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Multiplicity Results for Second Order Impulsive Differential Equations via Variational Methods

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Abstract

In this paper we investigate a class of impulsive differential equations with Dirichlet boundary conditions. Firstly, we define new inner product of $H_0^1[0,1]$ and prove that the norm which is deduced by the inner product is equivalent to the usual norm. Secondly, we construct the lower and upper solutions of (1.1). Thirdly, we obtain the existence of a positive solution, a negative solution and a sign-changing solution by using critical point theory and variational methods. Finally, an example is presented to illustrate the application of our main result.

Keywords

Impulsive Differential Equation, Sign-Changing Solution, Critical Point Theory, Variational Method

1. Introduction

This paper is mainly concerned with the following second order impulsive differential equations with Dirichlet boundary conditions

$$\begin{cases} -\left(p(t)x'(t)\right)' + q(t)x(t) = f\left(t, x(t)\right), \ t \in J, t \neq t_i, \\ -\Delta\left(p(t_i)x'(t_i)\right) = \alpha_i x(t_i), \quad i = 1, 2, \cdots, k, \\ x(0) = x(1) = 0, \end{cases} \tag{1.1}$$

where $J \subset [0,1]$, $p \in C^1\left[J, [\beta, +\infty)\right]$, $\beta > 0$, $q \in C\left[J, [0, +\infty)\right]$, $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = 1$, $\alpha_1, \alpha_2, \cdots, \alpha_k$ are nonnegative constants with $\sum_{i=1}^{k} \alpha_i < 4\beta$. $-\Delta\left(p(t_i)x'(t_i)\right) = p\left(t_i^+\right)x'\left(t_i^+\right) - p\left(t_i^-\right)x'\left(t_i^-\right)$. Here $x'\left(t_i^+\right)$ (respectively $x'\left(t_i^-\right)$) denotes the right limit (respectively left limit) of $x'(t)$ at

$t = t_i$, and $f(t,x) \in C[J \times \mathbb{R}]$ is locally Lipschitz continuous for x uniformly in $t \in J \setminus \{t_1, t_2, \cdots, t_k\}$.

The phenomena of sudden or discontinuous jumps are often seen in chemotherapy, population dynamics, optimal control, ecology, engineering, etc. The mathematical model that describes the phenomena is impulsive differential equations. Due to their significance, impulsive differential equations have been developing as an important area of investigation in recent years. For the theory and classical results, we refer to [1] [2] [3] [4]. Considerable effort has been devoted to impulsive differential equations due to their theoretical challenge and potential applications, for example [5]-[12]. We point out that in the motion of spacecraft one has to consider instantaneous impulses depending on the position that result in jump discontinuities in velocity, but with no change in position [13]. The impulses occur not only on the velocity but also in impulsive mechanics [14]. Second order impulsive differential Equation (1.1) of this paper happens to be the mathematical model of this kind of problem.

Motivated by the papers mentioned above, we study the existence of sign-changing solution for second order impulsive differential equations with Dirichlet boundary conditions. However, to the best of our knowledge, there are few papers concerned with the existence of sign-changing solution for impulsive differential equations. In this paper, we obtain the existence of a positive solution, a negative solution and a sign-changing solution of (1.1) by using critical point theory and variational methods. An example is presented to illustrate the application of our main result. In comparison with previous works such as [15] [16], this paper has several new features. Firstly, we consider the eigenvalues and the eigenfunctions for second order linear impulsive differential equations with Dirichlet boundary conditions. Secondly, we construct the lower and upper solutions of (1.1) by using the eigenfunction corresponding to the first eigenvalue. Finally, the existence of sign-changing solution of (1.1) is obtained by using critical point theory and variational methods.

