Engineering, 2023, 15, 632-646
https://www.scirp.org/journal/eng
ISSN Online: 1947-394X
ISSN Print: 1947-3931

Long-Time Behavior of Solution for Autonomous Suspension Bridge Equations with State-Dependent Delay

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How to cite this paper: Wang, S.P., Ma, Q.Z. and Shao, X.K. (2023) Long-Time Behavior of Solution for Autonomous Suspension Bridge Equations with State-Dependent Delay. *Engineering*, **15**, 632-646.
https://doi.org/10.4236/eng.2023.1510044

Received: September 27, 2023
Accepted: October 28, 2023
Published: October 31, 2023

Abstract

This work is devoted to the following suspension bridge with state-dependent delay: $\partial_{tt}u+\partial_{xxxx}u+\mu\partial_t u+ku^+ + f(u)+u\left(x,\left(t-\eta\left[u^t\right]\right)\right)=g(x)$. The main goal of this paper is to investigate the long-time behavior of the system. Under suitable hypothesis, the quasi-stability estimates of the system are established, based on which the existence of global attractor with finite fractal dimension is obtained. Furthermore, the existence of exponential attractor is proved.

Keywords

Suspension Bridge Equation, State-Dependent Delay, Global Attractor, Exponential Attractor, Quasi-Stability

1. Introduction

In this paper, we are concerned with the following autonomous suspension bridge with state-dependent delay

$$\begin{cases}\partial_{tt}u+\partial_{xxxx}u+\mu\partial_t u+ku^+ +u\left(x,\left(t-\eta\left[u^t\right]\right)\right)+f(u)=g, & x\in\Omega, t>0,\\ u(0,t)=u(L,t)=\partial_{xx}u(0,t)=\partial_{xx}u(L,t)=0, & t>0,\\ u(x,t)=\psi(x,t), & x\in\Omega, t\in[-h,0],\\ \partial_t u(x,t)=\partial_t\psi(x,t), & x\in\Omega, t\in[-h,0],\end{cases} \tag{1}$$

where $\Omega=[0,L]$, $u(x,t)$ denotes the deflection in the downward direction, η is a mapping defined on solutions with values in some interval $[0,h]$, $h>0$ presents the retardation time, $\mu\partial_t u(\mu>0)$ represents the viscous damping,

$u^+(x,t)=\max\{u(x,t),0\}$, ψ is the initial data on the interval $[-h,0]$, $u(t-\eta[u^t])$ denotes state-dependent delay term and $u^t \equiv u(t+\theta), \theta \in [-h,0]$, $g \in L^2(\Omega)$ is an external force term.

With regard to partial differential equation with delay (constant and time-dependent delay), there are many results [1]-[6]. For example, Wang in [3] studied dynamics of wave equation with delay by means of pullback asymptotically compactness for the multi-valued processes. Aouadi [5] proved the global and exponential attractors for extensible thermoelastic plate with time-varying delay by establishing quasi-stability estimates. Wang and Ma in [6] considered the existence of pullback attractors for suspension bridge equations with constant delay by using contractive function methods. In order to describe the real world, a new class of state-dependent delay models was introduced and studied recently. When the delay term depends on unknown variables in an equation, we call it a state-dependent delay differential equation. Partial differential equations with state-dependent delay have been essentially less investigated, see the discussions in the papers [7] [8] where they considered the parabolic case, and the results about the systems with state-dependent delay are not so rich as that for other kinds of delay differential equations so far. Chueshov and Rezounenko [9] considered dynamics of second order of in time evolution equations with state-dependent delay where they gave the abstracts results of system with state-dependent delay.

Inspired by the above-mentioned papers, our main goal is to study the existence of global attractor for autonomous suspension bridge equations with state-dependent delay. Compared with the dynamics of suspension bridge equation with constant or time-dependent delay, the new problem encountered in this paper is that the appearance of the state-dependent delay term firstly will lead to the solution of system is not unique, in order to guarantee the uniqueness of solutions, we prove the well-posedness of solution in a certain appropriate C^1-type space. Secondly, in the proof of the dissipative property, we need an additional term in energy functional as a compensator for the delay term. In the end, we obtain the existence of global and exponential attractor using quasi-stability method which is different from [10] where the authors considered the non-autonomous suspension bridge equations with state-dependent delay by using contractive function.

The rest of this article consists of four Sections. In the next Section, we give functions setting and iterate some useful lemmas and abstracts. In Section 3, we show the well-posedness of the solution for (1). Finally, the existence of global attractor and exponential attractor for (1) is proved in Sections 4 and 5.

2. Preliminaries

Firstly, define

$$D(A)=\{u \in V, Au \in H : u(0,t)=u(L,t)=\partial_{xx}u(0,t)=\partial_{xx}u(L,t)=0\},$$

where $A=\Delta^2=\partial_{xxxx}$, then $A: D(A) \to H$ is a strictly positive self-adjoint operator, and introduce the scale of Hilbert spaces generated the powers of A as follows:

$$V_s = D\left(A^{\frac{s}{4}}\right),\ (u,v)_{V_s} = \left(A^{\frac{s}{4}}u, A^{\frac{s}{4}}v\right),\ \|u\|_{V_s}^2 = \left\|A^{\frac{s}{4}}u\right\|^2, \forall s \in \mathbb{R}.$$

denote $H = V_0 = L^2(\Omega)$, $V_1 = H_0^1(\Omega)$, $V = V_2 = D\left(A^{\frac{1}{2}}\right) = H_0^1(\Omega)\cap H^2(\Omega)$, and the scalar product and the norm of H, V_1 and V as follows:

$$(u,v)_H = (u,v),\ |u|_H^2 = |u|^2,$$

$$(u,v)_{V_1} = (\nabla u, \nabla v),\ \|u\|_{V_1}^2 = \|\nabla u\|^2,$$

$$(u,v)_V = (\Delta u, \Delta v),\ \|u\|_V^2 = \|\Delta u\|^2.$$

By the Poincaré inequality, we have

$$\lambda_1 \|u\|_s^2 \le \|u\|_{s+1}^2,\ \forall u \in V_{s+1},$$

where λ_1^2 is the first eigenvalue of A.

We will denote by C_X the Banach space $C^0([-h,0];X)$, endowed with the sup-norm. For an element $u \in C_X$, its norm is $\|u\|_{C_X} = \sup_{\theta\in[-h,0]}\|u(\theta)\|_X$.

