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On the Chromatic Number of (P_5, C_5, Cricket)-Free Graphs

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Abstract

For a graph G, let $\chi(G)$ be the chromatic number of G. It is well-known that $\chi(G) \geq \omega$ holds for any graph G with clique number ω. For a hereditary graph class $\mathscr{G}$, whether there exists a function f such that $\chi(G) \leq f(\omega(G))$ holds for every $G \in \mathscr{G}$ has been widely studied. Moreover, the form of minimum such an f is also concerned. A result of Schiermeyer shows that every $(P_5, \text{cricket})$-free graph G with clique number ω has $\chi(G) \leq \omega^2$. Chudnovsky and Sivaraman proved that every (P_5, C_5)-free with clique number ω graph is $2^{\omega-1}$-colorable. In this paper, for any $(P_5, C_5, \text{cricket})$-free graph G with clique number ω, we prove that $\chi(G) \leq \left\lceil \frac{\omega^2}{2} \right\rceil + \omega$. The main methods in the proof are set partition and induction.

Keywords

P_5-Free Graphs, Chromatic Number, χ-Boundedness

1. Introduction

In this paper, we consider undirected, simple graphs. For a given graph H, a graph G is called H-free if G contains no induced subgraphs isomorphic to H. Let $H_1, H_2, \cdots, H_k$ $(k \geq 2)$ be different graphs. If for any $1 \leq i \leq k$, G is H_i-free, then we say that G is $(H_1, H_2, \cdots, H_k)$-free. A graph $G = (V, E)$ is k-colorable if there exists a function $\varphi: V(G) \mapsto \{1, 2, \cdots, k\}$ such that for any $uv \in E(G)$, there is $\varphi(u) \neq \varphi(v)$. The *chromatic number* of G is the minimum integer k such that G is k-colorable, denoted by $\chi(G)$. For a graph $G = (V, E)$, a subset S of $V(G)$ is called a clique if S induces a complete subgraph. We use $\omega(G)$ to denote the maximum size of cliques of G. It is well-known that

$\omega(G) \le \chi(G)$ for every graph *G*. A graph is *perfect* if for any induced subgraph G' of *G*, $\omega(G') = \chi(G')$. Chudnovsky *et al.* [1] gave an equivalent characterization of perfect graphs, which is also called as the Strong Perfect Graph Theorem.

Theorem 1.1. [1] *A graph is perfect if and only if it contains neither odd cycles of length at least five nor the complements of these odd cycles.*

We say a hereditary graph class $\mathscr{G}$ is χ-bounded, if there exists a function *f* such that for any $G \in \mathscr{G}$, $\chi(G) \le f(\omega(G))$. Moreover, *f* is called a χ-binding function of $\mathscr{G}$. Erdös [2] showed that for arbitrary integers $k, l \ge 3$, there exists a graph *G* with girth at least *l* and $\chi(G) \ge k$, which implies that the class of *H*-free graphs is not χ-bounded when *H* contains a cycle. Gyárfás conjectured that the graph class obtained by forbidding a tree (or forest) is χ-bounded.

Conjecture 1.2. [3] *Let T be a tree* (*or forest*), *then there exists a function* f_T *such that, for any T-free graph G,* $\chi(G) \le f_T(\omega(G))$.

Moreover, Gyárfás [3] verified this conjecture when $T = P_k$, and showed that $f_T \le (k-1)^{\omega(G)-1}$. When $T = P_5$, Esperet *et al.* [4] gave a χ-binding function of P_5-free graphs as following.

Theorem 1.3. [4] *Suppose G is a* P_5*-free graph with clique number* $\omega \ge 3$. *Then* $\chi(G) \le 5 \cdot 3^{\omega-3}$.

As far as we know, for $\omega \ge 3$, $f(\omega) = 5 \cdot 3^{\omega-3}$ is the optimal χ-binding function of P_5-free graphs at present. Furthermore, determining a polynomial χ-binding function of the class of P_5-free graphs is an open problem. A result in [5] shows that the class of *H*-free graphs has a linear χ-binding function *f*, if and only if $f(\omega) = \omega$ and *H* is an induced subgraph of P_4, which means that the class of P_5-free graphs has no linear χ-binding function.

In this paper, we focus on subclasses of P_5-free graphs. While the class of P_5-free graphs has no linear χ-binding function, some subclasses of P_5-free have linear χ-binding functions.

