$$\begin{aligned} d_{(k+2)^+}(e) &\leq \sum_{v \in e} \left\lfloor \frac{n - k - (k-1)(d_k(v) - 1)}{k+1} \right\rfloor \\ &\leq \left\lfloor \frac{\sum_{v \in e} (n - k - (k-1)(d_k(v) - 1))}{k+1} \right\rfloor \\ &= \left\lfloor \frac{k(n - k) - (k-1)\sum_{v \in e} (d_k(v) - 1)}{k+1} \right\rfloor \\ &= n - k + \left\lfloor \frac{-n + k - (k-1)d_k(e)}{k+1} \right\rfloor. \end{aligned}$$

Then we have

$$\begin{aligned} c_e &\leq d_{(k+2)^+}(e) + d_k(e) \\ &= n - k + \left\lfloor \frac{-n + k - (k-1)d_k(e)}{k+1} \right\rfloor + d_k(e) \\ &= n - k + \left\lfloor \frac{-n + k + 2d_k(e)}{k+1} \right\rfloor. \end{aligned}$$

Recalling that  $d_k(e) \leq \frac{k^2 + n - 1}{2}$ , there is

$$\begin{aligned} c_e &\leq n - k + \left\lfloor \frac{-n + k + k^2 + n - 1}{k+1} \right\rfloor \\ &= n - k + \left\lfloor \frac{k^2 + k - 1}{k+1} \right\rfloor \\ &= n - k + k - 1 = n - 1. \end{aligned}$$

As discussed above, in either of the cases,  $c_e \leq n - 1$  holds for each integer  $k \in [ar(H), r(H)]$  and each  $k$ -edge  $e \in E(H)$ , which means that there exists an available color to color every edge  $e \in E(H)$ . Then we have an edge coloring of  $H$  with at most  $n$  colors.  $\square$

For general hypergraphs, we have the following extension. Note that, for each odd-edge (even-edge) hypergraph  $H$ ,  $\|e\| - \|f\| \geq 2$  holds for any pair of distinct edges  $e, f \in E(H)$  with  $|e| \neq |f|$ . Inspired by the gap  $\|e\| - \|f\|$  in odd-edge (even-edge) hypergraphs, we can describe a more general class of linear hypergraphs by the following definition. Call

$L(H) = \min \{ \|e\| - \|f\| : e, f \in E(H), |e| \neq |f| \}$  is the edge gap of hypergraph  $H$ . Restricting  $d_k(e)$  by edge gap of hypergraph, we have the following result.

**Theorem 5.** Let  $H$  be a loopless linear hypergraph. If

$$d_k(e) \leq \max \left\{ k^2 - 1, n - \frac{n - k^2 + 1}{L(H)} \right\} \text{ holds for every integer } k \in [ar(H), r(H)]$$

and every  $k$ -edge  $e \in E(H)$ , then  $\chi'(H) \leq n$ .

*Proof.* We give an edge coloring of  $H$  with at most  $n$  colors in a non-increasing order of edge cardinality. For any integer  $k \in [ar(H), r(H)]$ , if  $H$  has no  $k$ -edges, then there is no further discussion. If  $H$  has  $k$ -edges, let  $e \in E(H)$  be a  $k$ -edge. We also use  $c_e$  to denote the number of colors appearing on intersecting edges of  $e$ . We discuss two cases according to the magnitude between  $n$  and  $k^2 - 1$ .

Case 1.  $n < k^2 - 1$ .

Noting that  $n - k^2 + 1 < 0$  in this case,  $n - \frac{n - k^2 + 1}{L(H)}$  is strictly monotonically decreasing with respect to  $L(H)$ . By  $L(H) \geq 1$ , there is

$$n - \frac{n - k^2 + 1}{L(H)} \leq n - n + k^2 - 1 = k^2 - 1. \text{ Therefore, we have } d_k(e) \leq k^2 - 1. \text{ As}$$

discussed above, when we are ready to color edge  $e$ , the colored edges are either