Introduce the phase space

$$Y \equiv C([-h,0];V)\cap C^1([-h,0];H),$$

its norm is

$$\|\psi\|_Y = \|\psi\|_{C_V} + \|\partial_t \psi\|_{C_H},\ \forall \psi \in Y.$$

Secondly, in order to prove the well-posedness of solution and dissipative of system (see [9] [10] for details), assume that nonlinear term f satisfies the following dissipative conditions (2) (3) and growth conditions (4):

$$\liminf_{|s|\to\infty} \frac{F(s)}{s^2} \ge 0, \tag{2}$$

where $F(s) = \int_0^s F(\tau)\,\mathrm{d}\tau$.

$$\liminf_{|s|\to\infty} \frac{sf(s) - C_0 F(s)}{s^2} \ge 0,\ \ C_0 > 0. \tag{3}$$

and

$$\limsup_{|s|\to\infty} \frac{|f'(s)|}{|s|^p} = 0, \tag{4}$$

where $0 \le p < \infty$.

For every $\nu > 0$, there exists $C_\nu > 0$, such that

$$\|u\|^2 \le C_\nu + \nu\left(\|\Delta u\|^2 + \int_\Omega F(u)\,\mathrm{d}x\right). \tag{5}$$

Finally, due to the appearance of the state delay term, while proving the well-posedness of solution, it is necessary to suppose that the mapping

$\eta : Y \to [0,h]$ is locally Lipschitz

$$\left|\eta(\psi_1)-\eta(\psi_2)\right| \le C_R \left\|\psi_1-\psi_2\right\|_Y, \tag{6}$$

for every $\psi_1,\psi_2 \in Y$, $\left\|\psi_j\right\|_Y \le R,\ j=1,2$.

Remark [9] The main example of state-depenent delay term is

$$M(\phi)=\phi(-\pi(\phi)),\ \phi \in C([-h,0];H),$$

where π maps $C([-h,0];H)$ into some interval $[0,h]$. We note that this delay term M is not locally Lipschitz in the classical space of continuous functions $C([-h,0];H)$, see [6] [9] [10] [11] [13] for details.

Definition 1 [12] [14] Let X,Y,Z be three reflexive Banach spaces with X compactly embedded in Y, $H = X\times Y\times Z$ and let $(S(t),H)$ be a dynamical system given by an evolution operator

$$S(t)y_i = y_i(t), i=1,2, \forall y_1,y_2 \in B, t>0.$$

We call the dynamical system $(S(t),H)$ quasi-stable on $B\subset H$. If there exists a compact seminorm n_X on X and nonnegative scalar functions $a(t)$ and $c(t)$, locally bounded in $[0,\infty)$, and $b(t)\in L_1(\mathbb{R}^+)$ with $\lim_{t\to\infty} b(t)=0$, such that

$$\left\|S(t)y_1-S(t)y_2\right\|_H^2 \le a(t)\left\|y_1-y_2\right\|_H^2, \tag{7}$$

and

$$\left\|S(t)y_1-S(t)y_2\right\|_H^2 \le b(t)\left\|y_1-y_2\right\|_H^2 + c(t)\sup_{s\in[0,t]}\left[n_X\left(u(s)-v(s)\right)\right]. \tag{8}$$

Proposition 2 [12] [14] Let the hypothesis in Definition 1 be in force. Suppose that the dynamical system $(S(t),H)$ is quasi-stable on every bounded forward invariant set B in H. Then $(S(t),H)$ is asymptotically smooth.

Theorem 3 [12] [14] A dissipative dynamical system $(S(t),H)$ has a compact global attractor if and only if it is asymptotically smooth.

Theorem 4 [12] [14] If dynamical system $(S(t),H)$ possesses a compact global attractor $\mathcal{A}$ and is quasi-stable on $\mathcal{A}$. Then the attractor $\mathcal{A}$ has finite fractal dimension.

Theorem 5 [12] [14] Suppose that dynamical system $(S(t),H)$ is dissipative and quasi-stable on some bounded absorbing set B. In addition, assume that there exists an extended space $\bar{H}\supseteq H$, such that mapping $t\to S(t)y$ is Hölder continuous in $\bar{H}$ for each $y\in B$, that is, there exist $0<\gamma\le 1$ and $C_{B,T}$, such that

$$\left\|S(t_1)y-S(t_2)y\right\|_{\bar{H}} \le C_{B,T}\left|t_1-t_2\right|^{\gamma}, t_1,t_2\in[0,T], y\in B.$$

Then the dynamical system $(S(t),H)$ possesses a generalized fractal exponential attractor whose dimension is finite in the space $\bar{H}$.

3. Well-Posedness

Definition 6 A vector function

$$u(t)\in C([-h,T];V_2)\cap C^1([-h,T];V_0)$$

is said to be a weak solution of the problem (1) on the interval $[0,T]$, if $u(x,t)$ satisfies:

1) $u(t)=\psi(t), \forall t\in[-h,0]$;
2) $\forall v\in V_2$, we have that

$$(\partial_{tt}u,v)+(\Delta u,\Delta v)+(\mu\partial_t u,v)+(ku^+,v)+(f(u),v)+(u(t-\pi[u^t]),v)=(g,v).$$

Lemma 7 Suppose that f satisfy (2) - (3) and (5), $g\in L^2(\Omega)$. For any μ_0, there exists $h_0=h(\mu_0)>0$, such that $(\mu,h)\in[\mu_0,+\infty)\times(0,h_0]$. Then the solution $(u,\partial_t u)$ of Equation (1) satisfies the following estimates

$$\|z(t)\|^2+\left\|A^{\frac{1}{2}}u(t)\right\|^2+k\|u^+(t)\|^2\le 4\mathrm{e}^{-\delta t}\left(E(0)+\alpha h\|\psi\|_Y^2\right)+\frac{4C}{\delta}, \tag{9}$$

where $C=\frac{2}{\mu}\|g\|^2+2\delta K_1+2\varepsilon K_2+\frac{4}{\mu}C_\rho$.