Theorem 1.4. [6] [7] [8] [9] *Suppose* $H \in \{diamond, gem, paraglider, paw\}$, *then the class of* (P_5, H)*-free graphs has a* χ*-binding function.*

More formally, Chudnovsky *et al.* [6] proved that the class of (P_5, gem)-free graphs has a χ-binding function $f(\omega) \le \left\lceil \frac{5}{4}\omega \right\rceil$. Huang and Karthick [7] showed that $(P_5, \text{paraglider})$ graphs have a χ-binding function $f(\omega) \le \left\lceil \frac{3}{2}\omega \right\rceil$. Karthick and Maffray [8] gave a χ-binding function $f(\omega) = \omega + 1$ for $(P_5, \text{diamond})$-free graphs. And Randerath [9] showed that (P_5, paw)-free graphs have a χ-binding function $f(\omega) = \omega + 1$ (diamond, gem, paraglider and paw are given in **Figure 1**).

It is worth noting that a result in [10] shows that when *H* contains an independent set with size at least 3, the class of (P_5, H)-free graphs has no linear χ-binding function.

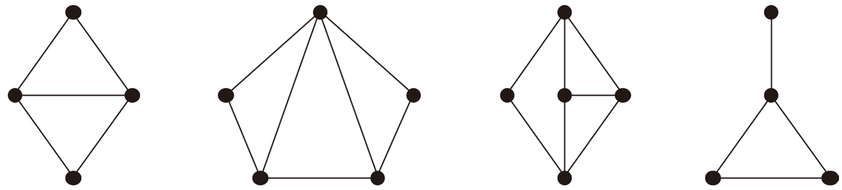

Figure 1. Examples of diamond, gem, paraglider and paw.

Theorem 1.5. [10] *The class of* $(2K_2, 3K_1)$ *-free graphs has no linear* χ *-binding function.*

Obviously, when H is a graph with independent number at least 3, (P_5, H) -free graphs is a superclass of $(2K_2, 3K_1)$ -free graphs. Thus the class of (P_5, H) -free graphs has no χ -binding function.

The following theorem shows that some subclasses of P_5 -free graphs have a χ -binding function $f(\omega) = \binom{\omega+1}{2}$ (The addition forbidden subgraphs are given in **Figure 2**).

Theorem 1.6. [10] [11] [12] *The class of* (P_5, H) *-free graphs has a* χ *-binding function* $f(\omega) = \binom{\omega+1}{2}$ *when* $H \in \{bull, house, hammer\}$.

In [13], Schiermeyer proved that the class of (P_5, H) -free graphs has a χ -binding function $f(\omega) = \omega^2$ for $H \in \{\text{claw}, \text{cricket}, \text{dart}, \text{gem}+\}$ (see **Figure 3**).

In addition to the subclasses of P_5 -free graphs we mentioned above, there are many subclasses had been proved that admit a polynomial χ -binding function, which is given in [14] and [15]. More results on χ -binding function, see [16].

The class of (P_5, C_5) -free graphs, which is a superclass of $(P_5, C_5, \text{cricket})$ -free graphs, has been studied by Chudnovsky and Sivaraman [11]. They showed that every (P_5, C_5) -free graph with clique number ω is $2^{\omega-1}$ -colorable. In this paper, we obtain the following result. In the next section, we will give the proof.

Theorem 1.7. *Every* $(P_5, C_5, \text{cricket})$ *-free graph* G *with clique number* ω *has* $\chi(G) \le \left\lceil \frac{\omega^2}{2} \right\rceil + \omega$.

2. The Proof of Main Result

For two vertex sets A and B, let $E(A,B) = \{uv \in E(G) : u \in A \text{ and } v \in B\}$. We say that A is complete to B, if for any $x \in A$ and $y \in B$, $xy \in E(G)$. For a given graph $G = (V, E)$, let $N(v)$ denote the neighborhood of $v \in V(G)$, and for a subset S of $V(G)$, set $N(S) = \bigcup_{v \in S} N(v)$. An induced subgraph D of G is called a *dominating* D, if there is $V(G) \setminus V(D) \subseteq N(V(D))$. In this paper, for an induced P_4: $P = v_1v_2v_3v_4$, we simply write $V(P)$ as P. First, we give a lemma based on the structure of a (P_5, C_5) -free graph.

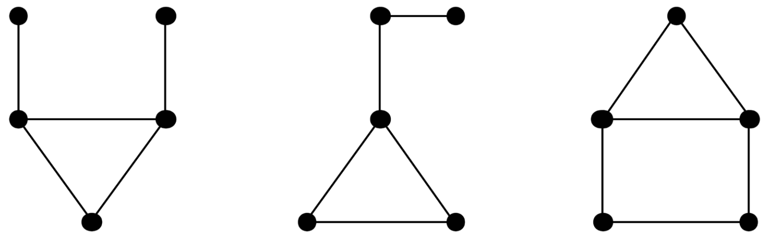

Figure 2. Examples of bull, hammer and house.

