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A Sufficient Condition for 2-Distance-Dominating Cycles

Xinman Wang, Linyu Li

School of Mathematics and Statistics, Shandong Normal University, Jinan, China
Email: 578580136@qq.com

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Abstract

A cycle C of a graph G is a m-distance-dominating cycle if for all vertices of $V(G)$, $d_G(x,C) \le m$. Defining $\sigma_k(G)$ denotes the minimum value of the degree sum of any k independent vertices of G. In this paper, we prove that if G is a 3-connected graph on n vertices, and if $\sigma_4(G) > 4n/3 - 4/3$, then every longest cycle is m-distance-dominating cycles.

Keywords

Degree Sums, Distance Dominating Cycles, Insertible Vertex

1. Introduction

Let $G = (V,E)$ be a graph and H be a subgraph of G, for a $S \subseteq V(G)$, let $N_H(S) = N(S) \cap V(H)$. For any $x, y \in V(G)$, $Y \subseteq V(G)$, xy denotes the edge with ends x and y, an (x,Y)-path denotes a path starting at x and ending at Y. We denote by $\alpha(G)$ and $\kappa(G)$ the independence number and the connectivity of G, respectively.

Let C be a cycle of G, and denote by $\vec{C}$ the cycle C with a given orientation. For $v \in V(C)$, define v^+ and v^- to be the successor and predecessor of v on C, define v^{+i} and v^{-i} to be the i-th successor and predecessor of v on C, respectively. In particular, we write $A^{+i} = \{a^{+i} \mid a \in A\}$ and $A^{-i} = \{a^{-i} \mid a \in A\}$. If $u, v \in V(G)$, we denote by $u\vec{C}v$ the consecutive vertices of C from u to v in the direction specified by $\vec{C}$. The same path, in reverse order, is denoted by $v\overleftarrow{C}u$. We will consider $u\vec{C}v$ and $v\overleftarrow{C}u$ both as paths and as vertex sets.

We use $\min\{d_G(v_1,v_2): v_1 \in V(H_1), v_2 \in V(H_2)\}$ to denote the distance $d_G(H_1,H_2)$ between H_1 and H_2, H_1 and H_2 are all the subgraphs of G, where $d_G(v_1,v_2)$ denotes the length of a shortest path between v_1 and v_2 in G. A subgraph H of G is m-dominating if for all $x \in V(G)$, $d_G(x,H) \le m$. For an integer $k \geq 2$, define

$$\sigma_k = \min\left\{\sum_{i=1}^{k} d(x_i) \mid \{x_1, \cdots, x_k\} \text{ is an independent set of } G\right\}$$

In 1987, Bondy [1] considered the existence of *k*-connected graphs of order *n*.

Theorem 1 [1] *Let G be a k-connected graph on n vertices, where* $k \geq 2$. *If any* $k+1$ *independent vertices* $x_i\ (0 \leq i \leq k)$ *with* $N(x_i) \cap N(x_j) = \varnothing$ $(0 \leq i \neq j \leq k)$ *have degree-sum* $\sum_{i=0}^{k} d(x_i) \geq n - 2k$, *then G has a* 1*-distance-dominating cycle.*

In 1988, Broersma [2] and Fraisse [3] proved some results about *m*-distance-dominating cycles.

Theorem 2 [2] [3] *Let G be a k-connected graph with no set of cardinality* $k+1$, *whose vertices are pairwise at distance at least* $2m+2$. *Then G has an m-distance-dominating cycle.*

The circumference $c(G)$ of a graph *G* is the length of the longest cycle on the graph. In 2021, Xiong [4] considered the relation between the graph circumference and *m*-distance-dominating cycle, and proved a sufficient condition that every longest cycle in *k*-connected graph is *m*-distance-dominating cycle.

Theorem 3 [4] *Let G be a graph with* $\kappa(G) = k \geq 2$. *If* $c(G) \geq (2m+2)k - 1$, *then every longest cycle of G is a m-distance-dominating cycle.*

A cycle *C* is *m*-edge-dominating if for all $e \in E(G)$, $d_G(e, C) \leq m$. Clearly, a cycle is 0-edge-dominating (or simply dominating) if every edge of *G* is incident with a vertex of *C*, $G - V(C)$ is edgeless. It is very popular to decide whether a longest cycle is (0-edge) dominating. Bondy [5] gave a sufficient condition such that every longest cycle of 2-connected graph is (0-edge) dominating.