Proof Similar to reference [9], we set

$$E(t)=\|z\|^2+\|\Delta u\|^2+k\|u^+\|^2+2\int_\Omega F(u)\mathrm{d}x+2K_1\ge 0,$$

$$V(t)=E(t)+\frac{\alpha}{h}\int_0^h\int_{t-s}^t\|\partial_t u(r)\|^2\mathrm{d}r\mathrm{d}s,$$

we can prove that

$$\|z(t)\|^2+\left\|A^{\frac{1}{2}}u(t)\right\|^2+k\|u^+(t)\|^2\le 4\mathrm{e}^{-\delta t}\left(E(0)+\alpha h\|\psi\|_Y^2\right)+\frac{4C}{\delta}, \tag{10}$$

where $C=\frac{2}{\mu}\|g\|^2+2\delta K_1+2\varepsilon K_2+\frac{4}{\mu}C_\rho$. Then (9) hold true.

Set $U(t)=(u(t);v(t))$, rewrite (1) as the following first order differential equation in the space $\mathcal{H}=V\times H$

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t)+\mathcal{A}U(t)=\mathcal{N}(U^t),\quad (x,t)\in\Omega\times(0,+\infty),$$

where $\Phi=(\psi;\partial_t\psi),\psi\in Y$, define the operator $\mathcal{A}$ and the mapping $\mathcal{N}$ as follows:

$$\mathcal{A}z=(-v(t);Au+\mu v(t)),\ \ U=(u;v)\in D(\mathcal{A})\equiv V\times H,$$
$$\mathcal{N}(\Phi)=(0;-f(\psi(0))-k(\psi(0))-\psi(-\eta[\psi])+g).$$

We can show that the operator $\mathcal{A}$ generates exponentially stable C_0-semigroup $\{\mathrm{e}^{-t\mathcal{A}}:t\ge 0\}$ in $\mathcal{H}$ (see [15]).

Definition 8 A mild solution of (1) on an interval $[0,T]$ is defined as a function

$$u\in C([-h,T];V)\cap C^1([-h,T];H),$$

where $u(\theta)=\psi(\theta),\theta\in[-h,0]$ and $U(t)=(u(t);\partial_t u(t))$ satisfies

$$U(t)=\mathrm{e}^{-t\mathcal{A}}U(0)+\int_0^t\mathrm{e}^{-(t-s)\mathcal{A}}\mathcal{N}(U^s)\mathrm{d}s,\quad t\in[0,T].$$

Theorem 9 [9] [10] Assume that f satisfy (4) and (6), $g \in L^2(\Omega)$. Then for any $\psi_i \in Y$, $\|\psi_i\|_Y \le C, i=1,2$, there exists $0 < T_{\max} \le \infty$, and a unique mild solution $U(t) \equiv (u(t); \partial_t u(t))$ of (1.1) on the interval $[0, T_{\max}]$, $T_{\max} = \infty$ or $\lim_{t \to T_{\max}^-} \|u^t\|_Y = \infty$.

Theorem 10 [9] [10] (Well-posedness) Let assumptions (2) - (6) hold true and $g \in L^2(\Omega)$. Then for any $\psi_1, \psi_2 \in Y$, $\|\psi_i\|_Y \le \mu, i=1,2$, there exists a unique global mild solution $U(t) \equiv (u(t); \partial_t u(t))$ of (1) on the interval $[0, +\infty]$. Moreover, for any $\mu > 0$ and $T > 0$ there exists a positive constant $C_{\mu,T}$, such that

$$\left\| A^{\frac{1}{2}} \left(u_1(t) - u_2(t) \right) \right\|^2 + \left\| \partial_t u_1(t) - \partial_t u_2(t) \right\|^2 \le C_{\mu,T} \|\psi_1 - \psi_2\|_Y^2, \quad t \in [0, T].$$

Owing to Theorem 10, we can define an evolution operator as following:

$$S(t): Y \to Y, \ \forall t > 0, \tag{11}$$

by the formula $S(t)\psi = u^t$, where $u(t)$ is the solution of (1), satisfying $u^0 = \psi$.

4. Global Attractor

In this section, firstly, we prove the system $(S(t), Y)$ has a bounded set; secondly, we will show that the semigroup $\{S(t)\}_{t \ge 0}$ corresponding to (1) is asymptotically compact; Finally, the existence of global attractor system (1) is obtained.

Theorem 11 (Dissipative) Assume that the assumptions in Lemma 7 be in force. Then dynamical system $(S(t), Y)$ is dissipative, *i.e.* there exists $R > 0$, $\forall \rho > 0$, such that

$$\|S(t)\psi\|_Y \le R, \forall \psi \in Y, \|\psi\|_Y \le \rho, t \ge t_\rho.$$

Proof Similar to Lemma 7, there holds

$$\|z(t)\|^2 + \left\| A^{\frac{1}{2}} u(t) \right\|^2 + k\|u^+(t)\|^2 \le 4e^{-\delta t} \left(E(0) + \alpha h \|\psi\|_Y^2 \right) + \frac{4C}{\delta}.$$

now setting $t + \theta$ instead of t (where $\theta \in [-h, 0]$) in (10),

$$\begin{aligned} &\left\| A^{\frac{1}{2}} u(t+\theta) \right\|^2 + \|z(t+\theta)\|^2 + k\|u^+(t+\theta)\|^2 \\ &\le 4e^{-\delta(t+\theta)} \left(E(0) + \alpha h \|\psi\|_Y^2 \right) + \frac{4C}{\delta} \le 4e^{-\delta(t-h)} \left(E(0) + \alpha h \|\psi\|_Y^2 \right) + \frac{4C}{\delta}. \end{aligned} \tag{12}$$

Hence, by (12)

$$\begin{aligned} \|u^t\|_Y^2 &= \max_{\theta \in [-h,0]} \|z(t+\theta)\|^2 + \max_{\theta \in [-h,0]} \left\| A^{\frac{1}{2}} u(t+\theta) \right\|^2 \\ &\le 2 \max_{\theta \in [-h,0]} \left(\|z(t+\theta)\|^2 + \left\| A^{\frac{1}{2}} u(t+\theta) \right\|^2 \right) \\ &\le 8e^{-\delta(t-h)} \left(E(0) + \alpha h \|\psi\|_Y^2 \right) + \frac{8C}{\delta}. \end{aligned}$$

It implies that there exists $t \ge t_\rho$, any ball $B = \bar{B}(0,R)$ with $R > \dfrac{2\sqrt{2C}}{\delta}$ is a bounded absorbing set B of $(S(t),Y)$. □

In order to prove $\{S(t)\}_{t\ge 0}$ is asymptotically smooth, furthermore assume that there exists $\delta > 0$, the delay term satisfies for any $R > 0$, $\psi_i, i = 1,2$, there exists $L_R > 0$, such that $\|\psi_i\|_Y \le R$, one has

$$\left|\eta(\psi_1) - \eta(\psi_2)\right| \le L_R \max_{\theta\in[-h,0]} \left\| A^{\frac{1}{2}-\delta} \left(\psi_1(\theta) - \psi_2(\theta)\right)\right\|. \tag{13}$$

Remark In this paper, the one-dimensional suspension bridge equation with state-dependent delay is considered, but we still need to assume that state-dependent delay term satisfies condition (13), when we establish quasi-stability estimates (8) to verify asymptotical compactness of semigroup $\{S(t)\}_{t\ge 0}$.

