Scientific Research Publishing

Engineering, 2023, 15, 114-133
https://www.scirp.org/journal/eng
ISSN Online: 1947-394X
ISSN Print: 1947-3931

Virtual Element Discretization of Optimal Control Problem Governed by Brinkman Equations

Yanwei Li

School of Mathematics and Statistics, Shandong Normal University, Jinan, China
Email: liyanwei_sdnu@163.com

How to cite this paper: Li, Y.W. (2023) Virtual Element Discretization of Optimal Control Problem Governed by Brinkman Equations. *Engineering*, **15**, 114-133.
https://doi.org/10.4236/eng.2023.152010

Received: January 27, 2023
Accepted: February 21, 2023
Published: February 24, 2023

Abstract

In this paper, we discuss virtual element method (VEM) approximation of optimal control problem governed by Brinkman equations with control constraints. Based on the polynomial projections and variational discretization of the control variable, we build up the virtual element discrete scheme of the optimal control problem and derive the discrete first order optimality system. A priori error estimates for the state, adjoint state and control variables in L^2 and H^1 norm are derived. The theoretical findings are illustrated by the numerical experiments.

Keywords

Virtual Element Method, Optimal Control Problem, Brinkman Equations, A Priori Error Estimate

1. Introduction

In this paper we consider virtual element discretization of the following optimal control problem: find $(\boldsymbol{y}, p, \boldsymbol{u}) \in \boldsymbol{V} \times Q \times \boldsymbol{U}_{ad}$ satisfying

$$\min_{\boldsymbol{u} \in \boldsymbol{U}_{ad}} J(\boldsymbol{y}, \boldsymbol{u}) = \frac{1}{2}\|\boldsymbol{y} - \boldsymbol{y}_d\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\gamma}{2}\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} \tag{1.1}$$

subject to

$$\begin{cases} -\Delta \boldsymbol{y} + \nabla p + \mathbb{K}^{-1}\boldsymbol{y} = \boldsymbol{f} + \boldsymbol{u} & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{y} = 0 & \text{in } \Omega, \\ \boldsymbol{y} = 0 & \text{on } \Gamma, \end{cases} \tag{1.2}$$

where $J(\boldsymbol{y}, \boldsymbol{u})$ is the objective functional, $\boldsymbol{y}_d$ is the desired state, $\gamma > 0$ is the regularization parameter, and Ω is a bounded domain in $\mathbb{R}^2$ with the boundary Γ. We suppose that the $\mathbb{K}$ is a uniformly symmetric positive definite tensor, *i.e.* there exist two positive constants $\lambda_1, \lambda_2 > 0$ such that

$$\lambda_1 \eta^{\mathrm{T}} \eta \le \eta^{\mathrm{T}} \mathbb{K}^{-1} \eta \le \lambda_2 \eta^{\mathrm{T}} \eta.$$

The admissible control set $\boldsymbol{U}_{ad}$ is defined by

$$\boldsymbol{U}_{ad} = \left\{ \boldsymbol{u} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{u}_a \le \boldsymbol{u}(\boldsymbol{x}) \le \boldsymbol{u}_b \text{ a.e. in } \Omega \right\}.$$

The quantities $\boldsymbol{u}_a, \boldsymbol{u}_b \in \mathbb{R}^2$ are constant vectors and the inequality $\boldsymbol{u}_a \le \boldsymbol{u}(\boldsymbol{x}) \le \boldsymbol{u}_b$ is understood componentwise.

Flow control problems have important applications in aerospace, chemical engineering and so on. The Brinkman equations can be viewed as a parameter-dependent combination of the Darcy and the Stokes equations [1]. In the past decades, developing numerical methods for optimal control model governed by Brinkman equations has become a hot topic. For example, a discontinuous finite volume method for the approximation of distributed optimal control problems governed by the Brinkman equations was derived in [2]. In [3] the author investigated adaptive hybridizable discontinuous Galerkin methods for the gradient-velocity-pressure formulation of Brinkman equations and extended to solve the Brinkman optimal control problem. In [4] the author studied an optimal control problem constrained by the unsteady Stokes-Brinkman equation involving random data. For more models, we can refer to [5] [6].

The virtual element method (VEM), first introduced in [7], is regarded as an extension of finite element method. Unlike finite element method, the VEM has the advantages including: it can deal with highly general polygonal/polyhedral meshes; the basis function needn't to be explicit expression, etc. VEM has been widely applied to approximate various PDEs [8] [9] [10] [11] [12]. There are many crucial literatures about the VEM framework for Brinkman problems. A mixed virtual element method for the Brinkamn equations was discussed in [13]. In [14], the divergence free virtual element space in [11] was extended to solve the Brinkman equations. In [15], the authors presented two stable virtual element methods for the Brinkman equations.

For the literature on the application of virtual element method to optimal control problem, we can refer to [16] and [17]. The authors study the virtual element discrete scheme of the elliptic optimal control problem and give a priori and a posteriori error analysis. There is still a gap in combination of the virtual element method and optimal control problem governed by Brinkman equations. Thus, in this paper, we aim to apply the VEM to approximate optimal control problem governed by Brinkman equations with pointwise control constraint. By making use of the virtual element projection operators the virtual element discrete scheme of the optimal control problem is developed, where the piecewise L^2 projection of the discrete state is used in the cost functional to guarantee the computability of the discrete adjoint state equation. Then, we derive a priori error estimates for state, adjoint state and control variables in L^2 and H^1 norm.

Finally numerical experiments on three polygonal meshes are given to verify the theoretical findings.

The structure of this paper is as follows. In Section 2, we give the continuous first order optimality condition of problem (1.1)-(1.2). Then, some basic concepts about VEM are introduced. In Section 3, we derive the virtual element discrete scheme for (1.1)-(1.2) and the discrete first order optimality condition. In Section 4, a priori error estimates of the state, adjoint state and control variables are proved. In Section 5, we show numerical results to verify the theoretical results.

Throughout this paper, for an open bounded domain *K*, we will denote scale and vector Sobolev space by $H^s(K)$ and $\boldsymbol{H}^s(K)$ equipped with seminorm $|\cdot|_{s,K}$ and norm $\|\cdot\|_{s,K}$, while $(\cdot,\cdot)_{0,K}$ will denote the $L^2(K)$ or $\boldsymbol{L}^2(K)$ inner product for scale and vector.

2. Preliminaries

In this section, we firstly recall the continuous first order optimality condition for problem (1.1)-(1.2). Then we introduce the definitions of virtual element space and two projection operators.

We consider the spaces:

$$\boldsymbol{V} := \boldsymbol{H}_0^1(\Omega), \quad Q := L_0^2(\Omega) = \left\{ q \in L^2(\Omega) \text{ s.t.} \int_\Omega q \mathrm{d}\Omega = 0 \right\}.$$

We endow the space $\boldsymbol{V}$ with the norm

$$|||\boldsymbol{v}|||_V := \|\nabla \boldsymbol{v}\|_0^2 + \left\|\mathbb{K}^{-1/2}\boldsymbol{v}\right\|_0^2$$

and the space Q with L^2-norm.

Then the weak formulation of the optimal control problem (1.1)-(1.2) is given by seeking $(\boldsymbol{y}, p, \boldsymbol{u}) \in \boldsymbol{V} \times Q \times \boldsymbol{U}_{ad}$ satisfying

$$\min_{\boldsymbol{u}\in \boldsymbol{U}_{ad}} J(\boldsymbol{y},\boldsymbol{u})$$

s.t.

$$\begin{cases} A(\boldsymbol{y},\boldsymbol{v}) - b(\boldsymbol{v}, p) = (\boldsymbol{f} + \boldsymbol{u}, \boldsymbol{v}) & \forall \boldsymbol{v} \in \boldsymbol{V}, \\ b(\boldsymbol{y}, q) = 0 & \forall q \in Q, \end{cases}$$

where

$$A(\boldsymbol{y},\boldsymbol{v}) = a(\boldsymbol{y},\boldsymbol{v}) + d(\boldsymbol{y},\boldsymbol{v}), \quad a(\boldsymbol{y},\boldsymbol{v}) = \int_\Omega \nabla \boldsymbol{y} : \nabla \boldsymbol{v} \mathrm{d}\Omega,$$

$$d(\boldsymbol{y},\boldsymbol{v}) = \int_\Omega \mathbb{K}^{-1}\boldsymbol{y}\cdot\boldsymbol{v}\mathrm{d}\Omega, \quad b(\boldsymbol{v}, p) = \int_\Omega p \operatorname{div}\boldsymbol{v}\mathrm{d}\Omega.$$

Additionally, we introduce the kernel:

$$\boldsymbol{Z} := \left\{ \boldsymbol{v} \in \boldsymbol{V} \text{ s.t.}\, b(\boldsymbol{v}, q) = 0 \ \forall q \in Q \right\}.$$

Following [14], we can obtain that:

- $A(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are continuous, *i.e.*

$$|A(\boldsymbol{u},\boldsymbol{v})| \le C |||\boldsymbol{u}|||_V \, |||\boldsymbol{v}|||_V \qquad \forall \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{V},$$

$$|b(\boldsymbol{v},q)| \le C |||\boldsymbol{v}|||_V \|q\|_Q \qquad \forall \boldsymbol{v} \in \boldsymbol{V}, q \in Q.$$

- $A(\cdot,\cdot)$ is coercive on the kernel $\boldsymbol{Z}$, *i.e.*

$$A(\boldsymbol{v},\boldsymbol{v}) \ge |||\boldsymbol{v}|||_V^2 \quad \forall \boldsymbol{v} \in \boldsymbol{Z}.$$

- $b(\cdot,\cdot)$ satisfies the inf-sup condition, *i.e.*

$$\exists \beta > 0, \text{ such that } \max_{\boldsymbol{v}\in\boldsymbol{V},\boldsymbol{v}\neq 0} \frac{b(\boldsymbol{v},p)}{|||\boldsymbol{v}|||_V} \ge \beta \|p\|_Q \quad \forall p \in Q.$$

We introduce the following Lagrangian functional:

$$\mathcal{L}(\boldsymbol{y},p,\boldsymbol{u},\boldsymbol{z},\phi) := J(\boldsymbol{y},\boldsymbol{u}) - A(\boldsymbol{y},\boldsymbol{z}) + b(\boldsymbol{z},p) - b(\boldsymbol{y},\phi) + (\boldsymbol{f}+\boldsymbol{u},\boldsymbol{z}).$$

Then the following continue first order optimality condition can be obtained by computing the derivatives of $\mathcal{L}(\cdot,\cdot,\cdot,\cdot,\cdot)$ with respect to $(\boldsymbol{y},p,\boldsymbol{u},\boldsymbol{z},\phi)$:

$$\begin{cases} A(\boldsymbol{y},\boldsymbol{w}) - b(\boldsymbol{w},p) = (\boldsymbol{f}+\boldsymbol{u},\boldsymbol{w}) & \forall \boldsymbol{w} \in \boldsymbol{V}, \\ b(\boldsymbol{y},\psi) = 0 & \forall \psi \in Q, \end{cases} \tag{2.1}$$

$$\begin{cases} A(\boldsymbol{w},\boldsymbol{z}) + b(\boldsymbol{w},\phi) = (\boldsymbol{y}-\boldsymbol{y}_d,\boldsymbol{w}) & \forall \boldsymbol{w} \in \boldsymbol{V}, \\ b(\boldsymbol{z},\psi) = 0 & \forall \psi \in Q, \end{cases} \tag{2.2}$$

$$(\gamma\boldsymbol{u}+\boldsymbol{z},\boldsymbol{v}-\boldsymbol{u}) \ge 0 \qquad \forall \boldsymbol{v} \in \boldsymbol{U}_{ad}, \tag{2.3}$$

where $(\boldsymbol{z},\phi)$ is the adjoint state variable. Following [18], the variational inequality (2.3) is equivalent to

$$\boldsymbol{u} = P_{U_{ad}}\left(-\frac{1}{\gamma}\boldsymbol{z}\right),$$

where

$$P_{U_{ad}}(\boldsymbol{u}) = \max\{\boldsymbol{u}_a, \min\{\boldsymbol{u},\boldsymbol{u}_b\}\}$$

denotes the projection onto the admissible set $\boldsymbol{U}_{ad}$.