Theorem 4 [5] *Let G be a* 2*-connected graph on n vertices. If* $\sigma_3(G) \geq n+2$, *then every longest cycle is dominating.*

Wu [6] considered the same problem for *k*-connected graphs and established the following.

Theorem 5 [6] *Let G be k-connected graph on n vertices with* $k \geq 2$. *If* $\sigma_{k+1}(G) > (n+1)(k+1)/3$, *then every longest cycle is dominating.*

In this paper, we consider the general version for degree sums condition that guarantees that every longest cycle is a 2-distance-dominating cycle in 3-connected graphs. Our main result is the following.

Theorem 6 *Let G be a* 3*-connected graph on n vertices. If* $\sigma_4(G) > 4n/3 - 4/3$, *then every longest cycle is a* 2*-distance-dominating cycle.*

1. Key Lemmas

Lemma 1 [6] *Let* $P = u_1 u_2 \cdots u_l$ *and* $Q_1, \cdots, Q_m$ *be* $m+1$ *pairwise vertex disjoint paths of a graph G. If for any* $v \in V(Q_i)$, *there are* $u_k, u_{k+1} \in N(v)$ *such that* $\{u_k, u_{k+1}\} \not\subseteq N(v')$ *for any* $v' \in V(Q_j)$ *with* $j \neq i$, *then G has a* (u_1, u_l)*-path with* $V(P) \cup \left(\bigcup_{i=1}^{m} V(Q_i)\right)$ *as its vertex set.*

A *k*-fan from *x* to *Y* is a family of *k* internal disjoints (x, Y)-paths whose terminal vertices are distinct. The following lemma known as Fan Lemma establishes an useful property of *k*-connected graphs.

Lemma 2 [7] *Let G be a k-connected graph, let x be a vertex of G, and let* $Y \subseteq V(G)\setminus x$ *be a set of at least k vertices of G. Then there exists a k-fan in G from x to Y.*

Next, we assume G be a *k*-connected non-hamiltonian graph of order *n*, $k \geq 2$. Let C be a longest cycle of G with a given orientation. Let $R = V(G) - V(C)$, assume H is a component of $G - V(C)$ and $N_C(H) = \{h_1, h_2, \cdots, h_t\}$, where the subscripts increase with the orientation of *C*.

A vertex $u \in h_i^+ \vec{C} h_{i+1}^-$ is insertible if there exist vertices $v, v^+ \in h_{i+1} \vec{C} h_i$ such that $uv, uv^+ \in E(G)$ and the edge $vv^+ \in E(G)$ is called an insertion edge of *u*, and noninsertible otherwise.

For any *i* with $1 \leq i \leq t$, if each vertex of $h_i^+ \vec{C} h_{i+1}^-$ is insertible, then by Lemma 1, G has an (h_i, h_{i+1})-path P such that $V(P) = V(C)$. Thus, there is a (h_i, h_{i+1})-path L with internal vertices in H and $|L| \geq 3$. We find $\mathbb{C} = h_i P h_{i+1} L h_i$ is a cycle longer than *C*, contradiction. Thus, $h_i^+ \vec{C} h_{i+1}^-$ contains at least one noninsertible vertex. Write a_i as the first noninsertible vertex occurring on $h_i^+ \vec{C} h_{i+1}^-$, $A_i = h_i^+ \vec{C} a_i$. For any $v \in V(H)$, we let $A_v = \{a_i \mid h_i \in N(v)\}$.

Lemma 3 [6] 1) *There is no* (x, y)*-path without internal vertices in* $V(C) \cup V(H)$ *for any* $x \in A_i$ *and* $y \in A_j$ *with* $i \neq j$;

2) $N_P^-(A_i) \cap N_P(A_j) = N_Q(A_i) \cap N_Q^-(A_j) = \varnothing$, where $P = a_i^+ \vec{C} h_j$ and $Q = a_j^+ \vec{C} h_i$.

Lemma 4 [6] *Suppose* $v_1, v_2 \in V(H)$ *with* $v_1 \neq v_2$, $a_i \in A_{v_1}$ *and* $a_j \in A_{v_2}$ *with* $i \neq j$. *Then* 1) $a_i^+ \notin N(A_j)$ *and* $a_j^+ \notin N(A_i)$;

2) $N_P^{-2}(A_i) \cap N_P(A_j) = N_Q(A_i) \cap N_Q^{-2}(A_j) = \varnothing$, where $P = a_i^+ \vec{C} h_j$ and $Q = a_j^+ \vec{C} h_i$.