Lemma 12 (Quasi-stability) Suppose that (2) - (6) and (13) hold true and $g \in L^2(\Omega)$. Then there exists $c_1(R), c_2(R)$ and $\bar{\lambda}$, such that solutions u_1, u_2 of (1), initial data ψ_1, ψ_2 satisfying

$$\left\|\partial_t u_i(t)\right\|^2 + \left\|A^{\frac{1}{2}} u_i(t)\right\|^2 \le R^2, t \ge -h, i = 1,2, \tag{14}$$

and there holds the following quasi-stability estimates

$$\begin{aligned} &\left\|\partial_t u_1(t) - \partial_t u_2(t)\right\|^2 + \left\|A^{\frac{1}{2}} u_1(t) - A^{\frac{1}{2}} u_2(t)\right\|^2 \\ &\le c_1(R) \mathrm{e}^{-\bar{\lambda} t} \left\|\psi_1 - \psi_2\right\|_Y^2 + c_2(R) \max_{r\in[0,t]} \left\| A^{\frac{1}{2}-\varepsilon}\left(u_1(r) - u_2(r)\right)\right\|^2, \end{aligned} \tag{15}$$

where $\varepsilon > 0$.

Proof Assume that u_1, u_2 are solutions of Equation (1), set $w = u_1(t) - u_2(t)$ is solution of the following equation

$$\begin{aligned} &\partial_{tt} w + Aw + \mu \partial_t w \\ &= -\left(k\left(u_1^+ - u_2^+\right)\right) - \left(f(u_1) - f(u_2)\right) - \left(u_1\left(t - \eta\left[u_1^t\right]\right) - u_2\left(t - \eta\left[u_2^t\right]\right)\right). \end{aligned} \tag{16}$$

According to Theorem 11, it is obviously that (14) *i.e.* true.

We define energy functional

$$E_w(t) = \frac{1}{2}\left(\left\|A^{\frac{1}{2}} w\right\|^2 + \left\|\partial_t w\right\|^2\right).$$

Multiplying (16) with $\partial_t w(t)$ and integrating it on $[t,T]$, it yields

$$\begin{aligned} &E_w(T) - E_w(t) + \mu \int_t^T \left\|\partial_t w(s)\right\|^2 \mathrm{d}s \\ &\le \int_t^T \left(f(u_2(s)) - f(u_1(s)), \partial_t w(s)\right) \mathrm{d}s + k \int_t^T \left(u_2^+(s) - u_1^+(s), \partial_t w(s)\right) \mathrm{d}s \\ &\quad + \int_t^T \left(u_2\left(s - \eta\left[u_2^s\right]\right) - u_1\left(s - \eta\left[u_1^s\right]\right), \partial_t w(s)\right) \mathrm{d}s, \end{aligned} \tag{17}$$

By Differential mean value theorem and $\left|f(u)\right|_{L^\infty} \le K_3, \left|f'(u)\right|_{L^\infty} \le K_3$.

$$\begin{aligned}&\left|\int_\Omega\left(f\left(u_2(t)\right)-f\left(u_1(t)\right)\right)\partial_t w(t)\mathrm{d}x\right|\\&\le K_3\int_\Omega\left|u_1(t)-u_2(t)\right|\left|\partial_t w(t)\right|\mathrm{d}x\le\frac{\varepsilon}{2}\left\|A^{\frac{1}{2}}w(t)\right\|^2+\frac{C_R}{2\varepsilon}\left\|\partial_t w\right\|^2,\end{aligned}\tag{18}$$

where $\varepsilon>0$, and

$$\begin{aligned}k\left|\int_\Omega\left(u_2^+(t)-u_1^+(t)\right)\partial_t w(t)\right|&\le kl\int_\Omega\left|u_1-u_2\right|\left|\partial_t w(t)\right|\mathrm{d}x\\&\le\frac{\varepsilon}{2}\left\|A^{\frac{1}{2}}w(t)\right\|^2+\frac{C_R}{2\varepsilon}\left\|\partial_t w(t)\right\|^2.\end{aligned}\tag{19}$$

Applying condition (13), we obtain

$$\begin{aligned}&\left|\int_\Omega\left(u_2\left(t-\eta\left[u_2^t\right]\right)-u_1\left(t-\eta\left[u_1^t\right]\right)\right)\partial_t w(t)\mathrm{d}x\right|\\&\le\left\|u_2\left(t-\eta\left[u_2^t\right]\right)-u_1\left(t-\eta\left[u_1^t\right]\right)\right\|\left\|\partial_t w(t)\right\|\\&\le\max_{\theta\in[-h,0]}\left\|A^{\frac{1}{2}-\varepsilon}w(t+\theta)\right\|^2+C_R\left\|\partial_t w(t)\right\|^2.\end{aligned}\tag{20}$$

Combining (18) - (20), from (17), there holds

$$\begin{aligned}&\left|E_w(T)-E_w(t)+\mu\int_t^T\left\|\partial_t w(s)\right\|^2\mathrm{d}s\right|\\&\le\varepsilon\int_t^T\left\|A^{\frac{1}{2}}w(s)\right\|^2\mathrm{d}s+\int_t^T\max_{\theta\in[-h,0]}\left\|A^{\frac{1}{2}-\varepsilon}w(s+\theta)\right\|^2\mathrm{d}s\\&\quad+C_R\left(1+\frac{1}{\varepsilon}\right)\int_t^T\left\|\partial_t w(s)\right\|^2\mathrm{d}s.\end{aligned}\tag{21}$$