Let $\mathcal{T}_h$ be a sequence of decompositions of Ω into general polygonal elements K with

$$h_K := \text{diameter}(K), \quad h := \max_{K\in\mathcal{T}_h} h_K.$$

Assumption 2.1. *We assume that there exists two positive constants c and* ρ *such that, every* $K \in \mathcal{T}_h$ *satisfies the following assumptions*:

(A1) Each element K is star-shaped with respect to a ball of radius $\ge ch_K$,

(A2) The distance D_h between any two points of each element K satisfies $D_h \ge \rho h_K$.

The bilinear forms $A(\cdot,\cdot)$ and $b(\cdot,\cdot)$, the norms $|||\cdot|||_V$ and $\|\cdot\|_Q$, can be decomposed into local contributions, *i.e.*:

$$A(\boldsymbol{y},\boldsymbol{v}) =: \sum_{K\in\mathcal{T}_h} A^K(\boldsymbol{y},\boldsymbol{v}) =: \sum_{K\in\mathcal{T}_h} \left(a^K(\boldsymbol{y},\boldsymbol{v}) + d^K(\boldsymbol{y},\boldsymbol{v})\right) \quad \forall \boldsymbol{y},\boldsymbol{v} \in \boldsymbol{V},$$

$$b(\boldsymbol{v},p) =: \sum_{K\in\mathcal{T}_h} b^K(\boldsymbol{v},p) \quad \forall \boldsymbol{v} \in \boldsymbol{V}, p \in Q,$$

and

$$|||\boldsymbol{v}|||_V =: \left(\sum_{K\in\mathcal{T}_h} |||\boldsymbol{v}|||_{V,K}\right)^{\frac{1}{2}} \quad \forall \boldsymbol{v}\in\boldsymbol{V}, \quad \|q\|_Q =: \left(\sum_{K\in\mathcal{T}_h} \|q\|_{Q,K}^2\right)^{\frac{1}{2}} \quad \forall q\in Q.$$

Definition 2.1. *For all* $K\in\mathcal{T}_h$, *we define the energy projection operator* $\boldsymbol{\Pi}_2^{\nabla}: \boldsymbol{H}^1(K)\to\left[\mathbb{P}_2(K)\right]^2$ *as follows*:

$$\begin{cases} a^K\left(\boldsymbol{v_h}-\boldsymbol{\Pi}_2^{\nabla}\boldsymbol{v_h},\boldsymbol{p}\right)=0, \quad \forall \boldsymbol{v_h}\in\boldsymbol{H}^1(K), \boldsymbol{p}\in\left[\mathbb{P}_2(K)\right]^2, \\ \int_K \boldsymbol{v_h}\mathrm{d}K = \int_K \boldsymbol{\Pi}_2^{\nabla}\boldsymbol{v_h}\mathrm{d}K. \end{cases}$$

It obviously holds $\boldsymbol{\Pi}_2^{\nabla}\boldsymbol{p}=\boldsymbol{p}$ for all $\boldsymbol{p}\in\left[\mathbb{P}_2(K)\right]^2$.

Definition 2.2. *For all* $K\in\mathcal{T}_h$, *we define the* L^2 *projection operator* $\boldsymbol{\Pi}_{\boldsymbol{k}}^0$: $\boldsymbol{L}^2(K)\to\left[\mathbb{P}_k(K)\right]^2$ *as follows*:

$$\left(\boldsymbol{P}_{\boldsymbol{k}}^0\boldsymbol{v_h}-\boldsymbol{v_h},\boldsymbol{p}\right)=0, \quad \forall \boldsymbol{v_h}\in\boldsymbol{L}^2(K), \boldsymbol{p}\in\left[\mathbb{P}_k(K)\right]^2.$$

For $k\in\mathbb{N}$, we define the following spaces:

- $\mathbb{P}_k(K)$: the set of polynomials on K of degree $\le k$, usually, $\mathbb{P}_{-1}=\{0\}$,
- $\mathbb{B}_k(K) := \left\{v\in C^0(\partial K) \text{ s.t } v_{|e}\in\mathbb{P}_k(e), \forall e\subset\partial K\right\}$,
- $\mathcal{G}_k(K) := \nabla\left(\mathbb{P}_{k+1}(K)\right)\subseteq\left[\mathbb{P}_k(K)\right]^2$,
- $\mathcal{G}_k(K)^{\perp}\subseteq\left[\mathbb{P}_k(K)\right]^2$ is the L^2-orthogonal complement to $\mathcal{G}_k(K)$.

In [14] the following local virtual element space was introduced

$$\boldsymbol{V}_{\boldsymbol{h}}^{\boldsymbol{k}} := \left\{\boldsymbol{v}\in\boldsymbol{U}_{\boldsymbol{h}}^{\boldsymbol{k}} \text{ s.t.} \left(\boldsymbol{v}-\boldsymbol{\Pi}_2^{\nabla}\boldsymbol{v},\boldsymbol{g}_2^{\perp}\right)=0, \forall \boldsymbol{g}_2^{\perp}\in\mathcal{G}_2(K)^{\perp}/\mathcal{G}_0(K)^{\perp}\right\},$$

where

$$\boldsymbol{U}_{\boldsymbol{h}}^{\boldsymbol{K}} := \left\{\boldsymbol{v}\in\boldsymbol{H}^1(K) \text{ s.t } \boldsymbol{v}_{|\partial K}\in\left[\mathbb{B}_2(\partial K)\right]^2, -\Delta\boldsymbol{v}+\nabla s\in\mathcal{G}_2(K)^{\perp} \right.$$
$$\left. \text{and } \operatorname{div}\boldsymbol{v}\in\mathbb{P}_1(K), \text{ for some } s\in L^2(K)\right\},$$

and $\mathcal{G}_2(K)^{\perp}/\mathcal{G}_0(K)^{\perp}$ denotes the polynomials in $\mathcal{G}_2(K)^{\perp}$ that are L^2-orthogonal to all polynomials in $\mathcal{G}_0(K)^{\perp}$.

For the pressure space we adopt the finite-dimensional space

$$Q_h^K := \mathbb{P}_1(K).$$

Then we define the global virtual element spaces:

$$\boldsymbol{V}_{\boldsymbol{h}} := \left\{\boldsymbol{v}\in\boldsymbol{V} \text{ s.t } \boldsymbol{v}|_K\in\boldsymbol{V}_{\boldsymbol{h}}^{\boldsymbol{K}}, \forall K\in\mathcal{T}_h\right\}$$

and

$$Q_h := \left\{q\in Q \text{ s.t } q|_K\in Q_h^K, \forall K\in\mathcal{T}_h\right\}.$$

We remark that the above spaces have the following relation

$$\operatorname{div}\boldsymbol{V}_{\boldsymbol{h}}\subseteq Q_h.$$

This implies an exactly divergence-free discrete velocity.

Lemma 2.3. (*See* [8]) *There exists a positive constant* C *such that, for all* $K\in\mathcal{T}_h$ *and all smooth enough functions* $\boldsymbol{\varphi}$ *defined on K, it holds*:

$$\left\|\boldsymbol{\varphi}-\boldsymbol{P}_{\boldsymbol{k}}^0\boldsymbol{\varphi}\right\|_{m,K}\le Ch_K^{s-m}|\boldsymbol{\varphi}|_{s,K}, \quad m=0,1, s\in\mathbb{N}, m\le s\le k+1.$$

3. Virtual Element Approximation

The virtual element discrete scheme of (1.2) can be defined as follows:

$$\begin{cases} A_h\left(\boldsymbol{y}_h(\boldsymbol{u}),\boldsymbol{v}_h\right)-b\left(\boldsymbol{v}_h,p_h(\boldsymbol{u})\right)=\sum_{K\in\mathcal{T}_h}\left(\boldsymbol{f}+\boldsymbol{u},\Pi_2^0\boldsymbol{v}_h\right)_{0,K} & \forall \boldsymbol{v}_h\in \boldsymbol{V}_h, \\ b\left(\boldsymbol{y}_h(\boldsymbol{u}),q_h\right)=0 & \forall q_h\in Q_h, \end{cases}$$

where

$$\begin{aligned} A_h\left(\boldsymbol{y}_h(\boldsymbol{u}),\boldsymbol{v}_h\right) &:= \sum_{K\in\mathcal{T}_h} A_h^K\left(\boldsymbol{y}_h(\boldsymbol{u}),\boldsymbol{v}_h\right) \\ &= \sum_{K\in\mathcal{T}_h}\left(a_h^K\left(\boldsymbol{y}_h(\boldsymbol{u}),\boldsymbol{v}_h\right)+d_h^K\left(\boldsymbol{y}_h(\boldsymbol{u}),\boldsymbol{v}_h\right)\right), \end{aligned}$$

$$a_h^K\left(\boldsymbol{y}_h(\boldsymbol{u}),\boldsymbol{v}_h\right):=a^K\left(\Pi_2^\nabla\boldsymbol{y}_h(\boldsymbol{u}),\Pi_2^\nabla\boldsymbol{v}_h\right)+S^K\left(\boldsymbol{y}_h(\boldsymbol{u})-\Pi_2^\nabla\boldsymbol{y}_h(\boldsymbol{u}),\boldsymbol{v}_h-\Pi_2^\nabla\boldsymbol{v}_h\right),$$

$$d_h^K\left(\boldsymbol{y}_h(\boldsymbol{u}),\boldsymbol{v}_h\right):=d^K\left(\Pi_2^0\boldsymbol{y}_h(\boldsymbol{u}),\Pi_2^0\boldsymbol{v}_h\right)+R^K\left(\boldsymbol{y}_h(\boldsymbol{u})-\Pi_2^0\boldsymbol{y}_h(\boldsymbol{u}),\boldsymbol{v}_h-\Pi_2^0\boldsymbol{v}_h\right).$$

Here, $R^K(\cdot,\cdot)$ and $S^K(\cdot,\cdot)$ are symmetric stabilizing bilinear forms satisfying

$$c_1a^K\left(\boldsymbol{v}_h,\boldsymbol{v}_h\right)\le S^K\left(\boldsymbol{v}_h,\boldsymbol{v}_h\right)\le c_2a^K\left(\boldsymbol{v}_h,\boldsymbol{v}_h\right), \forall \boldsymbol{v}_h\in\boldsymbol{V}_h \text{ with } \Pi_2^\nabla\boldsymbol{v}_h=0,$$

$$c_3d^K\left(\boldsymbol{v}_h,\boldsymbol{v}_h\right)\le R^K\left(\boldsymbol{v}_h,\boldsymbol{v}_h\right)\le c_4d^K\left(\boldsymbol{v}_h,\boldsymbol{v}_h\right), \forall \boldsymbol{v}_h\in\boldsymbol{V}_h \text{ with } \Pi_2^0\boldsymbol{v}_h=0.$$

where c_1, c_2, c_3 and c_4 are positive constants independent of h. One can refer to [14] for the example of construction of $R^K(\cdot,\cdot)$ and $S^K(\cdot,\cdot)$. Moreover, the bilinear form $A_h^K(\cdot,\cdot)$ satisfies:

- *Consistency*:

$$A_h^K\left(\boldsymbol{p},\boldsymbol{v}_h\right)=A^K\left(\boldsymbol{p},\boldsymbol{v}_h\right), \quad \forall \boldsymbol{p}\in\left[\mathbb{P}_2(K)\right]^2, \boldsymbol{v}_h\in\boldsymbol{V}_h^K.$$

- *Stability*:

$$\alpha_*A^K\left(\boldsymbol{v}_h,\boldsymbol{v}_h\right)\le A_h^K\left(\boldsymbol{v}_h,\boldsymbol{v}_h\right)\le\alpha^*A^K\left(\boldsymbol{v}_h,\boldsymbol{v}_h\right), \quad \forall \boldsymbol{v}_h\in\boldsymbol{V}_h^K.$$

Here, α_* and α^* are two positive constants independent of h. By the stability of $A_h^K(\cdot,\cdot)$ and the coercive of $A(\cdot,\cdot)$, we obtain that the bilinear form $A_h(\cdot,\cdot)$ is coercive, *i.e.*:

$$A_h\left(\boldsymbol{v}_h,\boldsymbol{v}_h\right)\ge C\left|\!\left|\!\left|\boldsymbol{v}_h\right|\!\right|\!\right|_V^2 \quad \forall \boldsymbol{v}_h\in\boldsymbol{V}_h. \tag{3.1}$$

Next, the bilinear $b(\cdot,\cdot)$ satisfies the inf-sup condition [14].

Lemma 3.1. *Given the discrete spaces* $\boldsymbol{V}_h$ *and* Q_h*, there exists a positive constant* $\tilde{\beta}$ *independent of h with*

$$\sup_{\boldsymbol{v}_h\in\boldsymbol{V}_h,\boldsymbol{v}_h\ne0}\frac{b\left(\boldsymbol{v}_h,q_h\right)}{\left|\!\left|\!\left|\boldsymbol{v}_h\right|\!\right|\!\right|_V}\ge\tilde{\beta}\left\|q_h\right\|_Q \quad \forall q_h\in Q_h.$$

Then the virtual element approximation of optimal control problem (1.1)-(1.2) is to find $\left(\boldsymbol{y}_h,p_h,\boldsymbol{u}_h\right)\in\boldsymbol{V}_h\times Q_h\times\boldsymbol{U}_{ad}$ such that

$$\min_{\boldsymbol{u}_h\in\boldsymbol{U}_{ad}} J\left(\boldsymbol{y}_h,\boldsymbol{u}_h\right):=\frac{1}{2}\sum_{K\in\mathcal{T}_h}\int_K\left(\Pi_2^0\boldsymbol{y}_h-\boldsymbol{y}_d\right)^2\mathrm{d}K+\frac{\gamma}{2}\int_\Omega\boldsymbol{u}_h^2\mathrm{d}\Omega$$

subject to

$$\begin{cases} A_h\left(\boldsymbol{y_h},\boldsymbol{v_h}\right)-b\left(\boldsymbol{v_h},p_h\right)=\sum_{K\in\mathcal{T}_h}\left(\boldsymbol{f}+\boldsymbol{u_h},\Pi_2^0\boldsymbol{v_h}\right)_{0,K} & \forall \boldsymbol{v_h}\in \boldsymbol{V_h},\\ b\left(\boldsymbol{y_h},q_h\right)=0 & \forall q_h\in Q_h. \end{cases} \tag{3.2}$$

Here the control variable is implicitly discretized (see [19]), and the minimization problem is defined on infinite dimensional set $\boldsymbol{U_{ad}}$, instead of virtual element space. In order to balance the convergence rates of state and control variables, in the discrete state equation we adopt the L^2 projection Π_2^0.

We introduce the following Lagrangian functional:

$$\begin{aligned}\mathcal{L}\left(\boldsymbol{y_h},p_h,\boldsymbol{u_h},\boldsymbol{z_h},\phi_h\right):=J\left(\boldsymbol{y_h},\boldsymbol{u_h}\right)-A_h\left(\boldsymbol{y_h},\boldsymbol{z_h}\right)+b\left(\boldsymbol{z_h},p_h\right)-b\left(\boldsymbol{y_h},\phi_h\right)\\ +\sum_{K\in\mathcal{T}_h}\left(\boldsymbol{f}+\boldsymbol{u_h},\Pi_2^0\boldsymbol{z_h}\right)_{0,K}.\end{aligned}$$

Then the following discrete first order optimality condition can be obtained by computing the derivatives of $\mathcal{L}(\cdot,\cdot,\cdot,\cdot,\cdot)$ with respect to $\left(\boldsymbol{y_h},p_h,\boldsymbol{u_h},\boldsymbol{z_h},\phi_h\right)$:

$$\begin{cases} A_h\left(\boldsymbol{y_h},\boldsymbol{w_h}\right)-b\left(\boldsymbol{w_h},p_h\right)=\sum_{K\in\mathcal{T}_h}\left(\boldsymbol{f}+\boldsymbol{u_h},\Pi_2^0\boldsymbol{w_h}\right)_{0,K} & \forall \boldsymbol{w_h}\in \boldsymbol{V_h},\\ b\left(\boldsymbol{y_h},\psi_h\right)=0 & \forall \psi_h\in Q_h, \end{cases} \tag{3.3}$$

$$\begin{cases} A_h\left(\boldsymbol{w_h},\boldsymbol{z_h}\right)+b\left(\boldsymbol{w_h},\phi_h\right)=\sum_{K\in\mathcal{T}_h}\left(\boldsymbol{y_h}-\boldsymbol{y_d},\Pi_2^0\boldsymbol{w_h}\right)_{0,K} & \forall \boldsymbol{w_h}\in \boldsymbol{V_h},\\ b\left(\boldsymbol{z_h},\psi_h\right)=0 & \forall \psi_h\in Q_h, \end{cases} \tag{3.4}$$

$$\sum_{K\in\mathcal{T}_h}\left(\gamma\boldsymbol{u_h}+\Pi_2^0\boldsymbol{z_h},\boldsymbol{v_h}-\boldsymbol{u_h}\right)_{0,K}\geq 0 \qquad \forall \boldsymbol{v_h}\in \boldsymbol{U_{ad}}. \tag{3.5}$$

4. A Priori Error Estimates

Lemma 4.1. (*See* [20]) *For the state equation, there exists a positive constant C, the state variables admit the following estimates*

$$\|\boldsymbol{y}\|_2+\|p\|_1\leq C\|\boldsymbol{f}\|_0.$$

To achieve a priori error estimates, we introduce some auxiliary problems: $\forall\left(\boldsymbol{w_h},\psi_h\right)\in \boldsymbol{V_h}\times Q_h$,

$$\begin{cases} A_h\left(\boldsymbol{y_h}(\boldsymbol{u}),\boldsymbol{w_h}\right)-b\left(\boldsymbol{w_h},p_h(\boldsymbol{u})\right)=\sum_{K\in\mathcal{T}_h}\left(\boldsymbol{f}+\boldsymbol{u},\Pi_2^0\boldsymbol{w_h}\right)_{0,K},\\ b\left(\boldsymbol{y_h}(\boldsymbol{u}),\psi_h\right)=0, \end{cases} \tag{4.1}$$

$$\begin{cases} A_h\left(\boldsymbol{w_h},\boldsymbol{z_h}(\boldsymbol{y})\right)+b\left(\boldsymbol{w_h},\phi_h(\boldsymbol{y})\right)=\sum_{K\in\mathcal{T}_h}\left(\boldsymbol{y}-\boldsymbol{y_d},\Pi_2^0\boldsymbol{w_h}\right)_{0,K},\\ b\left(\boldsymbol{z_h}(\boldsymbol{y}),\psi_h\right)=0, \end{cases} \tag{4.2}$$

$$\begin{cases} A_h\left(\boldsymbol{w_h},\boldsymbol{z_h}(\boldsymbol{u})\right)+b\left(\boldsymbol{w_h},\phi_h(\boldsymbol{u})\right)=\sum_{K\in\mathcal{T}_h}\left(\boldsymbol{y}_h(\boldsymbol{u})-\boldsymbol{y_d},\Pi_2^0\boldsymbol{w_h}\right)_{0,K},\\ b\left(\boldsymbol{z_h}(\boldsymbol{u}),\psi_h\right)=0. \end{cases} \tag{4.3}$$

Additionally, we introduce the discrete kernel:

$$\boldsymbol{Z_h}:=\left\{\boldsymbol{v_h}\in \boldsymbol{V_h}\ \text{s.t.}\ b\left(\boldsymbol{v_h},q_h\right)=0\ \ \forall q_h\in Q_h\right\}.$$

Lemma 4.2. (*See* [14]) *Let* $\boldsymbol{y},\boldsymbol{z}\in \boldsymbol{V}\cap \boldsymbol{H}^{s+1}(\Omega)$ *with* $0\leq s\leq 2$. *Under the*

Assumption (2.1) *on the decomposition* $\mathcal{T}_h$, *there exist* $\boldsymbol{y}_I, \boldsymbol{z}_I \in \boldsymbol{V}_h$ *such that*

$$\|\boldsymbol{y}-\boldsymbol{y}_I\|_{0,K} + h_K |||\boldsymbol{y}-\boldsymbol{y}_I|||_{V,K} \le C h_K^{s+1} |\boldsymbol{y}|_{s+1,K},$$

$$\|\boldsymbol{z}-\boldsymbol{z}_I\|_{0,K} + h_K |||\boldsymbol{z}-\boldsymbol{z}_I|||_{V,K} \le C h_K^{s+1} |\boldsymbol{z}|_{s+1,K},$$

where C is a positive constant independent of h.

Lemma 4.3. *Let* $(\boldsymbol{y}, p)$ *and* $(\boldsymbol{y}_h(\boldsymbol{u}), p_h(\boldsymbol{u}))$ *be the solutions* (2.1) *and* (4.1), *respectively. Under the Assumption* 2.1, *we have the following estimates*

$$|||\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u})|||_V \le Ch^2\left(|\boldsymbol{f}+\boldsymbol{u}|_1 + |\boldsymbol{y}|_3\right),$$

$$\|p-p_h(\boldsymbol{u})\|_0 \le Ch^2\left(|\boldsymbol{f}+\boldsymbol{u}|_1 + |\boldsymbol{y}|_3 + |p|_2\right),$$

$$\|\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u})\|_0 \le Ch^3\left(|\boldsymbol{f}+\boldsymbol{u}|_1 + |\boldsymbol{y}|_3 + |p|_2\right).$$

Proof. Note that $(\boldsymbol{y}_h(\boldsymbol{u}), p_h(\boldsymbol{u}))$ is the virtual element approximation of $(\boldsymbol{y}, p)$. We observe that, if $\boldsymbol{y} \in \boldsymbol{V}$ is the velocity solution to Equation (2.1), then it is also the solution to the following problem: find $\boldsymbol{y} \in \boldsymbol{Z}$, such that

$$A(\boldsymbol{y}, \boldsymbol{w}) = (\boldsymbol{f}+\boldsymbol{u}, \boldsymbol{w}) \quad \forall \boldsymbol{w} \in \boldsymbol{Z}.$$

Analogously, if $\boldsymbol{y}_h(\boldsymbol{u}) \in \boldsymbol{V}_h$ is the velocity solution to Equation (4.1), then it is also the solution to problem: find $\boldsymbol{y}_h(\boldsymbol{u}) \in \boldsymbol{Z}_h$, such that

$$A_h(\boldsymbol{y}_h(\boldsymbol{u}), \boldsymbol{w}_h) = \sum_{K\in\mathcal{T}_h} \left(\boldsymbol{f}+\boldsymbol{u}, \Pi_2^0 \boldsymbol{w}_h\right)_{0,K} \quad \forall \boldsymbol{w}_h \in \boldsymbol{Z}_h.$$

Therefore, by using the same techniques of Theorem 4.6 and Theorem 4.7 in [11], we can derive the first and second estimate in this lemma. Now we just give the proof of the last one.