$(k+1)^+$ -edges or  $k$ -edges. And similarly, for each vertex  $v \in e$ , we have  $d_{(k+1)^+}(v) \leq \left\lfloor \frac{n-k-(k-1)(d_k(v)-1)}{k} \right\rfloor$ . Recalling that  $|e|=k$  and  $d_{(k+1)^+}(e) = \sum_{v \in e} d_{(k+1)^+}(v)$ , there is

$$\begin{aligned}
 d_{(k+1)^+}(e) &\leq \sum_{v \in e} \left\lfloor \frac{n-k-(k-1)(d_k(v)-1)}{k} \right\rfloor \\
 &\leq \left\lfloor \frac{\sum_{v \in e} (n-k-(k-1)(d_k(v)-1))}{k} \right\rfloor \\
 &= \left\lfloor \frac{k(n-k)-(k-1)\sum_{v \in e} (d_k(v)-1)}{k} \right\rfloor \\
 &= n-k + \left\lfloor \frac{-(k-1)d_k(e)}{k} \right\rfloor.
 \end{aligned} \tag{4}$$

Since the value of  $c_e$  is no more than the number of colored edges which are intersecting with  $e$ , by inequation (4) and  $d_k(e) \leq k^2 - 1$ , we have

$$\begin{aligned}
 c_e &\leq d_{(k+1)^+}(e) + d_k(e) \\
 &\leq n-k + \left\lfloor \frac{-(k-1)d_k(e)}{k} \right\rfloor + d_k(e) \\
 &= n-k + \left\lfloor \frac{d_k(e)}{k} \right\rfloor \\
 &\leq n-k + \left\lfloor \frac{k^2-1}{k} \right\rfloor \\
 &= n-k+k-1 = n-1.
 \end{aligned}$$

Case 2.  $n \geq k^2 - 1$ .

Noting that  $n - k^2 + 1 \geq 0$  in this case,  $n - \frac{n - k^2 + 1}{L(H)}$  is monotonically increasing with respect to  $L(H)$ . Then there is

$$\begin{aligned}
 n - \frac{n - k^2 + 1}{L(H)} &\geq n - n + k^2 - 1 = k^2 - 1. \text{ So,} \\
 d_k(e) &\leq \max \left\{ k^2 - 1, n - \frac{n - k^2 + 1}{L(H)} \right\} = n - \frac{n - k^2 + 1}{L(H)}.
 \end{aligned}$$

By the definition of the edge gap  $L(H)$ , if there exist some  $k$ -edges in  $E(H)$ , then  $H$  must have no  $(k+i)$ -edges for any  $i \in [1, L(H)-1]$  when  $L(H) \geq 2$ . By the order of coloring edges, before we color  $k$ -edge  $e$ , the colored edges are either  $(k+L(H))^+$ -edges or  $k$ -edges. As discussed above, we first count the number of  $(k+L(H))^+$ -edges which intersect with  $e$ . And similarly, for each vertex  $v \in e$ , the number of its adjacent vertices contained in intersecting  $(k+L(H))^+$ -edges of  $e$  is at most  $n-k-(k-1)(d_k(v)-1)$ . Note that each  $(k+L(H))^+$ -edge incident with vertex  $v$  has at least  $k+L(H)-1$  vertices



which are adjacent to  $v$ . Then we have  $d_{(k+L(H))^+}(v) \leq \left\lfloor \frac{n-k-(k-1)(d_k(v)-1)}{k+L(H)-1} \right\rfloor$

for each vertex  $v \in e$ . Since  $|e|=k$  and  $d_{(k+L(H))^+}(e) = \sum_{v \in e} d_{(k+L(H))^+}(v)$ , the number of  $(k+L(H))^+$ -edges intersect with  $e$  satisfies that

$$\begin{aligned} d_{(k+L(H))^+}(e) &\leq \sum_{v \in e} \left\lfloor \frac{n-k-(k-1)(d_k(v)-1)}{k+L(H)-1} \right\rfloor \\ &\leq \left\lfloor \frac{\sum_{v \in e} (n-k-(k-1)(d_k(v)-1))}{k+L(H)-1} \right\rfloor \\ &= \left\lfloor \frac{k(n-k)-(k-1)\sum_{v \in e} (d_k(v)-1)}{k+L(H)-1} \right\rfloor \\ &= n-k + \left\lfloor \frac{-(L(H)-1)n+(L(H)-1)k-(k-1)d_k(e)}{k+L(H)-1} \right\rfloor. \end{aligned}$$