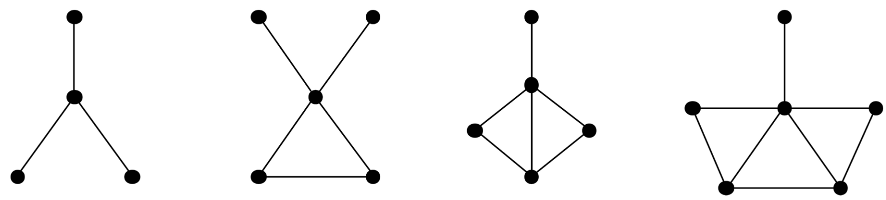

Figure 3. Examples of claw, cricket, dart and gem+.

Lemma 2.1. *If* $P = v_1v_2v_3v_4$ *is a dominating* P_4 *of a* (P_5, C_5)*-free graph G, then* v_2v_3 *is a dominating edge of G.*

Suppose, to the contrary, that there exists a vertex $u \notin N(v_2) \cup N(v_3)$. Since *P* is a dominating P_4, $u \in N(v_1) \cup N(v_4)$. By symmetry, we may assume that $uv_1 \in E(G)$. If $uv_4 \in E(G)$, then $uv_1v_2v_3v_4u$ would be an induced C_5. If $uv_4 \notin E(G)$, then $uv_1v_2v_3v_4$ would be an induced P_5. Either deduces a contradiction.

Next, we show that a subclass of $(P_5, C_5, \text{cricket})$-free graphs has a χ-binding function $f(\omega) = \left\lceil \frac{\omega^2}{2} \right\rceil$. Let $iK_1 + K_2$ be the graph consisted of one edge and *i* isolated vertices.

Lemma 2.2. *Every* $(P_5, C_5, 2K_1 + K_2)$*-free graph G with clique number* ω *has* $\chi(G) \le \left\lceil \frac{\omega^2}{2} \right\rceil$.

Apply induction on ω. If $\omega = 1$, it is obviously true. When $\omega = 2$, it is also true because every (P_5, C_5, K_3)-free graph is a bipartite graph. Moreover, when $\omega = 3$, by Theorem 1.3, $\chi(G) \le 5 = \left\lceil \frac{9}{2} \right\rceil$. Next, consider the cases $\omega \ge 4$. If *G* is P_4-free, then *G* is perfect by Theorem 1.1. So we may suppose that $P = v_1v_2v_3v_4$ is an induced P_4. We claim that *P* is a dominating P_4 of *G*. Otherwise, there would exist a vertex $u \in V(G) \setminus N(P)$. Noting that $P \subseteq N(P)$, $\{u, v_1, v_3, v_4\}$ induces a $2K_1 + K_2$, a contradiction. By Lemma 2.1, v_2v_3 is a dominating edge of *G*. Next, denote

$$V_2 = \{v : vv_2 \in E(G) \text{ and } vv_3 \notin E(G)\} \setminus \{v_3\},$$

$$V_3 = \{v : vv_2 \notin E(G) \text{ and } vv_3 \in E(G)\} \setminus \{v_2\},$$

$$V_{2,3} = N(v_2) \cap N(v_3).$$

For clarity, we give this partition in **Figure 4**. Let $G[S]$ denote the subgraph of G induced by S. Clearly, $G[V_2]$ is (P_5, C_5, K_1+K_2)-free. (Otherwise, let $\{u_1,u_2,u_3\}$ be an induced K_1+K_2 of $G[V_2]$. Then $\{u_1,u_2,u_3,v_3\}$ would induce a $2K_1+K_2$.) By Theorem 1.1, $G[V_2]$ is perfect. Noting that $\omega(G[V_2]) \le \omega-1$, we have $\chi(G[V_2]) \le \omega-1$. Similarly, $\chi(G[V_3]) \le \omega-1$. Moreover, there is $\omega(G[V_{2,3}]) \le \omega-2$. By induction, $\chi(G[V_{2,3}]) \le \left\lceil \frac{(\omega-2)^2}{2} \right\rceil$.