Let $(\boldsymbol{r}, t) \in \boldsymbol{V} \times Q$ be the solution to the dual problem

$$\begin{cases} -\Delta \boldsymbol{r} - \nabla t + \mathbb{K}^{-1}\boldsymbol{r} = \boldsymbol{y} - \boldsymbol{y}_h(\boldsymbol{u}) & \text{in } \Omega, \\ \nabla\cdot\boldsymbol{r} = 0 & \text{in } \Omega, \\ \boldsymbol{r} = 0 & \text{on } \Gamma. \end{cases} \tag{4.4}$$

From Lemma 4.1 we know that $\boldsymbol{r}$ satisfies the regularity bound

$$\|\boldsymbol{r}\|_2 \le C\|\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u})\|_0$$

and consequently for any interpolation $\boldsymbol{r}_I$ as in Lemma 4.2, we have

$$\begin{aligned} |||\boldsymbol{r}-\boldsymbol{r}_I|||_V &= \left(\sum_{K\in\mathcal{T}_h} |||\boldsymbol{r}-\boldsymbol{r}_I|||_{V,K}^2\right)^{\frac{1}{2}} \\ &\le C\left(\sum_{K\in\mathcal{T}_h} h_K^2 |\boldsymbol{r}|_{2,K}^2\right)^{\frac{1}{2}} \\ &\le Ch|\boldsymbol{r}|_2 \\ &\le Ch\|\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u})\|_0. \end{aligned}$$

Because of $\operatorname{div}\boldsymbol{V}_h \subseteq Q_h$, we obtain that the discrete velocity solution of state equation is divergence-free. Thus, we get $\nabla\cdot(\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u})) = 0$. Further, multiplying (4.4) by $\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u})$ and integrating leads to

$$
\begin{aligned}
\|\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u})\|_0^2 &= \left(\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u}),-\Delta\boldsymbol{r}-\nabla t+\mathbb{K}^{-1}\boldsymbol{r}\right)\\
&= A(\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u}),\boldsymbol{r})\\
&= A(\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u}),\boldsymbol{r}-\boldsymbol{r}_I)+A(\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u}),\boldsymbol{r}_I)\\
&= A(\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u}),\boldsymbol{r}-\boldsymbol{r}_I)+A(\boldsymbol{y},\boldsymbol{r}_I)-A(\boldsymbol{y}_h(\boldsymbol{u}),\boldsymbol{r}_I)\\
&= A(\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u}),\boldsymbol{r}-\boldsymbol{r}_I)+b(\boldsymbol{r}_I,p)-b(\boldsymbol{r}_I,p_h(\boldsymbol{u}))\\
&\quad +A_h(\boldsymbol{y}_h(\boldsymbol{u}),\boldsymbol{r}_I)-A(\boldsymbol{y}_h(\boldsymbol{u}),\boldsymbol{r}_I)\\
&\quad +(\boldsymbol{f}+\boldsymbol{u},\boldsymbol{r}_I)-\sum_{K\in\mathcal{T}_h}\left(\boldsymbol{\Pi}_2^0(\boldsymbol{f}+\boldsymbol{u}),\boldsymbol{r}_I\right)_{0,K}.
\end{aligned}
\tag{4.5}
$$

We label these as T_1, T_2, T_3 and T_f, respectively, and bound them separately. Firstly, we can bound T_1 as follows

$$
\begin{aligned}
T_1 &:= A(\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u}),\boldsymbol{r}-\boldsymbol{r}_I)\\
&\le C|||\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u})|||_V\,|||\boldsymbol{r}-\boldsymbol{r}_I|||_V\\
&\le Ch^2\left(|\boldsymbol{f}+\boldsymbol{u}|_1+|\boldsymbol{y}|_3\right)\cdot Ch\|\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u})\|_0\\
&\le Ch^3\left(|\boldsymbol{f}+\boldsymbol{u}|_1+|\boldsymbol{y}|_3\right)\|\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u})\|_0.
\end{aligned}
$$

Due to $\nabla\cdot\boldsymbol{r}=0$ the estimate of term T_2 follows

$$
\begin{aligned}
T_2 &:= b(\boldsymbol{r}_I,p)-b(\boldsymbol{r}_I,p_h(\boldsymbol{u}))\\
&= b(\boldsymbol{r}_I,p-p_h(\boldsymbol{u}))\\
&= b(\boldsymbol{r}_I-\boldsymbol{r},p-p_h(\boldsymbol{u}))\\
&\le |||\boldsymbol{r}_I-\boldsymbol{r}|||_V\,\|p-p_h(\boldsymbol{u})\|_0\\
&\le Ch\|\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u})\|_0\cdot Ch^2\left(|\boldsymbol{f}+\boldsymbol{u}|_1+|\boldsymbol{y}|_3+|p|_2\right)\\
&\le Ch^3\left(|\boldsymbol{f}+\boldsymbol{u}|_1+|\boldsymbol{y}|_3+|p|_2\right)\|\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u})\|_0.
\end{aligned}
$$

For the inconsistency term T_3 we have

$$
\begin{aligned}
T_3 &:= A_h(\boldsymbol{y}_h(\boldsymbol{u}),\boldsymbol{r}_I)-A(\boldsymbol{y}_h(\boldsymbol{u}),\boldsymbol{r}_I)\\
&= \sum_{K\in\mathcal{T}_h}\left(A_h^K(\boldsymbol{y}_h(\boldsymbol{u}),\boldsymbol{r}_I)-A^K(\boldsymbol{y}_h(\boldsymbol{u}),\boldsymbol{r}_I)\right)\\
&= \sum_{K\in\mathcal{T}_h}\left(A_h^K\left(\boldsymbol{y}_h(\boldsymbol{u})-\boldsymbol{\Pi}_2^0\boldsymbol{y},\boldsymbol{r}_I-\boldsymbol{\Pi}_1^0\boldsymbol{r}\right)-A^K\left(\boldsymbol{y}_h(\boldsymbol{u})-\boldsymbol{\Pi}_2^0\boldsymbol{y},\boldsymbol{r}_I-\boldsymbol{\Pi}_1^0\boldsymbol{r}\right)\right).
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\left(\sum_{K\in\mathcal{T}_h}|||\boldsymbol{y}_h(\boldsymbol{u})-\boldsymbol{\Pi}_2^0\boldsymbol{y}|||_{V,K}^2\right)^{\frac{1}{2}}\\
&=\left(\sum_{K\in\mathcal{T}_h}|||\boldsymbol{y}_h(\boldsymbol{u})-\boldsymbol{y}+\boldsymbol{y}-\boldsymbol{\Pi}_2^0\boldsymbol{y}|||_{V,K}^2\right)^{\frac{1}{2}}\\
&\le C\left(\sum_{K\in\mathcal{T}_h}|||\boldsymbol{y}_h(\boldsymbol{u})-\boldsymbol{y}|||_{V,K}^2+\sum_{K\in\mathcal{T}_h}|||\boldsymbol{y}-\boldsymbol{\Pi}_2^0\boldsymbol{y}|||_{V,K}^2\right)^{\frac{1}{2}}\\
&\le C\left(\sum_{K\in\mathcal{T}_h}|||\boldsymbol{y}_h(\boldsymbol{u})-\boldsymbol{y}|||_{V,K}^2\right)^{\frac{1}{2}}+C\left(\sum_{K\in\mathcal{T}_h}|||\boldsymbol{y}-\boldsymbol{\Pi}_2^0\boldsymbol{y}|||_{V,K}^2\right)^{\frac{1}{2}}
\end{aligned}
$$

$$\leq C |||\boldsymbol{y}_h(\boldsymbol{u})-\boldsymbol{y}|||_V + C\left(\sum_{K\in\mathcal{T}_h} h_K^4 |\boldsymbol{y}|_{3,K}^2\right)^{\frac{1}{2}}$$
$$\leq C |||\boldsymbol{y}_h(\boldsymbol{u})-\boldsymbol{y}|||_V + Ch^2 |\boldsymbol{y}|_3$$
$$\leq Ch^2\left(|\boldsymbol{f}+\boldsymbol{u}|_1 + |\boldsymbol{y}|_3\right)$$

and

$$\left(\sum_{K\in\mathcal{T}_h} |||\boldsymbol{r}_I - \Pi_1^0\boldsymbol{r}|||_{V,K}^2\right)^{\frac{1}{2}}$$
$$=\left(\sum_{K\in\mathcal{T}_h} |||\boldsymbol{r}_I - \boldsymbol{r} + \boldsymbol{r} - \Pi_1^0\boldsymbol{r}|||_{V,K}^2\right)^{\frac{1}{2}}$$
$$\leq C\left(\sum_{K\in\mathcal{T}_h} |||\boldsymbol{r} - \boldsymbol{r}_I|||_{V,K}^2 + \sum_{K\in\mathcal{T}_h} |||\boldsymbol{r} - \Pi_1^0\boldsymbol{r}|||_{V,K}^2\right)^{\frac{1}{2}}$$
$$\leq C\left(\sum_{K\in\mathcal{T}_h} |||\boldsymbol{r} - \boldsymbol{r}_I|||_{V,K}^2\right)^{\frac{1}{2}} + C\left(\sum_{K\in\mathcal{T}_h} |||\boldsymbol{r} - \Pi_1^0\boldsymbol{r}|||_{V,K}^2\right)^{\frac{1}{2}}$$
$$\leq C|||\boldsymbol{r}_I - \boldsymbol{r}|||_V + C\left(\sum_{K\in\mathcal{T}_h} h_K^2 |\boldsymbol{r}|_{2,K}^2\right)^{\frac{1}{2}}$$
$$\leq C|||\boldsymbol{r}_I - \boldsymbol{r}|||_V + Ch|\boldsymbol{r}|_2$$
$$\leq Ch\|\boldsymbol{y} - \boldsymbol{y}_h(\boldsymbol{u})\|_0 .$$