Let H be a Hilbert space and E a Banach space such that E is imbedded in H. Let φ be a C^{2-0} functional defined on H, that is, the differential φ' of φ is locally Lipschitz continuous from H to H. Assume that $\varphi'(x) = x - Bx$ and φ' is also locally Lipschitz continuous as an operator from E to E. Assume also that $K = \{x \in H \mid \varphi'(x) = 0\} \subset E$. For $x_0 \in E$, consider the initial value problem both in H and in E:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -x(t) + Bx(t), \\ x(0) = x_0. \end{cases} \tag{1.2}$$

Let $x(t,x_0)$ and $\tilde{x}(t,x_0)$ be the unique solution of this initial value problem considered in H and E respectively, with $[0,\eta(x_0))$ and $[0,\tilde{\eta}(x_0))$ the right maximal interval of existence. Because of the imbedding $E \mapsto H$, $\tilde{\eta}(x_0) \le \eta(x_0)$ and $\tilde{x}(t,x_0) = x(t,x_0)$ for $0 \le t < \tilde{\eta}(x_0)$. We assume that $\tilde{\eta}(x_0) = \eta(x_0)$ and

$\tilde{x}(t,x_0)=x(t,x_0)$ for $0\le t<\eta(x_0)$, and if $\lim\limits_{t\to\eta(x_0)} x(t,x_0)=x^*$ in H for some $x^*\in E$ then $\lim\limits_{t\to\eta(x_0)} x(t,x_0)=x^*$ in E.

Lemma 1.1 [17] Assume that the statement made in the last paragraph is valid. Assume that φ satisfies the (PS)-condition on H and there are two open convex subsets D_1 and D_2 of E with the properties that $D_1\cap D_2\ne\varnothing$, $B(\partial_E D_1)\subset D_1$, and $B(\partial_E D_2)\subset D_2$. If there exists a path $h:[0,1]\to E$ such that

$$h(0)\in D_1\setminus D_2,\quad h(1)\in D_2\setminus D_1,$$

and

$$\inf_{x\in\bar{D}_1^E\cap\bar{D}_2^E}\varphi(x)>\sup_{t\in[0,1]}\varphi(h(t)),$$

then φ has at least four critical points, one in $D_1\cap D_2$, one in $D_1\setminus\bar{D}_2^E$, one in $D_2\setminus\bar{D}_1^E$, and one in $E\setminus(\bar{D}_1^E\cup\bar{D}_2^E)$. Here $\partial_E D$ and $\bar{D}^E$ mean respectively the boundary and the closure of D relative to E.

Now we state our main result

Theorem 1.1 Assume

(f$_1$) $\lim\limits_{s\to0}\dfrac{|f(t,s)|}{\rho(t)|s|}<\lambda_1$ uniformly in $t\in[0,1]$, where $\rho\in C[J,(0,+\infty)]$,

(f$_2$) there exists a constant $M\ge0$ such that $f(t,s)+Ms$ is creasing in s,

(f$_3$) there exist $\mu>2$ and $r>0$ such that

$$0<\mu F(t,s)\le sf(t,s),\quad \forall t\in[0,1],|s|\ge r,$$

where $F(t,s)=\int_0^s f(t,\tau)\mathrm{d}\tau$.

Then problem (1.1) has at least three solutions: one positive, one negative, and one sign-changing.

2. Preliminaries

Let $H=H_0^1[0,1]$ be the Sobolev space endowed the norm

$$\|x\|_1=\left(\int_0^1 p(t)|x'(t)|^2+q(t)|x(t)|^2\,\mathrm{d}t-\sum_{i=1}^k\alpha_i x^2(t_i)\right)^{\frac{1}{2}},$$

which is equivalent to the usual norm $\|x\|_0=\left(\int_0^1|x'(t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}}$. Let

$$\begin{aligned}E=\Big\{&x\in C[0,1]\,|\,x(0)=x(1)=0,(px)'(\cdot)\in C([0,1]\setminus\{t_1,t_2,\cdots,t_k\}),\\ &x'(t_i^-)=x'(t_i),x'(t_i^+)\exists\Big\}\end{aligned}$$

with the norm

$$\|x\|_E=\max\left\{\max_{t\in[0,1]}|x(t)|,\ \sup_{t\in[0,1]}|x'(t)|\right\}.$$

Clearly E is a Banach space and densely embedded in H.