Also by  $c_e \leq d_{(k+L(H))^+}(e) + d_k(e)$  and  $d_k(e) \leq n - \frac{n-k^2+1}{L(H)}$ , there is

$$\begin{aligned} c_e &\leq n-k + \left\lfloor \frac{-(L(H)-1)n+(L(H)-1)k-(k-1)d_k(e)}{k+L(H)-1} \right\rfloor + d_k(e) \\ &= n-k + \left\lfloor \frac{-(L(H)-1)n+(L(H)-1)k+L(H)d_k(e)}{k+L(H)-1} \right\rfloor \\ &\leq n-k + \left\lfloor \frac{-(L(H)-1)n+(L(H)-1)k+k^2+(L(H)-1)n-1}{k+L(H)-1} \right\rfloor \\ &= n-k + \left\lfloor \frac{k^2+(L(H)-1)k-1}{k+L(H)-1} \right\rfloor \\ &= n-1. \end{aligned}$$

Then we have  $c_e \leq n-1$  for every integer  $k \in [ar(H), r(H)]$  and every  $k$ -edge  $e \in E(H)$ , which means that there exists an available color to color each edge  $e \in E(H)$  when we color the edge with  $n$  colors in an order of descending edge cardinality. So  $H$  has an edge coloring with at most  $n$  colors.  $\square$

Clearly, Theorem 5 implies the following result.

**Corollary 6.** Let  $H$  be a loopless linear hypergraph with  $n$  vertices. If  $d_k(e) \leq k^2 - 1$  holds for every integer  $k \in [ar(H), r(H)]$  and every  $k$ -edge  $e \in E(H)$ , then  $\chi'(H) \leq n$ .

Based on the fact that any pair of intersecting edges have at most one common vertex in linear hypergraphs, it is easy to see that the maximum possible number of  $k$ -edges in  $H$  decreases as  $k$  increases. On the other hand, if there is an edge  $e$  with large cardinality in linear hypergraph  $H$ , then there is no other edge of  $H$  that contains any pair of vertices in edge  $e$ . That is to say, if a linear hypergraph  $H$  has been with a large enough  $ar(H)$ , then  $H$  would satisfy the condition in Theorem 5.

Here is a detailed description. Note that, for each integer  $k \in [ar(H), r(H)]$ ,

$d_k(v) \leq \left\lfloor \frac{n-1}{k-1} \right\rfloor$  holds for each vertex  $v$  in any linear hypergraph with  $n$  vertices.

Let  $H$  be a loopless linear hypergraph with  $n$  vertices and  $e \in E(H)$  be a  $k$ -edge, where  $k \in [ar(H), r(H)]$ . Then there is

$d_k(e) = \sum_{v \in e} (d_k(v) - 1) \leq \sum_{v \in e} \left( \left\lfloor \frac{n-1}{k-1} \right\rfloor - 1 \right)$ . It is easy to see that, if

$$\left\lfloor \frac{n-1}{k-1} \right\rfloor - 1 \leq \max \left\{ k - \frac{1}{k}, \frac{(L(H)-1)n + k^2 - 1}{kL(H)} \right\} \text{ holds for every integer}$$

$k \in [ar(H), r(H)]$ , then  $d_k(e) \leq \max \left\{ k^2 - 1, n - \frac{n - k^2 + 1}{L(H)} \right\}$  is true for every

$k$ -edge  $e \in E(H)$ . By Theorem 5, there is  $\chi'(H) \leq n$ . Note that, when  $ar(H)$

is large enough,  $\left\lfloor \frac{n-1}{k-1} \right\rfloor - 1 \leq \max \left\{ k - \frac{1}{k}, \frac{(L(H)-1)n + k^2 - 1}{kL(H)} \right\}$  is true for every

integer  $k \in [ar(H), r(H)]$ . So,  $\chi'(H) \leq n$  holds for every loopless linear hypergraph  $H$  whose  $ar(H)$  is large enough.