Then applying the Cauchy-Schwarz inequality [21], the following conclusion can be drawn

$$\sum_{K\in\mathcal{T}_h}\left(A_h^K\left(\boldsymbol{y}_h(\boldsymbol{u})-\Pi_2^0\boldsymbol{y},\boldsymbol{r}_I-\Pi_1^0\boldsymbol{r}\right)-A^K\left(\boldsymbol{y}_h(\boldsymbol{u})-\Pi_2^0\boldsymbol{y},\boldsymbol{r}_I-\Pi_1^0\boldsymbol{r}\right)\right)$$
$$\leq \sum_{K\in\mathcal{T}_h}\left(|||\boldsymbol{y}_h(\boldsymbol{u})-\Pi_2^0\boldsymbol{y}|||_{V,K}\ |||\boldsymbol{r}_I-\Pi_1^0\boldsymbol{r}|||_{V,K}\right)$$
$$\leq\left(\sum_{K\in\mathcal{T}_h}|||\boldsymbol{y}_h(\boldsymbol{u})-\Pi_2^0\boldsymbol{y}|||_{V,K}^2\right)^{\frac{1}{2}}\left(\sum_{K\in\mathcal{T}_h}|||\boldsymbol{r}_I-\Pi_1^0\boldsymbol{r}|||_{V,K}^2\right)^{\frac{1}{2}}$$
$$\leq Ch^3\left(|\boldsymbol{f}+\boldsymbol{u}|_1+|\boldsymbol{y}|_3\right)\|\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u})\|_0 .$$

Finally, the definition and estimate of the L^2 projection operator leads to the estimate of T_f

$$T_f := (\boldsymbol{f}+\boldsymbol{u},\boldsymbol{r}_I) - \sum_{K\in\mathcal{T}_h}\left(\Pi_2^0(\boldsymbol{f}+\boldsymbol{u}),\boldsymbol{r}_I\right)_{0,K}$$
$$=\sum_{K\in\mathcal{T}_h}\left(\boldsymbol{f}+\boldsymbol{u}-\Pi_2^0(\boldsymbol{f}+\boldsymbol{u}),\boldsymbol{r}_I\right)_{0,K}$$
$$=\sum_{K\in\mathcal{T}_h}\left(\boldsymbol{f}+\boldsymbol{u}-\Pi_2^0(\boldsymbol{f}+\boldsymbol{u}),\boldsymbol{r}_I-\Pi_1^0\boldsymbol{r}_I\right)_{0,K}$$
$$\leq\sum_{K\in\mathcal{T}_h}\|\boldsymbol{f}+\boldsymbol{u}-\Pi_2^0(\boldsymbol{f}+\boldsymbol{u})\|_{0,K}\|\boldsymbol{r}_I-\Pi_1^0\boldsymbol{r}_I\|_{0,K}$$
$$\leq\left(\sum_{K\in\mathcal{T}_h}\|\boldsymbol{f}+\boldsymbol{u}-\Pi_2^0(\boldsymbol{f}+\boldsymbol{u})\|_{0,K}^2\right)^{\frac{1}{2}}\left(\sum_{K\in\mathcal{T}_h}\|\boldsymbol{r}_I-\Pi_1^0\boldsymbol{r}_I\|_{0,K}^2\right)^{\frac{1}{2}}$$

$$
\begin{aligned}
&\le C\left(\sum_{K\in\mathcal{T}_h} h_K^2 \left|\boldsymbol{f}+\boldsymbol{u}\right|_{1,K}^2\right)^{\frac{1}{2}}\cdot Ch^2\left|\boldsymbol{r}\right|_2\\
&\le Ch\left|\boldsymbol{f}+\boldsymbol{u}\right|_1\cdot Ch^2\left|\boldsymbol{r}\right|_2\\
&\le Ch^3\left|\boldsymbol{f}+\boldsymbol{u}\right|_1\left\|\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u})\right\|_0.
\end{aligned}
$$

Here, the estimate of $\left(\sum_{K\in\mathcal{T}_h}\left\|\boldsymbol{r}_I-\Pi_1^0\boldsymbol{r}_I\right\|_{0,K}^2\right)^{\frac{1}{2}}$ is derived as follows:

$$
\begin{aligned}
&\left(\sum_{K\in\mathcal{T}_h}\left\|\boldsymbol{r}_I-\Pi_1^0\boldsymbol{r}_I\right\|_{0,K}^2\right)^{\frac{1}{2}}\\
&=\left(\sum_{K\in\mathcal{T}_h}\left\|\boldsymbol{r}_I-\boldsymbol{r}+\boldsymbol{r}-\Pi_1^0\boldsymbol{r}+\Pi_1^0\boldsymbol{r}-\Pi_1^0\boldsymbol{r}_I\right\|_{0,K}^2\right)^{\frac{1}{2}}\\
&\le C\left(\sum_{K\in\mathcal{T}_h}\left\|\boldsymbol{r}_I-\boldsymbol{r}\right\|_{0,K}^2+\sum_{K\in\mathcal{T}_h}\left\|\boldsymbol{r}-\Pi_1^0\boldsymbol{r}\right\|_{0,K}^2+\sum_{K\in\mathcal{T}_h}\left\|\Pi_1^0\boldsymbol{r}-\Pi_1^0\boldsymbol{r}_I\right\|_{0,K}^2\right)^{\frac{1}{2}}\\
&\le C\left(2\sum_{K\in\mathcal{T}_h}\left\|\boldsymbol{r}_I-\boldsymbol{r}\right\|_{0,K}^2+\sum_{K\in\mathcal{T}_h}\left\|\boldsymbol{r}-\Pi_1^0\boldsymbol{r}\right\|_{0,K}^2\right)^{\frac{1}{2}}\\
&\le C\left(\sum_{K\in\mathcal{T}_h}\left\|\boldsymbol{r}_I-\boldsymbol{r}\right\|_{0,K}^2\right)^{\frac{1}{2}}+C\left(\sum_{K\in\mathcal{T}_h}\left\|\boldsymbol{r}-\Pi_1^0\boldsymbol{r}\right\|_{0,K}^2\right)^{\frac{1}{2}}\\
&\le C\left(\sum_{K\in\mathcal{T}_h}h_K^4\left|\boldsymbol{r}\right|_{2,D(K)}^2\right)^{\frac{1}{2}}+C\left(\sum_{K\in\mathcal{T}_h}h_K^4\left|\boldsymbol{r}\right|_{2,K}^2\right)^{\frac{1}{2}}\\
&\le Ch^2\left|\boldsymbol{r}\right|_2.
\end{aligned}
$$

Inserting above bounds into (4.5) yields the third estimate.

For the adjoint state variables, we have the following results.

Lemma 4.4. *Let* $(\boldsymbol{z},\phi)$ *and* $(\boldsymbol{z}_h(\boldsymbol{y}),\phi_h(\boldsymbol{y}))$ *be the solutions* (2.2) *and* (4.2), *respectively. Under the Assumption* 2.1, *we have the following estimates*

$$
\begin{aligned}
|||\boldsymbol{z}-\boldsymbol{z}_h(\boldsymbol{y})|||_V &\le Ch^2\left(\left|\boldsymbol{y}-\boldsymbol{y}_d\right|_1+\left|\boldsymbol{z}\right|_3\right),\\
\left\|\phi-\phi_h(\boldsymbol{y})\right\|_0 &\le Ch^2\left(\left|\boldsymbol{y}-\boldsymbol{y}_d\right|_1+\left|\boldsymbol{z}\right|_3+\left|\phi\right|_2\right),\\
\left\|\boldsymbol{z}-\boldsymbol{z}_h(\boldsymbol{y})\right\|_0 &\le Ch^3\left(\left|\boldsymbol{y}-\boldsymbol{y}_d\right|_1+\left|\boldsymbol{z}\right|_3+\left|\phi\right|_2\right).
\end{aligned}
$$

Proof. Note that $(\boldsymbol{z}_h(\boldsymbol{y}),\phi_h(\boldsymbol{y}))$ is the virtual element approximation of $(\boldsymbol{z},\phi)$. In a similar way to state variables, the Equation (2.2) can be rewritten as: finding $\boldsymbol{z}\in\boldsymbol{Z}$, such that

$$
A(\boldsymbol{w},\boldsymbol{z})=(\boldsymbol{y}-\boldsymbol{y}_d,\boldsymbol{w})\quad\forall\boldsymbol{w}\in\boldsymbol{Z},
$$

while, the Equation (4.2) can be rewritten as: finding $\boldsymbol{z}_h(\boldsymbol{y})\in\boldsymbol{Z}_h$, such that

$$
A_h(\boldsymbol{w}_h,\boldsymbol{z}_h(\boldsymbol{y}))=\sum_{K\in\mathcal{T}_h}\left(\boldsymbol{y}-\boldsymbol{y}_d,\Pi_2^0\boldsymbol{w}_h\right)_{0,K}\quad\forall\boldsymbol{w}_h\in\boldsymbol{Z}_h.
$$

Then by using the same techniques of Theorem 4.6 and Theorem 4.7 in [11], we can derive the first and second estimate in this lemma. The last one can be derived by the similar argument to Lemma 4.3.

Theorem 4.5. (*A priori error estimate*) *Suppose that* $(\boldsymbol{y}, p, \boldsymbol{u}, \boldsymbol{z}, \phi)$ *is the solution of* (2.1)-(2.3), *and* $(\boldsymbol{y}_h, p_h, \boldsymbol{u}_h, \boldsymbol{z}_h, \phi_h)$ *is the solution of* (3.3)-(3.5). *Under the Assumption* 2.1, *we derive*

$$\|\boldsymbol{y}-\boldsymbol{y}_h\|_0 + h\,|||\boldsymbol{y}-\boldsymbol{y}_h|||_V \le Ch^3, \quad \|p-p_h\|_0 \le Ch^2,$$

$$\|\boldsymbol{z}-\boldsymbol{z}_h\|_0 + h\,|||\boldsymbol{z}-\boldsymbol{z}_h|||_V \le Ch^3, \quad \|\phi-\phi_h\|_0 \le Ch^2,$$

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_0 \le Ch^3,$$

where C is a positive constant independent of h.

Proof. We decompose the errors $\boldsymbol{y}-\boldsymbol{y}_h, p-p_h, \boldsymbol{z}-\boldsymbol{z}_h$ and $\phi-\phi_h$ into

$$\boldsymbol{y}-\boldsymbol{y}_h = \boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u}) + \boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h, \quad p-p_h = p-p_h(\boldsymbol{u}) + p_h(\boldsymbol{u}) - p_h,$$

$$\boldsymbol{z}-\boldsymbol{z}_h = \boldsymbol{z}-\boldsymbol{z}_h(\boldsymbol{y}) + \boldsymbol{z}_h(\boldsymbol{y}) - \boldsymbol{z}_h, \quad \phi-\phi_h = \phi-\phi_h(\boldsymbol{y}) + \phi_h(\boldsymbol{y}) - \phi_h.$$

Recalling Lemma 4.3, we know

$$|||\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u})|||_0 \le Ch^3, \; |||\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u})|||_V \le Ch^2, \|p-p_h(\boldsymbol{u})\|_0 \le Ch^2.$$

Moreover, by the governing equations of $\boldsymbol{y}_h(\boldsymbol{u})$ and $\boldsymbol{y}_h$ we have: $\forall(\boldsymbol{w}_h, \psi_h) \in \boldsymbol{V}_h \times Q_h$,

$$\begin{cases} A_h(\boldsymbol{y}_h(\boldsymbol{u})-\boldsymbol{y}_h, \boldsymbol{w}_h) - b(\boldsymbol{w}_h, p_h(\boldsymbol{u})-p_h) = \sum_{K\in\mathcal{T}_h} (\boldsymbol{u}-\boldsymbol{u}_h, \Pi_2^0 \boldsymbol{w}_h)_{0,K}, \\ b(\boldsymbol{y}_h(\boldsymbol{u})-\boldsymbol{y}_h, \psi_h) = 0. \end{cases}$$