As is well known, for $\rho \in C\left[J,(0,+\infty)\right]$, the following linear eigenvalue problem

$$\begin{cases} -\left(p(t)x'(t)\right)' + q(t)x(t) = \lambda\rho(t)x(t), \ t \in J, t \neq t_i, \\ -\Delta\left(p(t_i)x'(t_i)\right) = \alpha_i x(t_i), \quad i = 1,2,\cdots,k, \\ x(0) = x(1) = 0, \end{cases}$$

possesses a sequence of positive eigenvalue $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j < \cdots \to \infty$ and the algebraic multiplicity of λ_j is equal to 1. Moreover

$$\lambda_1 = \inf_{x \in H\setminus\{0\}} \frac{\int_0^1 p(t)|x'(t)|^2 + q(t)|x(t)|^2 \,\mathrm{d}t - \sum_{i=1}^k \alpha_i x^2(t_i)}{\int_0^1 \rho(t)|x(t)|^2 \,\mathrm{d}t},$$

the eigenfunction e_1 with respect to λ_1 satisfies $\|e_1\|_1 = 1$, $e_1 > 0$ in $(0,1)$ and the eigenfunctions e_j corresponding to $\lambda_j (j \geq 2)$ are sign-changing in $(0,1)$ and $\|e_j\|_1 = 1$ (see [18]).

Let M be as in (f_2), we define new inner product of H as follows

$$(x,y) = \int_0^1 p(t)x'(t)y'(t) + \left[q(t)+M\right]x(t)y(t)\mathrm{d}t - \sum_{i=1}^k \alpha_i x(t_i)y(t_i). \quad (2.1)$$

The inner product induces the norm

$$\|x\| = \left(\int_0^1 p(t)|x'(t)|^2 + \left[q(t)+M\right]|x(t)|^2 \,\mathrm{d}t - \sum_{i=1}^k \alpha_i x^2(t_i)\right)^{\frac{1}{2}} \quad (2.2)$$

Define the functional $\varphi : H \to \mathbb{R}$ by

$$\varphi(x) = \frac{1}{2}\int_0^1 p(t)|x'(t)|^2 + q(t)|x(t)|^2 \,\mathrm{d}t - \frac{1}{2}\sum_{i=1}^k \alpha_i x^2(t_i) - \int_0^1 F(t,x(t))\mathrm{d}t. \quad (2.3)$$

It is clear that $\varphi \in C^{2-0}(H,\mathbb{R})$. By [12], we know that the solution x of problem (1.1) is equivalent to the critical point of φ, that is $\langle \varphi'(x), y\rangle = 0$ for $y \in H$, where

$$\begin{aligned} \langle \varphi'(x), y\rangle = &\int_0^1 p(t)x'(t)y'(t) + q(t)x(t)y(t)\mathrm{d}t \\ &- \sum_{i=1}^k \alpha_i x(t_i)y(t_i) - \int_0^1 f(t,x(t))y(t)\mathrm{d}t. \end{aligned} \quad (2.4)$$

Lemma 2.1 Two norms $\|x\|$ and $\|x\|_1$ defined on H are equivalent, that is, there exist positive constants γ_1, γ_2 such that

$$\gamma_1\|x\| \leq \|x\|_1 \leq \gamma_2\|x\|, \quad \forall x \in H. \quad (2.5)$$

Proof. For any $x \in H$, by Lemma 2.3 in [18], we have

$$\max_{t\in J} x^2(t) \leq \frac{1}{4}\int_0^1 |x'(t)|^2 \,\mathrm{d}t. \quad (2.6)$$

From this it is easy to see that

$$\|x\|_1^2 \geq \beta\int_0^1 |x'(t)|^2 \,\mathrm{d}t - \frac{1}{4}\sum_{i=1}^k \alpha_i \int_0^1 |x'(t)|^2 \,\mathrm{d}t \geq \left(\beta - \frac{1}{4}\sum_{i=1}^k \alpha_i\right)\|x\|_0. \quad (2.7)$$

Since $p \in C^1\left[J,[\beta,+\infty)\right]$, $q \in C\left[J,[0,+\infty)\right]$, we may assume that $\beta_1 = \max_{t\in J} p(t)$, $\beta_2 = \max_{t\in J} q(t)$. From (2.6) we have

$$\begin{aligned}\|x\|^2 &\le \beta_1 \int_0^1 |x'(t)|^2 \,\mathrm{d}t + \left(\beta_2 + M + \sum_{i=1}^k \alpha_i\right)\int_0^1 |x(t)|^2 \,\mathrm{d}t \\ &\le \left(\beta_1 + \frac{1}{4}\left(\beta_2 + M + \sum_{i=1}^k \alpha_i\right)\right)\|x\|_0.\end{aligned} \tag{2.8}$$

From (2.7) and (2.8), there exists a positive constant γ_1 such that $\gamma_1\|x\| \le \|x\|_1$. On the other hand, it is obvious that $\|x\|_1 \le \|x\|$. So, (2.5) holds.