2. Proof of Theorem 6

Let G be a graph of order n with connectivity $\kappa(G) \geq 3$, satisfying $\sigma_4(G) > (4n-4)/3$. Let C be a longest cycle with a given orientation and $|C| \geq 3$ since G is 3-connected. Assuming there is a vertex $u \in V(G) \setminus C$ such that $d(u, C) \geq 3$. By Lemma 2, there is a 3-fan $\mathfrak{B} = \{P_1, P_2, P_3\}$ in G from *u* to $V(C)$ and the length of P_i is at least 3, $i = 1,2,3$. Let $R = V(G) - V(C)$, and H is a component of $G - V(C)$ containing *u*. By the definition of *k*-fan, we get $|H| \geq 7$. Let $x_i = V(P_i) \cap V(C)$, $i = 1,2,3$, a_i is the noninsertible vertex as the same definition on the previous section, $A_i = x_i^+ \vec{C} a_i$, $i = 1,2,3$. And v_i is x_i's first neighbor on the path P_i, $i = 1,2,3$.

Claim 1 $A_i \cap N_C(H) = \varnothing$, $i = 1,2,3$.

proof Without loss of generality, suppose there is a vertex $x \in A_1$ such that $x \in N_C(H)$. Let $Q = x_1^+ \vec{C} x$, $P = x^+ \vec{C} x_1$, then all the vertex on Q are insertible. Thus we can get a (x_1, x)-path P' such that $V(P') = V(C)$ by Lemma 1. Let L denote a (x_1, x)-path with internal vertices in *H*, and $|L| \geq 3$. We can get a new cycle $\mathbb{C} = x_1 P' x L x_1$ longer than *C*. (Contradiction) □

Claim 2 $d(v_1) + d(a_1) + d(a_2) \leq n$.

proof We first find that $\{a_1, a_2\} \cap N_C(H) = \varnothing$ by Claim 1, a_1 and a_2

have no common neighbors on $R-H$ since Lemma 3(1). Thus, $N_R(v_1)$, $N_R(a_1)$, $N_R(a_2)$ are pairwise disjoint. And since $d(u,C)\ge 3$, $uv_1\notin E(G)$. Therefore, we have the inequality as follows.

$$d_R(v_1)+d_R(a_1)+d_R(a_2)\le |R|-2.$$

Similarly, by Claim 1 and Lemma 3(1), we have

$$d_{A_1\cup A_2}(v_1)+d_{A_1\cup A_2}(a_1)+d_{A_1\cup A_2}(a_2)\le |A_1|-1+|A_2|-1.$$

Next let $P=a_1^+\vec{C}x_2$, $Q=a_2^+\vec{C}x_1$, $U_1=A_{v_1}\cap V(P)$, $U_2=A_{v_1}\cap V(Q)$. Since $N_C(v_1)\cap A_i=\varnothing$, $i=1,2,3$, then $A_{v_1}\setminus\{a_1,a_2\}\subseteq U_1\cup U_2$. Note that $d_C(v_1)=|A_{v_1}|$, thus $|U_1|+|U_2|\ge d_C(v_1)-2$. Let $U_1=\{a_{v_{11}},a_{v_{12}},\cdots,a_{v_{1t}}\}$, $t\le n$. We will analyse $d_P(a_1)+d_P(a_2)$ by considering the following cases.

Case 1. For any $a_{v_{1j}}^+\in U_1^+$, $a_{v_{1j}}^+\notin N_P(a_1)$, which implies $a_{v_{1j}}\notin N_P^-(a_1)$, $j=1,2,\cdots,t$.

By Lemma 3(1), we have $a_{v_{1j}}\notin N_P(a_2)$, and thus for any $a_{v_{1j}}\in U_1$, $j=1,2,\cdots,t$, we have $a_{v_{1j}}\notin N_P^-(a_1)\cup N_P(a_2)$. And by Lemma 3(2), $N_P^-(a_1)\cap N_P(a_2)=\varnothing$. Hence $N_P^-(a_1)\cup N_P(a_2)\subseteq V(P)\cup\{a_1\}\setminus U_1$. Therefore,

$$d_P(a_1)+d_P(a_2)=|N_P^-(a_1)|+|N_P(a_2)|\le |P|+1-|U_1|.$$

Case 2. There exist some $a_{v_{1j}}^+\in U_1^+$, $a_{v_{1j}}^+\in N_P(a_1)$, say $\{a_{v_{i1}}^+,a_{v_{i2}}^+,\cdots,a_{v_{ir}}^+\}$, $r\le t$.