Setting $\boldsymbol{w}_h = \boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h$ and $\psi_h = p_h(\boldsymbol{u}) - p_h$ gives

$$A_h(\boldsymbol{y}_h(\boldsymbol{u})-\boldsymbol{y}_h, \boldsymbol{y}_h(\boldsymbol{u})-\boldsymbol{y}_h) = \sum_{K\in\mathcal{T}_h} (\boldsymbol{u}-\boldsymbol{u}_h, \Pi_2^0(\boldsymbol{y}_h(\boldsymbol{u})-\boldsymbol{y}_h))_{0,K}.$$

It follows from (3.1) that

$$\begin{aligned} C\,|||\boldsymbol{y}_h(\boldsymbol{u})-\boldsymbol{y}_h|||_V^2 &\le A_h(\boldsymbol{y}_h(\boldsymbol{u})-\boldsymbol{y}_h, \boldsymbol{y}_h(\boldsymbol{u})-\boldsymbol{y}_h) \\ &= \sum_{K\in\mathcal{T}_h} (\boldsymbol{u}-\boldsymbol{u}_h, \Pi_2^0(\boldsymbol{y}_h(\boldsymbol{u})-\boldsymbol{y}_h))_{0,K} \\ &\le \sum_{K\in\mathcal{T}_h} \|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,K} \|\boldsymbol{y}_h(\boldsymbol{u})-\boldsymbol{y}_h\|_{0,K} \\ &\le \|\boldsymbol{u}-\boldsymbol{u}_h\|_0 \,|||\boldsymbol{y}_h(\boldsymbol{u})-\boldsymbol{y}_h|||_V. \end{aligned}$$

We can deduce

$$|||\boldsymbol{y}_h(\boldsymbol{u})-\boldsymbol{y}_h|||_V \le C\|\boldsymbol{u}-\boldsymbol{u}_h\|_0.$$

Combining above inequalities leads to

$$\|\boldsymbol{y}-\boldsymbol{y}_h\|_0 \le C\left(h^3 + \|\boldsymbol{u}-\boldsymbol{u}_h\|_0\right) \tag{4.6}$$

and

$$|||\boldsymbol{y}-\boldsymbol{y}_h|||_V \le C\left(h^2 + \|\boldsymbol{u}-\boldsymbol{u}_h\|_0\right). \tag{4.7}$$

Next we estimate $\|p_h(\boldsymbol{u})-p_h\|_0$. Based on Lemma 3.1, we get

$$\begin{aligned}\tilde{\beta}\left\|p_h(\boldsymbol{u})-p_h\right\|_0 &\le \sup_{\boldsymbol{w_h}\in V_h,\boldsymbol{w_h}\neq 0}\frac{b\left(\boldsymbol{w_h},p_h(\boldsymbol{u})-p_h\right)}{|||\boldsymbol{w_h}|||_V}\\ &= \sup_{\boldsymbol{w_h}\in V_h,\boldsymbol{w_h}\neq 0}\frac{A_h\left(\boldsymbol{y_h}(\boldsymbol{u})-\boldsymbol{y_h},\boldsymbol{w_h}\right)+\sum_{K\in\mathcal{T}_h}\left(\boldsymbol{u_h}-\boldsymbol{u},\Pi_2^0\boldsymbol{w_h}\right)_{0,K}}{|||\boldsymbol{w_h}|||_V}\\ &\le C\left(|||\boldsymbol{y_h}(\boldsymbol{u})-\boldsymbol{y_h}|||_V+\left\|\boldsymbol{u_h}-\boldsymbol{u}\right\|_0\right)\\ &\le C\left\|\boldsymbol{u}-\boldsymbol{u_h}\right\|_0.\end{aligned}$$

By the triangle inequality, it holds

$$\left\|p-p_h\right\|_0 \le C\left(h^2+\left\|\boldsymbol{u}-\boldsymbol{u_h}\right\|_0\right). \tag{4.8}$$

In a similar way, from Lemma 4.4, we have

$$\left\|\boldsymbol{z}-\boldsymbol{z_h}(\boldsymbol{y})\right\|_0 \le Ch^3,\ |||\boldsymbol{z}-\boldsymbol{z_h}(\boldsymbol{y})|||_V \le Ch^2, \left\|\phi-\phi_h(\boldsymbol{y})\right\|_0 \le Ch^2.$$

By the governing equations of $\boldsymbol{z_h}(\boldsymbol{y})$ and $\boldsymbol{z_h}$ we have: $\forall\left(\boldsymbol{w_h},\psi_h\right)\in \boldsymbol{V_h}\times Q_h$,

$$\begin{cases}A_h\left(\boldsymbol{w_h},\boldsymbol{z_h}(\boldsymbol{y})-\boldsymbol{z_h}\right)+b\left(\boldsymbol{w_h},\phi_h(\boldsymbol{y})-\phi_h\right)=\sum_{K\in\mathcal{T}_h}\left(\boldsymbol{y}-\boldsymbol{y_h},\Pi_2^0\boldsymbol{w_h}\right)_{0,K},\\ b\left(\boldsymbol{z_h}(\boldsymbol{y})-\boldsymbol{z_h},\psi_h\right)=0.\end{cases}$$

Choosing $\boldsymbol{w_h}=\boldsymbol{z_h}(\boldsymbol{y})-\boldsymbol{z_h}$ and $\psi_h=\phi_h(\boldsymbol{y})-\phi_h$, we obtain

$$A_h\left(\boldsymbol{z_h}(\boldsymbol{y})-\boldsymbol{z_h},\boldsymbol{z_h}(\boldsymbol{y})-\boldsymbol{z_h}\right)=\sum_{K\in\mathcal{T}_h}\left(\boldsymbol{y}-\boldsymbol{y_h},\Pi_2^0\left(\boldsymbol{z_h}(\boldsymbol{y})-\boldsymbol{z_h}\right)\right)_{0,K}.$$

Further, it follows that

$$\begin{aligned}C\,|||\boldsymbol{z_h}(\boldsymbol{y})-\boldsymbol{z_h}|||_V^2 &\le A_h\left(\boldsymbol{z_h}(\boldsymbol{y})-\boldsymbol{z_h},\boldsymbol{z_h}(\boldsymbol{y})-\boldsymbol{z_h}\right)\\ &=\sum_{K\in\mathcal{T}_h}\left(\boldsymbol{y}-\boldsymbol{y_h},\Pi_2^0\left(\boldsymbol{z_h}(\boldsymbol{y})-\boldsymbol{z_h}\right)\right)_{0,K}\\ &\le \sum_{K\in\mathcal{T}_h}\left\|\boldsymbol{y}-\boldsymbol{y_h}\right\|_{0,K}\left\|\boldsymbol{z_h}(\boldsymbol{y})-\boldsymbol{z_h}\right\|_{0,K}\\ &\le \left\|\boldsymbol{y}-\boldsymbol{y_h}\right\|_0 |||\boldsymbol{z_h}(\boldsymbol{y})-\boldsymbol{z_h}|||_V.\end{aligned}$$

This implies

$$|||\boldsymbol{z_h}(\boldsymbol{y})-\boldsymbol{z_h}|||_V \le C\left\|\boldsymbol{y}-\boldsymbol{y_h}\right\|_0 \le C\left(h^3+\left\|\boldsymbol{u}-\boldsymbol{u_h}\right\|_0\right).$$

Combining above inequalities gives

$$\left\|\boldsymbol{z}-\boldsymbol{z_h}\right\|_0 \le C\left(h^3+\left\|\boldsymbol{u}-\boldsymbol{u_h}\right\|_0\right) \tag{4.9}$$

and

$$|||\boldsymbol{z}-\boldsymbol{z_h}|||_V \le C\left(h^2+\left\|\boldsymbol{u}-\boldsymbol{u_h}\right\|_0\right). \tag{4.10}$$

Using Lemma 3.1, we can derive

$$\begin{aligned}\tilde{\beta}\left\|\phi_h(\boldsymbol{y})-\phi_h\right\|_0 &\le \sup_{\boldsymbol{w_h}\in V_h,\boldsymbol{w_h}\neq 0}\frac{b\left(\boldsymbol{w_h},\phi_h(\boldsymbol{y})-\phi_h\right)}{|||\boldsymbol{w_h}|||_V}\\ &= \sup_{\boldsymbol{w_h}\in V_h,\boldsymbol{w_h}\neq 0}\frac{A_h\left(\boldsymbol{w_h},\boldsymbol{z_h}-\boldsymbol{z_h}(\boldsymbol{y})\right)+\sum_{K\in\mathcal{T}_h}\left(\boldsymbol{y}-\boldsymbol{y_h},\Pi_2^0\boldsymbol{w_h}\right)_{0,K}}{|||\boldsymbol{w_h}|||_V}\\ &\le C\left(|||\boldsymbol{z_h}-\boldsymbol{z_h}(\boldsymbol{y})|||_V+\left\|\boldsymbol{y}-\boldsymbol{y_h}\right\|_0\right)\\ &\le C\left(h^3+\left\|\boldsymbol{u}-\boldsymbol{u_h}\right\|_0\right).\end{aligned}$$

By the triangle inequality, we have

$$\|\phi-\phi_h\|_0 \le C\left(h^2+\|\boldsymbol{u}-\boldsymbol{u}_h\|_0\right). \tag{4.11}$$

Since the estimates of state and adjoint state variables both depend on the estimate of control variable, now it remains to estimate $\|\boldsymbol{u}-\boldsymbol{u}_h\|_0$. Define

$$\hat{J}_h'(\boldsymbol{u})(\boldsymbol{v}-\boldsymbol{u}) := \sum_{K\in\mathcal{T}_h}\int_K\left(\gamma\boldsymbol{u}+\Pi_2^0\boldsymbol{z}_h(\boldsymbol{u})\right)(\boldsymbol{v}-\boldsymbol{u})\,\mathrm{d}K.$$

We can prove that

$$\hat{J}_h'(\boldsymbol{v})(\boldsymbol{v}-\boldsymbol{u})-\hat{J}_h'(\boldsymbol{u})(\boldsymbol{v}-\boldsymbol{u}) \ge \gamma\|\boldsymbol{v}-\boldsymbol{u}\|_0^2. \tag{4.12}$$

Note that

$$\begin{aligned}
&\hat{J}_h'(\boldsymbol{v})(\boldsymbol{v}-\boldsymbol{u})-\hat{J}_h'(\boldsymbol{u})(\boldsymbol{v}-\boldsymbol{u})\\
&=\sum_{K\in\mathcal{T}_h}\int_K\left(\gamma\boldsymbol{v}+\Pi_2^0\boldsymbol{z}_h(\boldsymbol{v})-\gamma\boldsymbol{u}-\Pi_2^0\boldsymbol{z}_h(\boldsymbol{u})\right)(\boldsymbol{v}-\boldsymbol{u})\,\mathrm{d}K\\
&=\sum_{K\in\mathcal{T}_h}\int_K\gamma(\boldsymbol{v}-\boldsymbol{u})^2\,\mathrm{d}K+\sum_{K\in\mathcal{T}_h}\int_K\left(\Pi_2^0\left(\boldsymbol{z}_h(\boldsymbol{v})-\boldsymbol{z}_h(\boldsymbol{u})\right)\right)(\boldsymbol{v}-\boldsymbol{u})\,\mathrm{d}K\\
&=\gamma\int_\Omega(\boldsymbol{v}-\boldsymbol{u})^2\,\mathrm{d}\Omega+\sum_{K\in\mathcal{T}_h}\int_K\left(\Pi_2^0\left(\boldsymbol{z}_h(\boldsymbol{v})-\boldsymbol{z}_h(\boldsymbol{u})\right)\right)(\boldsymbol{v}-\boldsymbol{u})\,\mathrm{d}K.
\end{aligned}$$