Let $G(t,s)$ be the Green's function of

$$\begin{cases} Lx = 0, \ t \in J, \\ x(0) = x(1) = 0, \end{cases} \tag{2.9}$$

where $Lx(t) = -\left(p(t)x'(t)\right)' + \left[q(t)+M\right]x(t)$. From [19], we have the following result.

Lemma 2.2 [19] The Green function $G(t,s)$ of (2.9) possesses the following properties

1) $G(t,s)$ can be written by

$$G(t,s) = \begin{cases} \dfrac{m(t)n(s)}{\omega}, & 0 \le t \le s \le 1, \\ \dfrac{m(s)n(t)}{\omega}, & 0 \le s \le t \le 1. \end{cases}$$

2) $m(t) \in C^2([0,1],\mathbb{R})$ is increasing and $m(t) > 0, t \in (0,1]$.

3) $Lm(t) \equiv 0, m(0) = 0, m'(0) = 1$.

4) $n(t) \in C^2([0,1],\mathbb{R})$ is decreasing and $n(t) > 0, t \in [0,1)$.

5) $Ln(t) \equiv 0, n(1) = 0, n'(1) = -1$.

6) $p(t)\left(m'(t)n(t) - n'(t)m(t)\right) \equiv \omega$ is a positive constant.

7) $G(t,s)$ is continuous and symmetrical over $\{(t,s) \mid 0 \le t \le s \le 1\} \cup \{(t,s) \mid 0 \le s \le t \le 1\}$.

8) $G(t,s)$ has continuously partial derivative over $\{(t,s) \mid 0 \le t \le s \le 1\} \cup \{(t,s) \mid 0 \le s \le t \le 1\}$.

9) For fixed $s \in [0,1]$, $G(t,s)$ satisfies $LG(t,s) = 0$ for $t \ne s, t \in [0,1]$ and $G(0,s) = G(1,s) = 0$.

10) G_t' has discontinuous point of the first kind at $t = s$, $G_t'(s+0,s) - G_t'(s-0,s) = -\dfrac{1}{p(s)}$, $s \in (0,1)$.

Define an operator $B: H \to H$ by

$$Bx(t) = \int_0^1 G(t,s)\left(f(s,x(s)) + Mx(s)\right)\mathrm{d}s + \sum_{i=1}^k \alpha_i G(t,t_i)x(t_i). \tag{2.10}$$

Lemma 2.3 [16] $x \in H$ is a critical point of the functional φ if and only if

$x \in E$ is a fixed point of the operator B.

From Lemma 2.3, the critical point set $K = \{x \in H \mid \varphi'(x) = 0\} \subset E$. Notice that $f(t,x) \in C[J \times \mathbb{R}]$ is locally Lipschitz continuous for x uniformly in $t \in J \setminus \{t_1, t_2, \cdots, t_k\}$. It is easy to obtain that B defined by (2.10) is locally Lipschitz continuous both as an operator from H to H and as one from E to E. Let $x_0 \in E$ and consider the initial value problem (1.2) in both H and E.

Similar to Lemma 4.2 in [17], we have

Lemma 2.4 [17] 1) $\tilde{\eta}(x_0) = \eta(x_0)$ and $\tilde{x}(t, x_0) = x(t, x_0)$ for $0 \le t < \eta(x_0)$.

2) if $\lim_{t \to \eta(x_0)} x(t, x_0) = x^*$ in H for some $x^* \in K$ then $\lim_{t \to \eta(x_0)} x(t, x_0) = x^*$ in E.

3. Proof of Theorem

Lemma 3.1 The gradient of φ at a point $x \in H$ can be expressed as $grad\,\varphi(x) = x - Bx$.

This result is necessary, for the reader's convenience we present the proof in the Appendix.

Lemma 3.2 Assume that (f_3) holds, then the functional φ satisfies (PS)-condition.