Then we can note that $a_{v_{ij}}^{++}\notin N(a_1)$. Since a_1 is noninsertible, which implies $a_{v_{ij}}^+\notin N_P^-(a_1)$, $j=1,2,\cdots,r$, $r\le t$. And by Lemma 4(1), we know $a_{v_{ij}}^+\notin N_P(a_2)$. Thus $a_{v_{ij}}^+\notin N_P^-(a_1)\cup N_P(a_2)$. On the other hand, for the remaining vertices, $a_{v_{1j}}\in U_1\setminus\{a_{v_{i1}},a_{v_{i2}},\cdots,a_{v_{ir}}\}$, similar to case 1, we have $a_{v_{1j}}\notin N_P^-(a_1)\cup N_P(a_2)$. In addition, $a_{v_{1j}}^+\ne a_{v_{1j+1}}$, since there are some $x\in N_C(H)$ on $a_{v_{1j}}\vec{C}a_{v_{1j+1}}$. And by Lemma 3(2), $N_P^-(a_1)\cap N_P(a_2)=\varnothing$. Therefore, we have the inequality as follows.

$$d_P(a_1)+d_P(a_2)=|N_P^-(a_1)|+|N_P(a_2)|\le |P|+1-|U_1|.$$

By Lemma 3(1) and Lemma 4(1), for any $a_{1j}\in U_2$, we have $a_{1j}\notin N_Q(a_1)\cup N_Q^-(a_2)$. So we have the inequality as follows.

$$d_Q(a_1)+d_Q(a_2)=|N_Q(a_1)|+|N_Q^-(a_2)|\le |Q|+1-|U_2|.$$

Therefore,

$$\begin{aligned}
&d(v_1)+d(a_1)+d(a_2)\\
&\le (d_P(v_1)+|P|+1-|U_1|)+(d_Q(v_1)+|Q|+1-|U_2|)+(|A_1|+|A_2|-2)+(|R|-2)\\
&=d_C(v_1)+(|P|+|Q|+|A_1|+|A_2|+|R|)-(|U_1|+|U_2|)-2\\
&\le n+d_C(v_1)-(d_C(v_1)-2)-2\\
&=n.
\end{aligned}$$

□

By a similar argument as Claim 2, Claim 3 holds.

Claim 3 $d(v_1)+d(a_1)+d(a_3)\le n$.

Claim 4 $d(v_1)+d(a_2)+d(a_3)\le n$.

proof Similarly, by Claim 1 and Lemma 3(1), we have

$$d_R(v_1)+d_R(a_2)+d_R(a_3)\le |R|-2.$$

$$d_{A_2\cup A_3}(v_1)+d_{A_2\cup A_3}(a_2)+d_{A_2\cup A_3}(a_3)\le |A_2|-1+|A_3|-1.$$

Let $P=a_2^+\vec{C}x_3$, $Q=a_3^+\vec{C}x_2$. $U_1=A_{v_1}\cap V(P)$, $U_2=A_{v_1}\cap V(Q)$. Then we have $|U_1|+|U_2|\ge d_C(v_1)-2$. And by Lemma 3(2), $N_P^-(a_2)\cap N_P(a_2)=\varnothing$.

Let $U_1=\{a_{v_{11}},a_{v_{12}},\cdots,a_{v_{1t}}\}$, for any $a_{v_{1j}}\in U_1$, we have $a_{v_{1j}}^+\notin N_P(a_2)$ by Lemma 4(1), that is $a_{v_{1j}}\notin N_P^-(a_2)$, $j=1,2,\cdots,t$. And for any $a_{v_{1j}}\in U_1$, we have $a_{v_{1j}}\notin N_P(a_3)$ by Lemma 3(1). Therefore, $a_{v_{1j}}\notin N_P^-(a_2)\cup N_P(a_3)$, $j=1,2,\cdots,t$.