Using (4.1), we can derive: $\forall(\boldsymbol{w}_h,\psi_h)\in\boldsymbol{V}_h\times Q_h$,

$$\begin{cases}
A_h\left(\boldsymbol{y}_h(\boldsymbol{v})-\boldsymbol{y}_h(\boldsymbol{u}),\boldsymbol{w}_h\right)-b\left(\boldsymbol{w}_h,p_h(\boldsymbol{v})-p_h(\boldsymbol{u})\right)=\sum_{K\in\mathcal{T}_h}\left(\boldsymbol{v}-\boldsymbol{u},\Pi_2^0\boldsymbol{w}_h\right)_{0,K},\\
b\left(\boldsymbol{y}_h(\boldsymbol{v})-\boldsymbol{y}_h(\boldsymbol{u}),\psi_h\right)=0.
\end{cases}$$

Let $\boldsymbol{w}_h=\boldsymbol{z}_h(\boldsymbol{v})-\boldsymbol{z}_h(\boldsymbol{u})$ and $\psi_h=\phi_h(\boldsymbol{v})-\phi_h(\boldsymbol{u})$, then we obtain

$$\begin{cases}
A_h\left(\boldsymbol{y}_h(\boldsymbol{v})-\boldsymbol{y}_h(\boldsymbol{u}),\boldsymbol{z}_h(\boldsymbol{v})-\boldsymbol{z}_h(\boldsymbol{u})\right)-b\left(\boldsymbol{z}_h(\boldsymbol{v})-\boldsymbol{z}_h(\boldsymbol{u}),p_h(\boldsymbol{v})-p_h(\boldsymbol{u})\right)\\
\quad=\sum_{K\in\mathcal{T}_h}\left(\boldsymbol{v}-\boldsymbol{u},\Pi_2^0\left(\boldsymbol{z}_h(\boldsymbol{v})-\boldsymbol{z}_h(\boldsymbol{u})\right)\right)_{0,K},\\
b\left(\boldsymbol{y}_h(\boldsymbol{v})-\boldsymbol{y}_h(\boldsymbol{u}),\phi_h(\boldsymbol{v})-\phi_h(\boldsymbol{u})\right)=0.
\end{cases} \tag{4.13}$$

Using (4.3), we can obtain: $\forall(\boldsymbol{w}_h,\psi_h)\in\boldsymbol{V}_h\times Q_h$,

$$\begin{cases}
A_h\left(\boldsymbol{w}_h,\boldsymbol{z}_h(\boldsymbol{v})-\boldsymbol{z}_h(\boldsymbol{u})\right)+b\left(\boldsymbol{w}_h,\phi_h(\boldsymbol{v})-\phi_h(\boldsymbol{u})\right)=\sum_{K\in\mathcal{T}_h}\left(y_h(\boldsymbol{v})-y_h(\boldsymbol{u}),\Pi_2^0\boldsymbol{w}_h\right)_{0,K},\\
b\left(\boldsymbol{z}_h(\boldsymbol{v})-\boldsymbol{z}_h(\boldsymbol{u}),\psi_h\right)=0.
\end{cases}$$

Taking $\boldsymbol{w}_h=\boldsymbol{y}_h(\boldsymbol{v})-\boldsymbol{y}_h(\boldsymbol{u})$ and $\psi_h=p_h(\boldsymbol{v})-p_h(\boldsymbol{u})$ yields

$$\begin{cases}
A_h\left(\boldsymbol{y}_h(\boldsymbol{v})-\boldsymbol{y}_h(\boldsymbol{u}),\boldsymbol{z}_h(\boldsymbol{v})-\boldsymbol{z}_h(\boldsymbol{u})\right)+b\left(\boldsymbol{y}_h(\boldsymbol{v})-\boldsymbol{y}_h(\boldsymbol{u}),\phi_h(\boldsymbol{v})-\phi_h(\boldsymbol{u})\right)\\
\quad=\sum_{K\in\mathcal{T}_h}\left(y_h(\boldsymbol{v})-y_h(\boldsymbol{u}),\Pi_2^0\left(\boldsymbol{y}_h(\boldsymbol{v})-\boldsymbol{y}_h(\boldsymbol{u})\right)\right)_{0,K},\\
b\left(\boldsymbol{z}_h(\boldsymbol{v})-\boldsymbol{z}_h(\boldsymbol{u}),p_h(\boldsymbol{v})-p_h(\boldsymbol{u})\right)=0.
\end{cases} \tag{4.14}$$

According to (4.13)-(4.14) and the property of the L^2 projection, we deduce

$$\begin{aligned}
&\sum_{K\in\mathcal{T}_h}\left(\boldsymbol{v}-\boldsymbol{u},\Pi_2^0\left(\boldsymbol{z}_h(\boldsymbol{v})-\boldsymbol{z}_h(\boldsymbol{u})\right)\right)_{0,K}\\
&=A_h\left(\boldsymbol{y}_h(\boldsymbol{v})-\boldsymbol{y}_h(\boldsymbol{u}),\boldsymbol{z}_h(\boldsymbol{v})-\boldsymbol{z}_h(\boldsymbol{u})\right)
\end{aligned}$$

$$= \sum_{K\in\mathcal{T}_h} \left(y_h(v)-y_h(u), \Pi_2^0\left(y_h(v)-y_h(u)\right)\right)_{0,K}$$
$$= \sum_{K\in\mathcal{T}_h} \left(\Pi_2^0\left(y_h(v)-y_h(u)\right), \Pi_2^0\left(y_h(v)-y_h(u)\right)\right)_{0,K} \geq 0.$$

Therefore, from (4.12), it follows

$$\begin{aligned}
\gamma\|u-u_h\|_0^2 &\leq \hat{J}_h'(u)(u-u_h) - \hat{J}_h'(u_h)(u-u_h) \\
&= \sum_{K\in\mathcal{T}_h} \int_K \left(\gamma u + \Pi_2^0 z_h(u) - \gamma u_h - \Pi_2^0 z_h(u_h)\right)(u-u_h)\,\mathrm{d}K \\
&= \sum_{K\in\mathcal{T}_h} \int_K \left(\gamma u + z + \Pi_2^0 z_h(u) - z\right)(u-u_h)\,\mathrm{d}K \\
&\quad + \sum_{K\in\mathcal{T}_h} \int_K \left(\gamma u_h + \Pi_2^0 z_h(u_h)\right)(u_h-u)\,\mathrm{d}K \\
&= (\gamma u + z, u-u_h) + \sum_{K\in\mathcal{T}_h} \left(\Pi_2^0 z_h(u) - z, u-u_h\right)_{0,K} \\
&\quad + \sum_{K\in\mathcal{T}_h} \left(\gamma u_h + \Pi_2^0 z_h(u_h), u_h-u\right)_{0,K} \\
&\leq 0 + \sum_{K\in\mathcal{T}_h} \left(\Pi_2^0 z_h(u) - z, u-u_h\right)_{0,K} + 0.
\end{aligned}$$

This shows

$$\|u-u_h\|_0 \leq C\left(\sum_{K\in\mathcal{T}_h} \left\|\Pi_2^0 z_h(u) - z\right\|_{0,K}^2\right)^{\frac{1}{2}}. \tag{4.15}$$

Note that

$$\begin{aligned}
\left\|\Pi_2^0 z_h(u) - z\right\|_{0,K} &\leq \left\|\Pi_2^0 z_h(u) - \Pi_2^0 z\right\|_{0,K} + \left\|\Pi_2^0 z - z\right\|_{0,K} \\
&\leq \left\|z_h(u) - z\right\|_{0,K} + \left\|\Pi_2^0 z - z\right\|_{0,K}.
\end{aligned} \tag{4.16}$$

By Lemma 2.3, we have $\left\|\Pi_2^0 z - z\right\|_{0,K} \leq Ch^3$. We decompose the error $z_h(u)-z$ into

$$z_h(u) - z = z_h(u) - z_h(y) + z_h(y) - z.$$

Applying Lemma 4.4 yields

$$\left\|z - z_h(y)\right\|_0 \leq Ch^3.$$

By governing equations of $z_h(u)$ and $z_h(y)$ we have: $\forall (w_h, \psi_h) \in V_h \times Q_h$

$$\begin{cases}
A_h\left(w_h, z_h(u) - z_h(y)\right) + b\left(w_h, \phi_h(u) - \phi_h(y)\right) = \sum_{K\in\mathcal{T}_h} \left(y_h(u) - y, \Pi_2^0 w_h\right)_{0,K}, \\
b\left(z_h(u) - z_h(y), \psi_h\right) = 0.
\end{cases}$$

Setting $w_h = z_h(u) - z_h(y)$ and $\psi_h = \phi_h(u) - \phi_h(y)$, we obtain

$$\begin{cases}
A_h\left(z_h(u) - z_h(y), z_h(u) - z_h(y)\right) + b\left(z_h(u) - z_h(y), \phi_h(u) - \phi_h(y)\right) \\
\quad = \sum_{K\in\mathcal{T}_h} \left(y_h(u) - y, \Pi_2^0\left(z_h(u) - z_h(y)\right)\right)_{0,K}, \\
b\left(z_h(u) - z_h(y), \phi_h(u) - \phi_h(y)\right) = 0.
\end{cases}$$

It can be deduced immediately by (3.1)

$$
\begin{aligned}
C\left|\left|\left|z_h(u)-z_h(y)\right|\right|\right|_V^2 &\le A_h\left(z_h(u)-z_h(y),z_h(u)-z_h(y)\right)\\
&=\sum_{K\in\mathcal{T}_h}\left(y_h(u)-y,\Pi_2^0\left(z_h(u)-z_h(y)\right)\right)_{0,K}\\
&\le\sum_{K\in\mathcal{T}_h}\left\|y_h(u)-y\right\|_{0,K}\left\|z_h(u)-z_h(y)\right\|_{0,K}\\
&\le\left\|y_h(u)-y\right\|_0\left\|z_h(u)-z_h(y)\right\|_1.
\end{aligned}
$$

Then we can derive

$$\left\|z_h(u)-z_h(y)\right\|_0\le\left|\left|\left|z_h(u)-z_h(y)\right|\right|\right|_V\le C\left\|y_h(u)-y\right\|_0\le Ch^3.$$

Using the triangle inequality leads to

$$\left\|z-z_h(u)\right\|_0\le Ch^3. \tag{4.17}$$

Combining (4.15), (4.16) and (4.17) results in

$$\left\|u-u_h\right\|_0\le Ch^3.$$

Inserting above estimate into the estimates of state and adjoint state yields the final results.

5. Numerical Results

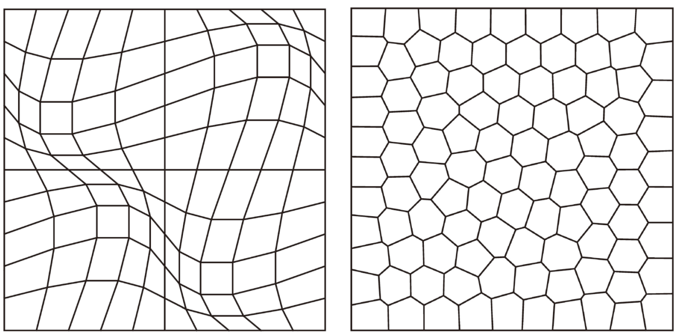
In this section, we present an example on domain $\Omega:=[0,1]\times[0,1]$ to validate the performance of our error analysis presented in this paper.