Proof. Suppose that $\{x_n\}$ is a (PS)-sequence, namely such that for some constant $c > 0$

$$|\varphi(x_n)| \le c, \text{ and } \varphi'(x_n) \to 0 \text{ as } n \to \infty.$$

This implies that there is a constant $C > 0$ such that

$$|\varphi(x_n)| \le C, \text{ and } \left|\langle \varphi'(x_n), x_n \rangle\right| \le C\|x_n\|_1, \quad \forall n \in \mathbb{N}. \tag{3.1}$$

Moreover, thanks to $f(t,x) \in C[J \times \mathbb{R}]$ we know that $f(t,x)$ is bounded on $J \times [-r, r]$. By (3.1) we have

$$\begin{aligned} C(1 + \|x_n\|_1) &\ge \varphi(x_n) - \frac{1}{\mu}\langle \varphi'(x_n), x_n \rangle \\ &= \left(\frac{1}{2} - \frac{1}{\mu}\right)\|x_n\|_1^2 + \int_0^1 \frac{1}{\mu} f(t, x_n(t)) x_n(t) - F(t, x_n(t))\,\mathrm{d}t \end{aligned} \tag{3.2}$$

Then from (f_3) we get

$$\begin{aligned} &\int_0^1 \frac{1}{\mu} f(t, x_n(t)) x_n(t) - F(t, x_n(t))\,\mathrm{d}t \\ &= \int_{J_0} \frac{1}{\mu} f(t, x_n(t)) x_n(t) - F(t, x_n(t))\,\mathrm{d}t \\ &+ \int_{J \setminus J_0} \frac{1}{\mu} f(t, x_n(t)) x_n(t) - F(t, x_n(t))\,\mathrm{d}t \\ &\ge \int_{J_0} \frac{1}{\mu} f(t, x_n(t)) x_n(t) - F(t, x_n(t))\,\mathrm{d}t \ge -\kappa, \end{aligned} \tag{3.3}$$

where $\kappa > 0$ and $J_0 = \{t \in J \mid |x_n(t) \le r|\}$. By (3.2) and (3.3), one has

$$C\left(1+\|x_n\|_1\right) \geq \left(\frac{1}{2}-\frac{1}{\mu}\right)\|x_n\|_1^2 - \kappa,$$

which of course implies that $\{x_n\}$ is bounded by means of Lemma 2.1.

Since H is a reflexive Banach space, we can assume that, up to a subsequence, there exists $x \in H$ such that $x_n \xrightarrow{w} x$. By (2.4) we have

$$\begin{aligned}
&\left\langle \varphi'(x_n)-\varphi'(x), x_n - x\right\rangle \\
&= \int_0^1 p(t)\left|\left(x_n(t)-x(t)\right)'\right|^2 + q(t)\left|x_n(t)-x(t)\right|^2 \mathrm{d}t \\
&\quad -\sum_{i=1}^{k}\alpha_i\left|x_n(t_i)-x(t_i)\right|^2 - \int_0^1\left[f\left(t,x_n(t)\right)-f\left(t,x(t)\right)\right]\left(x_n(t)-x(t)\right)\mathrm{d}t \\
&= \|x_n - x\|_1^2 - \int_0^1\left[f\left(t,x_n(t)\right)-f\left(t,x(t)\right)\right]\left(x_n(t)-x(t)\right)\mathrm{d}t.
\end{aligned} \tag{3.4}$$

By $\varphi'(x_n) \to 0$ and $x_n \xrightarrow{w} x$, we have

$$\left\langle \varphi'(x_n)-\varphi'(x), x_n - x\right\rangle \to 0 \text{ as } n \to \infty. \tag{3.5}$$

By $x_n \xrightarrow{w} x$ in *H*, we see that $\{x_n\}$ uniformly converges to x in $C[0,1]$. it is easy to obtain that

$$\int_0^1\left[f\left(t,x_n(t)\right)-f\left(t,x(t)\right)\right]\left(x_n(t)-x(t)\right)\mathrm{d}t \to 0 \text{ as } n \to \infty. \tag{3.6}$$

Then (3.4)-(3.6) and Lemma 2.1 yield that $\|x_n - x\| \to 0$ in *H*, that is, $\{x_n\}$ strongly converges to x in *H*.