By Lemma 3(2), $N_P^-(a_2)\cap N_P(a_3)=\varnothing$. Hence,

$$N_P^-(a_2)\cup N_P(a_3)\subseteq V(P)\cup\{a_2\}\setminus U_1.$$

Thus we have the inequality as follows,

$$d_P(a_2)+d_P(a_3)=|N_P^-(a_2)|+|N_P(a_3)|\le |P|+1-|U_1|.$$

Furthermore, according to the symmetry of *P* and *Q*,

$$d_Q(a_2)+d_Q(a_3)\le |Q|+1-|U_2|.$$

Therefore,

$$\begin{aligned}
&d(v_1)+d(a_2)+d(a_3)\\
&\le (d_P(v_2)+|P|+1-|U_1|)+(d_Q(v_1)+|Q|+1-|U_2|)+(|A_2|+|A_3|-2)+(|R|-2)\\
&=d_C(v_1)+(|P|+|Q|+|A_2|+|A_3|+|R|)-(|U_1|+|U_2|)-2\\
&\le n+d_C(v_1)-(d_C(v_1)-2)-2\\
&=n.
\end{aligned}$$

□

Claim 5 $d(a_1)+d(a_2)+d(a_3)\le n-4$.

proof Let $P=a_1^+\vec{C}x_2$, $Q=a_2^+\vec{C}x_3$, $M=a_3^+\vec{C}x_1$. By Lemma 3(2) and Lemma 4(2), note that $N_P^{-2}(a_1)$, $N_P(a_2)$, $N_P^-(a_3)$ are pairwise disjoint. So $N_P^{-2}(a_1)\cup N_P(a_2)\cup N_P^-(a_3)\subseteq a_1^-\vec{C}x_2$, which implies

$$d_P(a_1)+d_P(a_2)+d_P(a_3)\le |P|+2.$$

According to the symmetry of *P*, *Q* and *R*, we have

$$d_Q(a_1)+d_Q(a_2)+d_Q(a_3)\le |Q|+2.$$

$$d_M(a_1)+d_M(a_2)+d_M(a_3)\le |M|+2.$$

By Lemma 3(1), we have

$$d_{A_1\cup A_2\cup A_3}(a_1)+d_{A_1\cup A_2\cup A_3}(a_2)+d_{A_1\cup A_2\cup A_3}(a_3)\le |A_1|-1+|A_2|-1+|A_3|-1.$$

At last, by Claim 1 and Lemma 3(1), we have

$$d_R(a_1)+d_R(a_2)+d_R(a_3)\le |R|-|H|.$$

Note that $|H|\ge 7$, thus

$$\begin{aligned}
& d(a_1)+d(a_2)+d(a_3) \\
& \le (|P|+2)+(|Q|+2)+(|M|+2)+(|A_1|+|A_2|+|A_3|-3)+(|R|-|H|) \\
& = (|P|+|Q|+|M|+|A_1|+|A_2|+|A_3|+|R|)+3-|H| \\
& = n+3-|H| \\
& \le n-4.
\end{aligned}$$

□

By Lemma 3(1) and Claim 1, $\{v_1, a_1, a_2, a_3\}$ is an independent set in *G*. By Claim 2-5, we have

$$d(v_1)+d(a_1)+d(a_2)+d(a_3) \le \frac{4}{3}n-\frac{4}{3},$$

a contradiction.

This completes the proof of Theorem 6.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Bondy, J.A. and Fan, G. (1987) A Sufficient Condition for Dominating Cycles. *Discrete Mathematics*, **67**, 205-208. https://doi.org/10.1016/0012-365X(87)90029-X

[2] Broersma, H.J. (1988) Existence of Δ_λ-Cycles and Δ_λ-Paths. *Journal of Graph Theory*, **12**, 499-507. https://doi.org/10.1002/jgt.3190120405

[3] Fraisse, P. (1988) A Note on Distance-Dominating Cycles. *Discrete Math*, **71**, 89-92. https://doi.org/10.1016/0012-365X(88)90033-7

[4] Fang, Y. and Xiong, L. (2021) Circumference of a Graph and Its Distance Dominating Longest Cycles. *Discrete Mathematics*, **344**, Article ID: 112196. https://doi.org/10.1016/j.disc.2020.112196

[5] Bondy, J.A. (1980) Longest Paths and Cycles in Graphs of High Degree. Research Report CORR 80-16, University of Waterloo, Ontario.

[6] Wu, Y., Chen, Y. and Edein Cheng, T.C. (2021) Degree Sums and Dominating Cycles. *Discrete Mathematics*, **344**, Article ID: 112224. https://doi.org/10.1016/j.disc.2020.112224

[7] Bondy, J.A. and Murty, U.S.R. (2008) Graph Theory. Springer, Berlin. https://doi.org/10.1007/978-1-84628-970-5