For the convergence test we consider the following two sequences of meshes that are shown in Figure 1. The first sequence of meshes (labeled Distorted square) is the distorted square mesh. The second sequence of meshes (labeled Lloyd) is obtained by the Voronoi mesh generator (see [22]).

Example 5.1. *Consider the optimal control problem* (1.1)-(1.2) *on the square domain* Ω. *Let* $u_a=(-0.015,-0.015)^{\mathrm{T}}$, $u_b=(0.015,0.015)^{\mathrm{T}}$, $\mathbb{K}=I$, $\gamma=1$. *The exact solutions are chosen to be*

$$y=\begin{pmatrix}y_1\\y_2\end{pmatrix}=\begin{pmatrix}10x_1^2(1-x_1)^2x_2(1-x_2)(1-2x_2)\\-10x_1(1-x_1)(1-2x_1)x_2^2(1-x_2)^2\end{pmatrix},$$

$$p=10(2x_1-1)(2x_2-1),$$



(a) Distorted square (b) Lloyd

Figure 1. Two meshes.

$$z = 0.5y, \quad \phi = -0.5p.$$

The control variable is given $u = P_{U_{ad}}\left(-\frac{1}{\gamma}z\right)$. f and y_d can be determined from the exact solutions y, p, z, ϕ, u.

In Tables 1-3, we show the numerical results about the state variables y, p, the adjoint state variables z, ϕ and the control variable u on three different meshes. We can observe that the convergence rate is consistent with the previous theoretical analysis. In the following tables, *NE* is the number of mesh elements. We observe that both errors have optimal convergence rates, which satisfies the

Table 1. Errors and convergence rates of state variables on two meshes.

Distorted square mesh							
NE	h	$\lVert y-y_h\rVert_0$	Rate	$\lvert\lvert\lvert y-y_h\rvert\rvert\rvert_V$	Rate	$\lVert p-p_h\rVert_0$	Rate
100	1.000E−01	9.31208E−04		3.55874E−02		5.30223E−03	
400	5.000E−02	1.06371E−04	3.13	8.65357E−03	2.04	8.65357E−03	2.07
900	3.333E−02	3.20232E−05	2.96	3.89233E−03	1.97	3.89233E−03	1.97
1600	2.500E−02	1.33977E−05	3.03	2.18359E−03	2.01	2.18359E−03	2.02
Lloyd mesh							
NE	h	$\lVert y-y_h\rVert_0$	Rate	$\lvert\lvert\lvert y-y_h\rvert\rvert\rvert_V$	Rate	$\lVert p-p_h\rVert_0$	Rate
100	1.000E−01	8.13421E−04		8.51484E−02		2.51368E−03	
400	5.000E−02	9.54153E−05	3.09	2.05802E−02	2.05	6.45030E−04	1.96
900	3.333E−02	2.89133E−05	2.94	8.99788E−03	2.04	2.80872E−04	2.05
1600	2.500E−02	1.22585E−05	2.98	5.04953E−03	2.01	1.58005E−04	2.00

Table 2. Errors and convergence rates of adjoint state variables on two meshes.

Distorted square mesh							
NE	h	$\lVert z-z_h\rVert_0$	Rate	$\lvert\lvert\lvert z-z_h\rvert\rvert\rvert_V$	Rate	$\lVert \psi-\psi_h\rVert_0$	Rate
100	1.000E−01	2.53147E−03		1.23694E−01		5.24139E−03	
400	5.000E−02	2.97297E−04	3.09	3.04978E−02	2.02	1.27452E−03	2.04
900	3.333E−02	8.95017E−05	2.96	1.36622E−02	1.98	5.73271E−04	1.97
1600	2.500E−02	3.74454E−05	3.03	7.66446E−03	2.01	3.21603E−04	2.01
Lloyd mesh							
NE	h	$\lVert z-z_h\rVert_0$	Rate	$\lvert\lvert\lvert z-z_h\rvert\rvert\rvert_V$	Rate	$\lVert \psi-\psi_h\rVert_0$	Rate
100	1.000E−01	6.51423E−04		2.54301E−02		1.48532E−03	
400	5.000E−02	7.69442E−05	3.08	6.14640E−03	2.05	3.81145E−04	1.96
900	3.333E−02	2.32217E−05	2.95	2.69819E−03	2.03	1.66370E−04	2.04
1600	2.500E−02	9.84543E−06	2.98	1.51420E−03	2.01	9.33767E−05	2.01

Table 3. Errors and convergence rates of control variable on two meshes.

	Distorted square mesh		
NE	*h*	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_0$	Rate
100	1.000E−01	5.82184E−03	
400	5.000E−02	6.98085E−04	3.06
900	3.333E−02	2.03451E−04	3.04
1600	2.500E−02	8.66004E−05	2.97
	Lloyd mesh		
NE	*h*	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_0$	Rate
100	1.000E−01	2.65768E−03	
400	5.000E−02	3.18677E−04	3.06
900	3.333E−02	9.55500E−05	2.97
1600	2.500E−02	4.04384E−05	2.99

conclusion in Theorem 4.5.

Acknowledgements

The author would like to thank editor and referees for their valuable advices for the improvement of this article.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Wilfrid, H. (2021) A New Unified Stabilized Mixed Finite Element Method of the Stokes-Darcy Coupled Problem: Isotropic Discretization. *Journal of Applied Physics*, **9**, 1673-1706. https://doi.org/10.4236/jamp.2021.97112

[2] Kumar, S., Ruiz-Baier, R. and Sandilya, R. (2019) Error Bounds for Discontinuous Finite Volume Discretisations of Brinkman Optimal Control Problems. *Journal of Scientific Computing*, **78**, 64-93. https://doi.org/10.1007/s10915-018-0749-z

[3] Leng, H.T. and Chen, L. (2021) Adaptive HDG Methods for the Brinkman Equations with Application to Optimal Control. *Journal of Scientific Computing*, **87**, 46. https://doi.org/10.1007/s10915-021-01450-x

[4] Benner, P., Dolgov, S., Onwunta, A. and Stoll, M. (2016) Low-Rank Solvers for Unsteady Stokes-Brinkman Optimal Control Problem with Random Data. *Computer Methods in Applied Mechanics and Engineering*, **304**, 26-54. https://doi.org/10.1016/j.cma.2016.02.004

[5] Mohan, M.T. (2021) The Time Optimal Control of Two Dimensional Convective Brinkman-Forchheimer Equations. *Applied Mathematics & Optimization*, **84**, 3295-3338. https://doi.org/10.1007/s00245-021-09748-w

[6] You, B. and Li, F. (2019) Optimal Distributed Control of the Cahn-Hilliard-Brink-

man System with Regular Potential. *Nonlinear Analysis*, **182**, 226-247. https://doi.org/10.1016/j.na.2018.12.014

[7] Beirão da Veiga, L., Brezzi, F., Cangiani, A., *et al.* (2013) Basic Principles of Virtual Element Methods. *Mathematical Models and Methods in Applied Sciences*, **23**, 199-214. https://doi.org/10.1142/S0218202512500492

[8] Beirão da Veiga, L., Brezzi, F., Marini, L.D. and Russo, A. (2016) Virtual Element Method for General Second Order Elliptic Problems on Polygonal Meshes. *Mathematical Models and Methods in Applied Sciences*, **26**, 729-750. https://doi.org/10.1142/S0218202516500160

[9] Beirão da Veiga, L., Brezzi, F. and Marini, L.D. (2013) Virtual ELEMENTS for Linear Elasticity Problems. *SIAM Journal on Numerical Analysis*, **51**, 794-812. https://doi.org/10.1137/120874746

[10] Brezzi, F. and Marini, L.D. (2013) Virtual Element Methods for Plate Bending Problems. *Computer Methods in Applied Mechanics and Engineering*, **253**, 455-462. https://doi.org/10.1016/j.cma.2012.09.012

[11] Beirão da Veiga, L., Lovadina, C. and Vacca, G. (2017) Divergence Free Virtual Elements for the Stokes Problem on Polygonal Meshes. *ESAIM*: *Mathematical Modelling and Numerical Analysis*, **51**, 509-535. https://doi.org/10.1051/m2an/2016032

[12] Beirão da Veiga, L., Lovadina, C. and Vacca, G. (2018) Virtual Elements for the Navier-Stokes Problem on Polygonal Meshes. *SIAM Journal on Numerical Analysis*, **56**, 1210-1242. https://doi.org/10.1137/17M1132811

[13] Cáceres, E., Gatica, G.N. and Sequeira, F.A. (2017) A Mixed Virtual Element Method for the Brinkman Problem. *Mathematical Models and Methods in Applied Sciences*, **27**, 707-743. https://doi.org/10.1142/S0218202517500142

[14] Vacca, G. (2018) An H^1-Conforming Virtual Element for Darcy and Brinkman Equations. *Mathematical Models and Methods in Applied Sciences*, **28**, 159-194. https://doi.org/10.1142/S0218202518500057

[15] Wang, G., Wang, Y. and He, Y.N. (2021) Two Robust Virtual Element Methods for the Brinkman Equations. *Calcolo*, **58**, 49. https://doi.org/10.1007/s10092-021-00442-5

[16] Wang, Q.M. and Zhou, Z.J. (2021) Adaptive Virtual Element Method for Optimal Control Problem Governed by General Elliptic Equation. *Journal of Scientific Computing*, **88**, 14. https://doi.org/10.1007/s10915-021-01528-6

[17] Wang, Q.M. and Zhou, Z.J. (2022) A Priori and a Posteriori Error Analysis for Virtual Element Discretization of Elliptic Optimal Control Problem. *Numerical Algorithms*, **90**, 989-1015. https://doi.org/10.1007/s11075-021-01219-1

[18] Becker, R., Kapp, H. and Rannacher, R. (2000) Adaptive Finite Element Methods for Optimal Control of Partial Differential Equations: Basic Concept. *SIAM Journal on Control and Optimization*, **39**, 113-132. https://doi.org/10.1137/S0363012999351097

[19] Hinze, M. (2005) A Variational Discretization Concept in Control Constrained Optimization: The Linear-Quadratic Case. *Computational Optimization and Applications*, **30**, 45-61. https://doi.org/10.1007/s10589-005-4559-5

[20] Mu, L., Wang, J.P. and Ye, X. (2014) A Stable Numerical Algorithm for the Brinkman Equations by Weak Galerkin Finite Element Methods. *Journal of Computational Physics*, **273**, 327-342. https://doi.org/10.1016/j.jcp.2014.04.017

[21] Bhatia, R. and Davis, C. (1995) A Cauchy-Schwarz Inequality for Operators with

Applications. *Linear Algebra and Its Applications*, **223**, 119-129. https://doi.org/10.1016/0024-3795(94)00344-D

[22] Talischi, C., Paulino, G.H., Pereira, A. and Menezes, I.F.M. (2012) PolyMesher: A General-Purpose Mesh Generator for Polygonal Elements Written in Matlab. *Structural and Multidisciplinary Optimization*, **45**, 309-328. https://doi.org/10.1007/s00158-011-0706-z