For any $\varepsilon>0$, choosing μ is enough large, such that

$$C_R\left(1+\frac{1}{\varepsilon}\right)<\frac{\mu}{2}.\tag{22}$$

Multiplying (16) with $w(t)$ and integrating it on $[0,T]$, it yields

$$\begin{aligned}&\left(\partial_t w(T),w(T)\right)-\left(\partial_t w(0),w(0)\right)-\int_0^T\left\|\partial_t w(s)\right\|^2\mathrm{d}s\\&+\int_0^T\left\|A^{\frac{1}{2}}w(s)\right\|^2\mathrm{d}s+\mu\int_0^T\left(\partial_t w(s),w(s)\right)\mathrm{d}s\\&\le\int_0^T\left(f\left(u_2(s)\right)-f\left(u_1(s)\right),w(s)\right)\mathrm{d}s+k\int_0^T\left(u_2^+(s)-u_1^+(s),w(s)\right)\mathrm{d}s\\&+\int_0^T\left(u_2\left(s-\eta\left[u_2^s\right]\right)-u_1\left(s-\eta\left[u_1^s\right]\right),w(s)\right)\mathrm{d}s.\end{aligned}\tag{23}$$

Similar to (18) - (20), we have

$$\int_\Omega\left(f\left(u_2(t)\right)-f\left(u_1(t)\right)\right)w(t)\mathrm{d}x\le\frac{1}{4}\left\|A^{\frac{1}{2}}w(t)\right\|^2+C_R\left\|w(t)\right\|^2,$$

$$k\int_\Omega\left(u_2^+(t)-u_1^+(t)\right)w(t)\mathrm{d}x\le\frac{1}{4}\left\|A^{\frac{1}{2}}w(t)\right\|^2+C_R\left\|w(t)\right\|^2,$$

and

$$
\begin{aligned}
&\int_{\Omega}\left(u_2\left(t-\eta\left[u_2^t\right]\right)-u_1\left(t-\eta\left[u_1^t\right]\right)\right)w(t)\mathrm{d}x \\
&\le \left\|u_2\left(t-\eta\left[u_2^t\right]\right)-u_1\left(t-\eta\left[u_1^t\right]\right)\right\|\left\|w(t)\right\| \\
&\le C_R' \max_{\theta\in[-h,0]}\left\|A^{\frac{1}{2}-\varepsilon}w(t+\theta)\right\|^2+C_R\left\|w(t)\right\|^2.
\end{aligned}
$$

Substitute above inequalities into (23),

$$
\begin{aligned}
&\left(\partial_t w(T),w(T)\right)-\left(\partial_t w(0),w(0)\right)-\int_0^T\left\|\partial_t w(s)\right\|^2\mathrm{d}s \\
&+\int_0^T\left\|A^{\frac{1}{2}}w(s)\right\|^2\mathrm{d}s+\mu\int_0^T\left(\partial_t w(s),w(s)\right)\mathrm{d}s \\
&\le\frac{1}{2}\int_0^T\left\|A^{\frac{1}{2}}w(s)\right\|^2\mathrm{d}s+C_R''\int_0^T\max_{\theta\in[-h,0]}\left\|A^{\frac{1}{2}-\varepsilon}w(t+\theta)\right\|^2\mathrm{d}s+C_R''\int_0^T\left\|w(s)\right\|^2\mathrm{d}s,
\end{aligned}
\tag{24}
$$

Furthermore, by using Young and Hölder inequalities, we have

$$
\mu\int_0^T\left(\partial_t w(s),w(s)\right)\mathrm{d}s\ge-\frac{1}{2}\int_0^T\left\|\partial_t w(s)\right\|^2\mathrm{d}s-\frac{\mu^2}{2}\int_0^T\left\|w(s)\right\|^2\mathrm{d}s.
$$

By the definition of energy functional and from (24), it implies that

$$
\begin{aligned}
\frac{1}{2}\int_0^T\left\|A^{\frac{1}{2}}w(s)\right\|^2\mathrm{d}s\le&\frac{3}{2}\int_0^T\left\|\partial_t w(s)\right\|^2\mathrm{d}s+C\left(E_w(0)+E_w(T)\right) \\
&+C_R''(\mu)\int_0^T\max_{\theta\in[-h,0]}\left\|A^{\frac{1}{2}-\varepsilon}w(s+\theta)\right\|^2\mathrm{d}s.
\end{aligned}
\tag{25}
$$

Integrating (21) on $[0,T]$, combining with (22), we obtain

$$
TE_w(T)\le\int_0^T E_w(s)\mathrm{d}s+\varepsilon T\int_0^T\left\|A^{\frac{1}{2}}w(s)\right\|^2\mathrm{d}s+T\int_0^T\max_{\theta\in[-h,0]}\left\|A^{\frac{1}{2}-\varepsilon}w(s+\theta)\right\|^2\mathrm{d}s. \tag{26}
$$

Set $t=0$ in (21) and combining with (22), we can see that

$$
\begin{aligned}
E_w(0)\le E_w(T)+\frac{3\mu}{2}\int_0^T\left\|\partial_t w(s)\right\|^2\mathrm{d}s+\varepsilon\int_0^T\left\|A^{\frac{1}{2}}w(s)\right\|^2\mathrm{d}s \\
+\int_0^T\max_{\theta\in[-h,0]}\left\|A^{\frac{1}{2}-\varepsilon}w(s+\theta)\right\|^2\mathrm{d}s
\end{aligned}
\tag{27}
$$

and

$$
\frac{\mu}{2}\int_0^T\left\|\partial_t w(s)\right\|^2\mathrm{d}s\le E_w(0)+\varepsilon\int_0^T\left\|A^{\frac{1}{2}}w(s)\right\|^2\mathrm{d}s+\int_0^T\max_{\theta\in[-h,0]}\left\|A^{\frac{1}{2}-\varepsilon}w(s+\theta)\right\|^2\mathrm{d}s. \tag{28}
$$