Proof of Theorem 1.1. By (f_1), there exist $0<\sigma<\lambda_1$ and $\delta>0$ such that

$$-\sigma\rho(t)|s| \leq f(t,s) \leq \sigma\rho(t)|s|, \quad \forall t\in[0,1], s\in[-\delta,\delta]\setminus\{0\}. \tag{3.7}$$

Let $v(t) = -\delta e_1(t), w(t) = \delta e_1(t)$, then

$$\begin{aligned}
&-\left(p(t)v'(t)\right)' + q(t)v(t) \\
&= -\delta\lambda_1\rho(t)e_1(t) = \lambda_1\rho(t)v(t) < \sigma\rho(t)v(t) \leq f\left(t,v(t)\right), \ t\in J, t\neq t_i, \\
&\qquad -\Delta\left(p(t_i)v'(t_i)\right) = \delta\Delta\left(p(t_i)e_1'(t_i)\right) = \alpha_i v(t_i), \quad i=1,2,\cdots,k.
\end{aligned}$$

So, we have

$$\begin{cases}
-\left(p(t)v'(t)\right)' + q(t)v(t) = \lambda_1\rho(t)v(t) < f\left(t,v(t)\right), \ t\in J, t\neq t_i, \\
-\Delta\left(p(t_i)v'(t_i)\right) = \alpha_i v(t_i), \quad i=1,2,\cdots,k, \\
v(0)=v(1)=0.
\end{cases} \tag{3.8}$$

Similarly, we also have

$$\begin{cases}
-\left(p(t)w'(t)\right)' + q(t)w(t) = \lambda_1\rho(t)w(t) > f\left(t,w(t)\right), \ t\in J, t\neq t_i, \\
-\Delta\left(p(t_i)w'(t_i)\right) = \alpha_i w(t_i), \quad i=1,2,\cdots,k, \\
w(0)=w(1)=0.
\end{cases} \tag{3.9}$$

Hence, $v(t)$ and $w(t)$ are lower and upper solutions to (1.1), respectively.

Let $D_1 = \{x\in E \mid x > v \text{ in } (0,1)\}$ and $D_2 = \{x\in E \mid x < w \text{ in } (0,1)\}$, it is clear that D_1 and D_2 are open convex sets in E and $D_1 \cap D_2 \neq \varnothing$. If $x \in \partial_E D_1$, then by (f_2)

$$\begin{aligned}Bx(t) &= \int_0^1 G(t,s)\big(f(s,x(s))+Mx(s)\big)\mathrm{d}s+\sum_{i=1}^{k}\alpha_i G(t,t_i)x(t_i)\\ &\geq \int_0^1 G(t,s)\big(f(s,v(s))+Mv(s)\big)\mathrm{d}s+\sum_{i=1}^{k}\alpha_i G(t,t_i)v(t_i)=Bv(t).\end{aligned} \tag{3.10}$$

From Lemma 2.3 and (3.8), we easily know that

$$v(t)=\int_0^1 G(t,s)\big(\lambda_1\rho(s)v(s)+Mv(s)\big)\mathrm{d}s+\sum_{i=1}^{k}\alpha_i G(t,t_i)v(t_i).$$

So,

$$Bv(t)-v(t)=\int_0^1 G(t,s)\big(f(s,v(s))-\lambda_1\rho(s)v(s)\big)\mathrm{d}s>0. \tag{3.11}$$

Since B defined by (2.10) is also an operator from E to E. Hence $Bx\in E$, by (3.10) and (3.11), $Bx>v$ and $B(\partial_E D_1)\subset D_1$. Similarly, $B(\partial_E D_2)\subset D_2$. ($f_3$) implies that there exist two positive constants C_1, C_2 such that

$$F(t,s)\geq C_1|s|^{\mu}-C_2,\quad \forall\, t\in[0,1],\, s\in\mathbb{R}.$$

Let $E_1=span\{e_1,e_2\}$, then E_1 is a finitely dimensional subspace of E, if $x\in E_1$, then we have, for some $C_3>0$,

$$\begin{aligned}\varphi(x) &= \frac{1}{2}\int_0^1 p(t)|x'(t)|^2+q(t)|x(t)|^2\,\mathrm{d}t-\frac{1}{2}\sum_{i=1}^{k}\alpha_i x^2(t_i)-\int_0^1 F(t,x(t))\mathrm{d}t\\ &\leq \frac{1}{2}\|x\|_1^2-C_1|x|_\mu^\mu+C_2\leq\frac{1}{2}\|x\|_1^2-C_3\|x\|_1^\mu+C_2.\end{aligned} \tag{3.12}$$