Now we color G. Let $K = \left\{1,2,\cdots,\left\lceil \frac{\omega^2}{2} \right\rceil\right\}$ be a color set. First, we color v_2 and v_3 by colors 1 and 2, respectively. Noting that $E(V_2,\{v_3\}) = \varnothing$, V_2 can be colored by $\{2,3,\cdots,\omega\}$. Similarly, V_3 can be colored by $\{1,\omega+1,\cdots,2\omega-2\}$. Thus, $\chi(G[V_2 \cup V_3 \cup \{v_2,v_3\}]) \le 2\omega-2$. Since v_2v_3 is a dominating edge of G, $V(G) = \{v_2,v_3\} \cup V_2 \cup V_3 \cup V_{2,3}$. So we have

$$\chi(G) \le \chi(G[V_2 \cup V_3 \cup \{v_2,v_3\}]) + \chi(G[V_{2,3}]) \le 2\omega-2+\left\lceil \frac{(\omega-2)^2}{2} \right\rceil = \left\lceil \frac{\omega^2}{2} \right\rceil.$$

Note that the bound given by Lemma 2.2 is tight for $\omega=2$, and C_4 is a $(P_5, C_5, \text{cricket})$-free graph with clique number 2 and chromatic number 2.

Proof of Theorem 1.7

When $\omega \le 3$, it is obviously true. Next, assume that $\omega \ge 4$. If G is P_4-free, then $\chi(G)=\omega$ by Theorem 1.1. So we may suppose that $P = v_1v_2v_3v_4$ is an induced P_4 of G. Let $N_2(P) = N(N(P)) \setminus N(P)$ and $N_3(P) = N(N_2(P)) \setminus N(P)$. Moreover, for arbitrary different $i,j,k \in \{1,2,3,4\}$, denote

$$U_i = \{v \in N(P) \setminus P : N(v) \cap P = \{v_i\}\},$$

$$U_{i,j} = \{v \in N(P) \setminus P : N(v) \cap P = \{v_i,v_j\}\},$$

$$U_{i,j,k} = \{v \in N(P) \setminus P : N(v) \cap P = \{v_i,v_j,v_k\}\},$$

$$A = \{v \in N(P) \setminus P : N(v) \cap P = P\}.$$

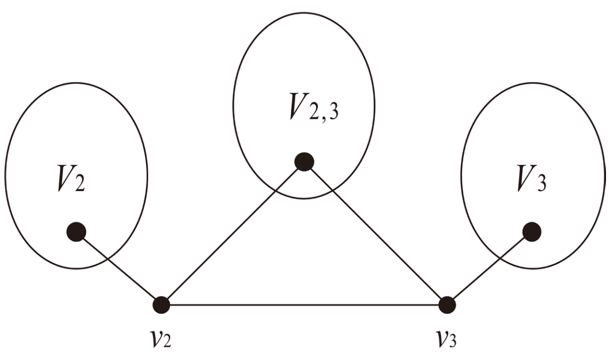

Figure 4. A partition of $V(G)$.

Clearly, $U_{i,j}=U_{j,i}$ and $U_{i,k,j}=U_{i,j,k}=U_{j,i,k}$. Since G is (P_5,C_5)-free, $U_1=U_4=U_{1,4}=\varnothing$. So

$$A\cup U_2\cup U_3\cup U_{1,2}\cup U_{1,3}\cup U_{2,3}\cup U_{2,4}\cup U_{3,4}\cup U_{1,2,3}\cup U_{1,2,4}\cup U_{1,3,4}\cup U_{2,3,4}=N(P)\setminus P.$$

The partition is shown in **Figure 5**. Since G is P_5-free, there is no vertex with a distance of 4 to *P*. So we can partition $V(G)$ into $N(P)$, $N_2(P)$, $N_3(P)$, and color these sets respectively. Next, we give two claims based on $N_3(P)$ and $N_2(P)$.

Claim 1 $N_3(P)=\varnothing$.

Otherwise, suppose there are vertices $x_3\in N_3(P)$ and $x_2\in N_2(P)$ such that $x_2x_3\in E(G)$. Let $u\in N(P)\setminus P$ be a neighbor of x_2. If $u\in A$, then $\{x_2,u,v_1,v_2,v_4\}$ would induce a cricket, a contradiction. So there exists v_i and v_j ($i,j\in\{1,2,3,4\}$) such that $v_iv_j\in E(G)$, $uv_i\in E(G)$ and $uv_j\notin E(G)$. Now $x_3x_2uv_iv_j$ is an induced P_5, a contradiction.

Claim 2 *Let T be a connected component of $G[N_2(P)]$ with $|V(T)|\geq 2$, then then at least one vertex of $U_{2,3}$ is complete to $V(T)$.*

First, we show that every edge xy in T has $N(x)\cap N(P)=N(y)\cap N(P)$. Suppose, to the contrary, that there exists a vertex $u\in(N(x)\cap N(P))\setminus(N(y)\cap N(P))$. Similar to the proof of Claim 1, there is an induced cricket or induced P_5, a contradiction. So, for each $xy\in E(T)$, *x* and *y* have same neighborhood in $N(P)$. By connectivity and transitivity, all vertices in T have same neighborhood in $N(P)$. Then there is at least one vertex, say *u*, in $N(P)\setminus P$ such that $V(T)$ is complete to $\{u\}$.