Adding (25) (28), and supposing that $\mu\ge 6$, it yields

$$
\begin{aligned}
&\left(\frac{\mu}{2}-2\right)\int_0^T\left\|\partial_t w(s)\right\|^2\mathrm{d}s+\int_0^T E_w(s)\mathrm{d}s \\
&\le\varepsilon\int_0^T\left\|A^{\frac{1}{2}}w(s)\right\|^2\mathrm{d}s+C\left(E_w(0)+E_w(T)\right)+C_R''(\mu)\int_0^T\max_{\theta\in[-h,0]}\left\|A^{\frac{1}{2}-\varepsilon}w(s+\theta)\right\|^2\mathrm{d}s.
\end{aligned}
\tag{29}
$$

Adding $\frac{1}{2}TE_w(T)$ to (29) and substituting (21) into (29),

$$\left(\frac{\mu}{2}-2\right)\int_0^T\left\|\partial_t w(s)\right\|^2\mathrm{d}s+\frac{1}{2}\int_0^T E_w(s)\mathrm{d}s+\frac{1}{2}TE_w(T)$$
$$\le\varepsilon\left(\frac{T}{2}+1\right)\int_0^T\left\|A^{\frac{1}{2}}w(s)\right\|^2\mathrm{d}s+C\left(E_w(0)+E_w(T)\right) \tag{30}$$
$$+C_R''(\mu)\left(1+\frac{T}{2}\right)\int_0^T\max_{\theta\in[-h,0]}\left\|A^{\frac{1}{2}-\varepsilon}w(s+\theta)\right\|^2\mathrm{d}s.$$

Valuation $E_w(0)+E_w(T)$, from (27)

$$E_w(0)+E_w(T)\le 2E_w(T)+\frac{3\mu}{2}\int_0^T\left\|\partial_t w(s)\right\|^2\mathrm{d}s+\varepsilon\int_0^T\left\|A^{\frac{1}{2}}w(s)\right\|^2\mathrm{d}s$$
$$+\int_0^T\max_{\theta\in[-h,0]}\left\|A^{\frac{1}{2}-\varepsilon}w(s+\theta)\right\|^2\mathrm{d}s. \tag{31}$$

substituting (31) into (30),

$$\frac{1}{2}\int_0^T E_w(s)\mathrm{d}s+\left(\frac{1}{2}T-2C\right)E_w(T)$$
$$\le C_\mu\int_0^T\left\|\partial_t w(s)\right\|^2\mathrm{d}s+c\varepsilon\left(\frac{T}{2}+1\right)\int_0^T\left\|A^{\frac{1}{2}}w(s)\right\|^2\mathrm{d}s \tag{32}$$
$$+C_R''(\mu)\left(1+\frac{T}{2}\right)\int_0^T\max_{\theta\in[-h,0]}\left\|A^{\frac{1}{2}-\varepsilon}w(s+\theta)\right\|^2\mathrm{d}s,$$

where $C_\mu>0$ depends on μ.

Set

$$\frac{1}{2}T-2C>1,$$

from (32),

$$E_w(T)+\frac{1}{2}\int_0^T E_w(s)\mathrm{d}s\le C_R''(\mu)\left(1+\frac{T}{2}\right)\int_0^T\max_{\theta\in[-h,0]}\left\|A^{\frac{1}{2}-\varepsilon}w(s+\theta)\right\|^2\mathrm{d}s$$
$$+2\varepsilon\left(\frac{T}{2}+1\right)\int_0^T\left\|A^{\frac{1}{2}}w(s)\right\|^2\mathrm{d}s+C_\mu\int_0^T\left\|\partial_t w(s)\right\|^2\mathrm{d}s. \tag{33}$$

Subsequently, we compute the last term, set $t=0$ in (21) and combining with (22),

$$\frac{\mu}{2}\int_0^T\left\|\partial_t w(s)\right\|^2\mathrm{d}s\le E_w(0)-E_w(T)+\varepsilon\int_0^T\left\|A^{\frac{1}{2}}w(s)\right\|^2\mathrm{d}s$$
$$+\int_0^T\max_{\theta\in[-h,0]}\left\|A^{\frac{1}{2}-\varepsilon}w(s+\theta)\right\|^2\mathrm{d}s.$$

Substituting above inequality into (33),

$$E_w(T)+\frac{1}{2}\int_0^T E_w(s)\mathrm{d}s \le C_\mu\left(E_w(0)-E_w(T)\right)+2C_\mu\varepsilon\left(\frac{T}{2}+1\right)\int_0^T\left\|A^{\frac{1}{2}}w(s)\right\|^2\mathrm{d}s$$
$$+C_\mu C_R''(\mu)\left(1+\frac{T}{2}\right)\int_0^T\max_{\theta\in[-h,0]}\left\|A^{\frac{1}{2}-\varepsilon}w(s+\theta)\right\|^2\mathrm{d}s.$$

By the definition of energy functional $E_w(t)$, we have $\left\|A^{\frac{1}{2}}w(s)\right\|^2\le 2E_w(s)$, choosing $\varepsilon>0$, such that

$$2C_\mu\varepsilon\left(\frac{T}{2}+1\right)<\frac{1}{4}.$$

So we obtain that

$$E_w(T)\le C_\mu\left(E_w(0)-E_w(T)\right)+C_R''(\mu)\left(1+\frac{T}{2}\right)\int_0^T\max_{\theta\in[-h,0]}\left\|A^{\frac{1}{2}-\varepsilon}w(s+\theta)\right\|^2\mathrm{d}s.$$

Furthermore,

$$E_w(T)\le\frac{C_\mu}{1+C_\mu}E_w(0)+C_R''(T,\mu)\int_0^T\max_{\theta\in[-h,0]}\left\|A^{\frac{1}{2}-\varepsilon}w(s+\theta)\right\|^2\mathrm{d}s.$$

set $\omega=\frac{C_\mu}{1+C_\mu}<1$, there exists $\beta>0$, such that

$$E_w(T)\le \mathrm{e}^{-\beta t}E_w(0)+C_R''(T,\mu)\int_0^T\max_{\theta\in[-h,0]}\left\|A^{\frac{1}{2}-\varepsilon}w(s+\theta)\right\|^2\mathrm{d}s.$$

By using reference [16], Repeat the above steps, we conclude that (15) is true. □

Remark Maximize (15) on $[t-h,t]$, it has

$$\begin{aligned}&\left\|S(t)u_1-S(t)u_2\right\|_Y^2\\&\le c_1(R)h\mathrm{e}^{\bar{\lambda}h}\mathrm{e}^{-\bar{\lambda}t}\left\|\psi_1-\psi_2\right\|_Y^2+c_2(R)h\max_{s\in[0,t]}\left[n_Y\left(u_{1s}-u_{2s}\right)\right]^2, t\ge h,\end{aligned}\tag{34}$$

where n_Y is compact semi-norm in *Y*, taking $b(t)=c_1(R)h\mathrm{e}^{\bar{\lambda}h}\mathrm{e}^{-\bar{\lambda}t}$, $c(t)=c_2(R)h$.