By Corollary 3,  $\chi'(H) \leq n$  holds for each loopless linear hypergraph  $H$  with  $n$  vertices and  $ar(H) \geq \sqrt{n}$ . For a hypergraph  $H$  and an edge set  $E' \subset E(H)$ , the subgraph induced by  $E'$  of  $H$  is a hypergraph with vertex set  $V(H)$  and edge set  $E'$ . Therefore, by Corollary 3, we can get an edge coloring of the subgraph induced by all  $(\sqrt{n})^+$ -edges in  $H$  with at most  $n$  colors, which means that we do not need to consider the coloring of  $(\sqrt{n})^+$ -edges in  $H$  when color  $H$  with  $n$  colors. So, the edge gap between any pair of  $(\sqrt{n})^+$ -edges in  $H$  does not affect whether  $\chi'(H) \leq n$  or not. Therefore, we can obtain a more general result by relaxing the constraint conditions in the definition of  $L(H)$ . For a loopless linear hypergraph  $H$  with  $n$  vertices, call

$$L_p(H) = \min \{ \|e - f\| : e, f \in E(H), |e| \neq |f|, |e| < \sqrt{n} \} \text{ the partial edge gap of } H.$$

Note that we count the gap between  $k$ -edges ( $k < \sqrt{n}$ ) and  $(\sqrt{n})^+$ -edges in the process of the calculation of partial edge gap  $L_p(H)$  of hypergraph  $H$ .

**Definition 2.2. (Gap-restricted hypergraphs)** A loopless linear hypergraph

$H$  with  $n$  vertices is called *gap-restricted* if  $d_k(e) \leq \max \left\{ k^2 - 1, n - \frac{n - k^2 + 1}{L_p(H)} \right\}$

holds for every integer  $k \in [ar(H), \sqrt{n}]$  and every  $k$ -edge  $e \in E(H)$ .

Based on the partial edge gap, we give a result for gap-restricted hypergraphs.

**Theorem 7.** Let  $H$  be a gap-restricted hypergraph with  $n$  vertices. There is  $\chi'(H) \leq n$ .

*Proof.* By Corollary 3, we have an edge coloring of the subgraph of  $H$  which is induced by all  $(\sqrt{n})^+$ -edges with at most  $n$  colors, which means that we can color all  $(\sqrt{n})^+$ -edges in  $H$  with no more than  $n$  colors. Now, we will color remaining edges which are not colored in  $H$  in a non-increasing order of edge cardinality. For any integer  $k \in [ar(H), \sqrt{n}]$ , if  $H$  has no  $k$ -edges, then there is no further discussion. If  $H$  has  $k$ -edges, let  $e$  be a  $k$ -edge in  $E(H)$ . We also use  $c_e$

to denote the number of colors which appear on intersecting edges of  $e$ .

When  $n < k^2 - 1$ , i.e.  $n - k^2 + 1 < 0$ ,  $n - \frac{n - k^2 + 1}{L_p(H)}$  is strictly monotonically decreasing with respect to  $L_p(H)$ . Recalling that  $L_p(H) \geq 1$ , there is

$$n - \frac{n - k^2 + 1}{L_p(H)} \leq n - n + k^2 - 1 = k^2 - 1.$$

Since  $H$  is a gap-restricted hypergraph, there is  $d_k(e) \leq k^2 - 1$ . By inequation (4) in the proof of Theorem 5 (Case 1),

$$\text{we have } d_{(k+1)^+}(e) \leq n - k + \left\lfloor \frac{-(k-1)d_k(e)}{k} \right\rfloor. \text{ Note that } c_e \leq d_{(k+1)^+}(e) + d_k(e).$$

Then there is

$$\begin{aligned} c_e &\leq n - k + \left\lfloor \frac{-(k-1)d_k(e)}{k} \right\rfloor + d_k(e) \\ &= n - k + \left\lfloor \frac{d_k(e)}{k} \right\rfloor \\ &\leq n - k + \left\lfloor \frac{k^2 - 1}{k} \right\rfloor \\ &= n - 1. \end{aligned}$$

When  $n \geq k^2 - 1$ , i.e.  $n - k^2 + 1 \geq 0$ ,  $n - \frac{n - k^2 + 1}{L_p(H)}$  is monotonically increasing with respect to  $L_p(H)$ . Recalling that  $L_p(H) \geq 1$ , there is

$$n - \frac{n - k^2 + 1}{L_p(H)} \geq n - n + k^2 - 1 = k^2 - 1. \text{ Therefore, we have } d_k(e) \leq n - \frac{n - k^2 + 1}{L_p(H)}.$$