Next, we pick an arbitrary edge xy in *T*. Then xuy is a triangle. If $u\in U_2\cup U_{1,2}$, then $xuv_2v_3v_4$ would be an induced P_5. And if $u\in A\cup U_{1,3}\cup U_{1,2,3}\cup U_{1,3,4}$, then $\{x,y,u,v_1,v_3\}$ would induce a cricket. Up to symmetry, there must be $u\in U_{2,3}$.

By Claim 2, for an arbitrary connected component T of $G[N_2(P)]$, there exists a vertex $u\in U_{2,3}$ such that $\{u\}$ is complete to $V(T)$. If there exists $x,y\in V(T)$ such that $xy\notin E(G)$, then $\{x,y,u,v_2,v_3\}$ would induce a cricket. Thus $V(T)$ is a clique with size at most $\omega-1$, which implies that

$$\chi(G[N_2(P)])\leq\omega-1. \tag{1}$$

Let $G'=G[N(P)]$. Note that P is a dominating P_4 of G'. By Lemma 2.1, v_2v_3 is a dominating edge of G'. Thus $V(G')\setminus\{v_2,v_3\}$ can be partitioned into $\{V_2,V_3,V_{2,3}\}$, which is defined as in Lemma 2.2. Since G' is $(P_5,C_5,\text{cricket})$-free, both $G[V_2]$ and $G[V_3]$ are (P_5,C_5,K_1+K_2)-free. Thus, by the coloring described in Lemma 2.2, there is $\chi(G[V_2\cup V_3\cup\{v_2,v_3\}])\leq 2\omega-2$. Moreover, noting that $G[V_{2,3}]$ is complete to $\{v_2,v_3\}$, we have that $G[V_{2,3}]$ is $(P_5,C_5,2K_1+K_2)$-free and $\omega(G[V_{2,3}])\leq\omega-2$. By Lemma 2.2, $\chi(G[V_{2,3}])\leq\left\lceil\frac{(\omega-2)^2}{2}\right\rceil$. Thus,

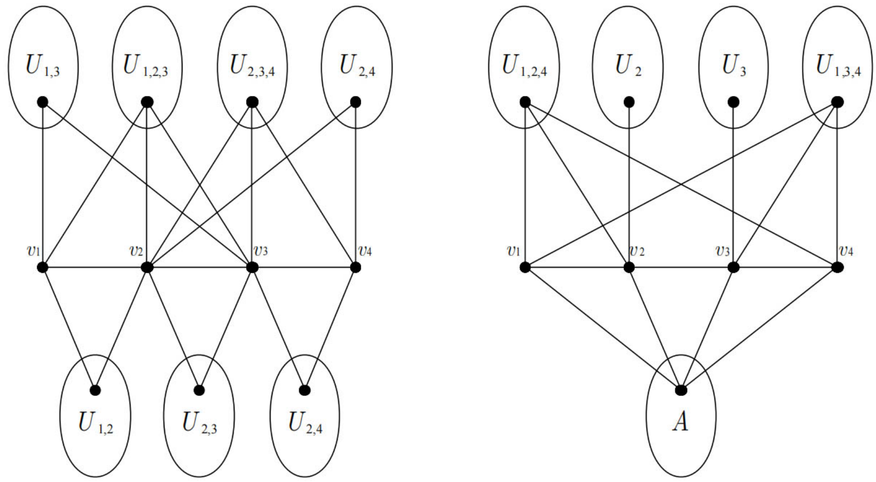

Figure 5. A partition of $N(P)\setminus P$.

$$\chi(G') \le \chi\left(G\left[V_2 \cup V_3 \cup \{v_2, v_3\}\right]\right) + \chi\left(G\left[V_{2,3}\right]\right) \le 2\omega - 2 + \left\lceil \frac{(\omega-2)^2}{2} \right\rceil \le \left\lceil \frac{\omega^2}{2} \right\rceil. \quad (2)$$

By Claim 1, $V(G) = N(P) \cup N_2(P)$. Hence, by Inequality (1) and (2), there is

$$\chi(G) \le \chi(G') + \chi\left(G\left[N_2(P)\right]\right) \le \left\lceil \frac{\omega^2}{2} \right\rceil + \omega.$$

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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