Theorem 13 (Global attractor) Suppose that (2) - (6) and (13) hold true and $g\in L^2(\Omega)$. Then the dynamical system $(S(t),Y)$ generated by problem (1) has a compact global attractor with finite fractal dimension.

Proof By Theorem 11, $(S(t),Y)$ is dissipative. So we only need to prove $(S(t),Y)$ is asymptotically smooth. Firstly, by Lemma 12 and (34), quasi-stability inequality (8) holds true, that is $(S(t),Y)$ is quasi-stable on any bounded positively invariant sets. According to Proposition 2, then $(S(t),Y)$ is asymptotically smooth. It follows from Theorem 3 that dynamical system $(S(t),Y)$ has a compact global attractor.

Secondly, we introduce the auxiliary space

$$Y(-h,T)=C([-h,T];V_2)\cap C^1([-h,T];H),T>0,$$

its norm

$$\|\psi\|_{Y(-h,T)} = \max_{s\in[-h,T]}\left\|A^{\frac{1}{2}}\psi(s)\right\| + \max_{s\in[-h,T]}\left\|\partial_t\psi(s)\right\|.$$

Notice that when $T=0$, $Y(-h,0)=Y$, so the space $Y(-h,T)$ is an extended space *Y*.

Let $B\subset Y$, denote B_T the set of functions $u\in Y(-h,T)$ which solve (1) with initial data $u_{t\in[-h,0]}=\psi\in B$. We also define the mapping $\mathcal{R}_T: B_T\mapsto Y(-h,T)$ by the formula

$$(\mathcal{R}_T u)(t)=u(T+t), t\in[-h,T], \tag{35}$$

where u is the solution of (1) with initial data from *B*. Then the following inequality holds true

$$\begin{aligned}&\|\mathcal{R}_T\psi_1-\mathcal{R}_T\psi_2\|_{Y(-h,T)}\\&\le c_1(R)\mathrm{e}^{-\tilde{\lambda}(T-h)}\|\psi_1-\psi_2\|_{Y(-h,T)}+c_2(R)\left[n(\psi_1-\psi_2)+n(\mathcal{R}_T\psi_1-\mathcal{R}_T\psi_2)\right],\end{aligned} \tag{36}$$

where $\psi_1,\psi_2\in B_T$, $n(\psi)=\sup_{r\in[0,T]}\left\|A^{\frac{1}{2}-\delta}\psi(r)\right\|$ is a compact seminorm on the space $Y(-h,T)$. The proof of inequality (36) can see reference [9]. We take $B=\mathcal{A}$ and choose $T>h$ such that $\xi_T=c_1(R)\mathrm{e}^{-\tilde{\lambda}(T-h)}<1$, where $\mathcal{A}$ is the global attractor. One can see that set $\mathcal{A}_T$ is strictly invariant. So we can get the finite dimensionality of the set $\mathcal{A}_T$ in $Y(-h,T)$ ([16], Theorem 2.15). Finally, we consider the restriction mapping

$$r_h:\{u(t),t\in[-h,T]\}\mapsto\{u(t),t\in[-h,0]\},$$

it is clear that mapping r_h is Lispschitz continuous. Since $r_h\mathcal{A}_T=\mathcal{A}$ and Lispschitz do not increase fractal dimension of a set, we can deduce that

$$\dim_f^Y\mathcal{A}\le\dim_f^{Y(-h,T)}\mathcal{A}_T<\infty.$$

□

5. Exponential Attractor

Theorem 14 (Exponential attractor) Let the assumptions of Theorem 13 be in force. Then the dynamical system $(S(t),Y)$ possesses a generalized fractal exponential attractor $\mathcal{A}_{exp}$ whose dimension is finite in extended space $\bar{Y}$

$$\bar{Y}\equiv C\left([-h,T];D\left(A^{\frac{1}{2}-\delta}\right)\right)\cap C^1([-h,T];H_{-\delta}),\ \forall\delta>0,$$

where $H_{-s}, s>0$, denotes the closure of H with respect to the norm $\|A^{-s}\cdot\|$.

Proof Let B be a forward invariant bounded absorbing set for $(S(t),Y)$. Then according to (36), we can obtain quasi-stability property for the mapping $\mathcal{R}_T$ defined in (35) on B_T. Choosing $T>h$ in (36) such that $\xi_T=c_1(R)\mathrm{e}^{-\tilde{\lambda}(T-h)}<1$ and deduce that the mapping $\mathcal{R}_T$ possesses a fractal exponential attractor $\mathcal{A}_T$ ([16], Corollary 2.23). Subsequently, using (1) we can see that $\|\partial_{tt}u\|_{-2}<c_R$ for all $t\in\mathbb{R}$. This allows us to show that $S(t)\psi$ is a

Hölder continuous in t in the space $\bar{Y}$,

$$\left|S(t_1)\psi - S(t_2)\psi\right|_{\bar{Y}} \le C_{B,T}\left|t_1 - t_2\right|^{\gamma}, t_1, t_2 \in [0,t], y \in B, 0 < \gamma \le 1. \quad (37)$$

Now we consider the restriction mapping r_h and sets $r_h\mathcal{A}_T = \mathcal{A} \subset Y$, $\mathcal{A}_{exp} \equiv \bigcup\left\{S(t)\mathcal{A} : t \in [0,T]\right\} \subset Y$. On can see that $\mathcal{A}_{exp}$ is forward invariant. Since $\mathcal{A}$ is finite dimensionality, r_h is Lipschitz from $Y(-h,T)$ into Y. So the property in (37) implies that $\mathcal{A}_{exp}$ has a finite fractal dimension in $\bar{Y}$ and $\mathcal{A}_{exp}$ is an exponential attracting set for $(S(t),Y)$. □