By the definition of  $L_p(H)$  and  $|e| = k$ , if  $L_p(H) \geq 2$ , then  $H$  has no  $(k+i)$ -edges for any  $i \in [1, L_p(H) - 1]$ . When we are ready to color  $k$ -edge  $e$  ( $k < \sqrt{n}$ ), since the edge coloring is performed in a non-increasing order, the colored edges are either  $(k + L_p(H))^+$ -edges or  $k$ -edges. As discussed above, for each vertex  $v \in e$ , the number of adjacent vertices of  $v$  contained by intersecting  $(k + L_p(H))^+$ -edges of  $e$  is at most  $n - k - (k-1)(d_k(v) - 1)$ . Note that each  $(k + L_p(H))^+$ -edge which is incident with vertex  $v$  contains at least  $k + L_p(H) - 1$  vertices which are adjacent to  $v$ . So,

$$d_{(k+L_p(H))^+}(v) \leq \left\lfloor \frac{n - k - (k-1)(d_k(v) - 1)}{k + L_p(H) - 1} \right\rfloor \text{ is true for each vertex } v \in e. \text{ Consi-}$$

dering that  $|e| = k$  and  $d_{(k+L_p(H))^+}(e) = \sum_{v \in e} d_{(k+L_p(H))^+}(v)$ , we have

$$\begin{aligned} d_{(k+L_p(H))^+}(e) &\leq \sum_{v \in e} \left\lfloor \frac{n - k - (k-1)(d_k(v) - 1)}{k + L_p(H) - 1} \right\rfloor \\ &\leq \left\lfloor \frac{k(n - k) - (k-1) \sum_{v \in e} (d_k(v) - 1)}{k + L_p(H) - 1} \right\rfloor \\ &= n - k + \left\lfloor \frac{-(L_p(H) - 1)n + (L_p(H) - 1)k - (k-1)d_k(e)}{k + L_p(H) - 1} \right\rfloor. \end{aligned}$$

By  $c_e \leq d_{(k+L_p(H))^+}(e) + d_k(e)$  and  $d_k(e) \leq n - \frac{n-k^2+1}{L_p(H)}$ , there is

$$\begin{aligned} c_e &\leq n - k + \left\lfloor \frac{-(L_p(H)-1)n + (L_p(H)-1)k - (k-1)d_k(e)}{k + L_p(H) - 1} \right\rfloor + d_k(e) \\ &= n - k + \left\lfloor \frac{-(L_p(H)-1)n + (L_p(H)-1)k + L_p(H)d_k(e)}{k + L_p(H) - 1} \right\rfloor \\ &\leq n - k + \left\lfloor \frac{k(k + L_p(H) - 1) - 1}{k + L_p(H) - 1} \right\rfloor \\ &= n - 1. \end{aligned}$$

There is an available color to color each edge  $e \in E(H)$  when we color  $H$  with  $n$  colors in an order of descending edge cardinality. Then we can get an edge coloring of  $H$  with at most  $n$  colors.  $\square$

### 3. Concluding Remarks

If a loopless linear hypergraph  $H$  with  $n$  vertices has at most  $k^2$   $k$ -edges for every  $k \in [2, \sqrt{n})$ , then  $d_k(e) \leq k^2 - 1 \leq \max \left\{ k^2 - 1, n - \frac{n-k^2+1}{L_p(H)} \right\}$  is true for every

integer  $k \in [ar(H), \sqrt{n})$  and every  $k$ -edge  $e \in E(H)$ . So, any hypergraph which satisfies the condition in Theorem 2 is a gap-restricted hypergraph. It is not vice versa. For example, for any integers  $k, h \geq 3$ , we construct a linear hypergraph  $H$  as follows. It is formed by  $h$   $k$ -stars (a linear  $k$ -uniform hypergraph with  $k^2$   $k$ -edges which have exactly one common vertex) and  $h$  2-edges such that the  $h$  common vertices of  $k$ -stars form an  $h$  cycle. Clearly,  $H$  satisfies the condition in Theorem 7 but does not satisfy the condition in Theorem 2.

Based on the proof of Theorem 7, if there exists a non-increasing sequence of all edges in  $H$  such that, for every integer  $k \in [ar(H), \sqrt{n})$  and every  $k$ -edge  $e \in E(H)$ , the number of intersecting  $k$ -edges of  $e$  is no more than

$$\max \left\{ k^2 - 1, n - \frac{n-k^2+1}{L_p(H)} \right\} \text{ before } e \text{ in the sequence, then } \chi'(H) \leq n.$$

### Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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