6. Conclusion and Suggestions

In the last several decades, many engineers, physicists and mathematicians intensely focused on studying the collapse of the Tacoma narrow bridge. They tried their best to explain such an amazing event. Lazer and McKenna [17] suggested that a one-dimensional simply supported beam suspended by hangers was modelled as a suspension bridge, which described the vibration of the roadbed in the vertical direction, and the long-time behavior of this suspension bridge model without delay effects were studied by many authors. But we considered the long-time behavior of suspension bridge equation model with state-dependent delay in the paper, compared with constant delay and time-varying delay, differential equations with state-dependent delay are more complex, but they are closer to simulating the real phenomena. However, the theoretical methods of differential equations with state-dependent delay are not as rich as those of other types of delay differential equations, so there are relatively few studies on PDE with state-dependent delay, and they mainly investigated the long-time behavior of the solution of parabolic equations with state-dependent delay. Under suitable assumptions, we consider the long-time behavior of the system by establishing quasi-stability inequality, and obtain the existence of global attractor, exponential attractor, and also discuss the fractal dimension of the attractor in this paper. Therefore, our work can provide theoretical support for the numerical calculation and simulation of suspension bridge equations, viscoelastic beam equations and nonlinear hyperbolic equations in engineering and mathematical physics, and ensure that the numerical calculation and simulation of the problems studied can be carried out smoothly. However, when proving the existence of the global attractor, compared with the contractive function method used in [10], the damping coefficient (22) needs to be large enough. On the other hand, the existence of attractors is proved in weak topological spaces, but the regularity or asymptotic structure of attractors in strong topological Spaces needs further consideration. Finally, in this paper, we only consider one of the factors affecting the suspension bridge—time delay factor, and then we should consider other factors affecting the stability of the suspension bridge subsequently, such as random factors.

Acknowledgements

We thank the referees for their insightful comments and suggestions all of which improved the presentation. This work was partly supported by the NSFC (11961059, 12161071). Natural Science Foundation of Gansu Province (22JR11RM165, 23JRRM730). Doctor research funding of Longdong University (XYBYZK2112, XYBYZK2113).

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Caraballo, T., Kloeden, P.E. and Real, J. (2004) Pullback and Forward Attractors for a Damped Wave Equation with Delays. *Stochastics and Dynamics*, **4**, 405-423. https://doi.org/10.1142/S0219493704001139

[2] Marin-Rubio, P. and Real, J. (2010) Pullback Attractors for 2D-Navier-Stokes Equation with Delay in Continuous and Sub-linear Operators. *Discrete and Continuous Dynamical Systems*, **26**, 989-1006. https://doi.org/10.3934/dcds.2010.26.989

[3] Wang, Y.J. (2015) Pullback Attractors for a Damped Wave Equation with Delays. *Stochastics and Dynamics*, **15**, Article ID: 1550003. https://doi.org/10.1142/S0219493715500033

[4] Wang, S.P. and Ma, Q.Z. (2019) Uniform Attractors for the Non-autonomous Suspension Bridge Equation with Time Delay. *Journal of Inequalities and Applications*, **180**, 1-17. https://doi.org/10.1186/s13660-019-2133-4

[5] Aouadi, M. (2020) Global and exponential Attractors for Extensible Thermoelastic Plate with Timevarying Delay. *Journal of Differential Equations*, **5**, 4079-4115. https://doi.org/10.1016/j.jde.2020.03.026

[6] Wang, S.P. and Ma, Q.Z. (2020) Existence of Pullback Attractors for the Non-autonomous Suspension Bridge Equation with Time Delay. *Discrete and Continuous Dynamical Systems, Series B*, **4**, 1299-1316. https://doi.org/10.3934/dcdsb.2019221

[7] Rezounenko, A.V. (2009) Differential Equations with Discrete State-dependent Delay: Uniqueness and Well-posedness in the Space of Continuous Functions. *Nonlinear Analysis, Theory, Methods and Applications*, **70**, 3978-3986. https://doi.org/10.1016/j.na.2008.08.006

[8] Chueshov, I. and Rezounenko, A. (2015) Finite-dimensional Global Attractors for Parabolic Nonlinear Equations with State-dependent Delay. *Communications on Pure and Applied Analysis*, **14**, 1685-1704. https://doi.org/10.3934/cpaa.2015.14.1685

[9] Chueshov, I. and Rezounenko, A. (2015) Dynamics of Second Order in Time Evolution Equations with State-dependent Delay. *Nonlinear Analysis, Theory, Methods and Applications*, **123-124**, 126-149. https://doi.org/10.1016/j.na.2015.04.013

[10] Wang, S.P., Ma, Q.Z. and Shao, X.K. (2022) Dynamics of Suspension Bridge Equation with Delay. *Journal of Dynamics and Differential Equations*. https://doi.org/10.1007/s10884-022-10130-9

[11] Hartung, F., Krisztin, T., Walther H.-O. and Wu, J.H. (2006) Chapter 5 Functional Differential Equations with State-dependent Delays: Theory and Applications. *Handbook of Differential Equations: Ordinary Differential Equations*, **3**, 435-545. https://doi.org/10.1016/S1874-5725(06)80009-X

[12] Chueshov, I. and Lasiecka, I. (2010) Von Karman Evolution Equations: Well-Posedness

and Long Time Dynamics (Springer Monographs in Mathematics). Springer, New York, https://doi.org/10.1007/978-0-387-87712-9

[13] Walther, H.-O. (2003) The Solution Manifold and C 1-Smoothness for Differential Equations with State-Dependent Delay. *Journal of Differential Equations*, **1**, 46-65. https://doi.org/10.1016/j.jde.2003.07.001

[14] Chueshov, I. (2015) Dynamics of Quasi-Stable Dissipative Systems. Springer, New York. https://doi.org/10.1007/978-3-319-22903-4

[15] Chueshov, I.D. (2009) Introduction to the Theory of Infinite-Dimensional Dissipative Systems. Biswas Hope Press, Kharkov.

[16] Chueshov, I. and Lasiecka, I. (2008) Long-time Behavior of Second Order Evolution Equatons with Nonlinear Damping. *Memoirs of the American Mathematical Society*, **912**, 1-183.

[17] Lazer, A.C. and McKenna, P.J. (1990) Large-amplitude Periodic Oscillations in Suspension Bridge: Some New Connections with Nonlinear Analysis. *SIAM Review*, **4**, 537-578. https://doi.org/10.1137/1032120