We define $h_R:[0,1]\to E$ by

$$h_R(s)=Re_1\cos(\pi s)+Re_2\sin(\pi s),$$

where R will be determined later. Then we have

$\|h_R(s)\|_1^2=\|Re_1\cos(\pi s)\|_1^2+\|Re_2\sin(\pi s)\|_1^2=R^2$, by (3.12),

$$\varphi(h_R(s))\leq\frac{1}{2}\|h_R(s)\|_1^2-C_3\|h_R(s)\|_1^\mu+C_2=\frac{1}{2}R^2-C_3R^\mu+C_2.$$

Since

$$\inf_{x\in\bar{D}_1^E\cap\bar{D}_2^E}\varphi(x)>-\infty,$$

we see that

$$h_R(0)\in D_1\setminus D_2,\quad h_R(1)\in D_2\setminus D_1,$$

and

$$\inf_{x\in\bar{D}_1^E\cap\bar{D}_2^E}\varphi(x)>\sup_{s\in[0,1]}\varphi(h_R(s)),$$

if R is sufficiently large. Applying Lemma 2.4 and Lemma 1.1, problem (1.1) has at least four solutions, $x_1\in D_1\cap D_2$, $x_2\in D_1\setminus\bar{D}_2^E$, $x_3\in D_2\setminus\bar{D}_1^E$, and $x_4\in E\setminus(\bar{D}_1^E\cup\bar{D}_2^E)$. It is clear that x_2 is positive, x_3 is negative, and x_4 is sign-changing.

4. An Example

To illustrate the application of our main result we present the following example.

Example 4.1 Consider the following second order impulsive differential equations with Dirichlet boundary condition:

$$\begin{cases} x''(t) = 10x^3 - 2x, \ t \in [0,1], t \neq \dfrac{1}{2}, \\ -\Delta x'\left(\dfrac{1}{2}\right) = x\left(\dfrac{1}{2}\right), \\ x(0) = x(1) = 0, \end{cases} \tag{4.1}$$

Then (4.1) has at least three solutions: one positive, one negative, and one sign-changing.

Proof. It is clear that (4.1) has the form of (1.1). Let $p(t) = \rho(t) = 1$, $q(t) = 0$, $f(t,x) = 10x^3 - 2x$, $k = 1$, $\alpha_1 = 1$ in (1.1). By (1.4) and (2.6), we can obtain that $\lambda_1 = \inf\limits_{x \in H\setminus\{0\}} \dfrac{\int_0^1 |x'(t)|^2 \,\mathrm{d}t - x^2\left(\frac{1}{2}\right)}{\int_0^1 |x(t)|^2 \,\mathrm{d}t} \geq 3$.

Taking $M = 2, \mu = 3$ and $r = 1$, by simple calculations, the conditions in Theorem 1.1 are satisfied.

Hence, (4.1) has at least three solutions: one positive, one negative, and one sign-changing.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Appendix

In this Appendix , for the reader's convenience we give the proof of Lemma 3.1.

Proof of Lemma 3.1. For any $h \in H$,

$$\begin{aligned}
&\frac{\varphi(x+\tau h)-\varphi(x)}{\tau} \\
&=\frac{1}{2}\int_0^1 p(t)\frac{|x'(t)+\tau h'(t)|^2-|x'(t)|^2}{\tau}+q(t)\frac{|x(t)+\tau h(t)|^2-|x(t)|^2}{\tau}\mathrm{d}t \\
&\quad-\frac{1}{2}\sum_{i=1}^k \alpha_i \frac{|x(t_i)+\tau h(t_i)|^2-|x(t_i)|^2}{\tau}-\int_0^1 \frac{F(t,x(t)+\tau h(t))-F(t,x(t))}{\tau}\mathrm{d}t.
\end{aligned}$$

By the Lagrange Theorem there exists θ with $0<\theta<1$ such that

$$\begin{aligned}
\frac{\varphi(x+\tau h)-\varphi(x)}{\tau} &= \int_0^1 p(t)x'(t)h'(t)+q(t)x(t)h(t)\mathrm{d}t-\sum_{i=1}^k \alpha_i x(t_i)h(t_i) \\
&\quad-\int_0^1 f(t,x(t)+\theta\tau h(t))h(t)\mathrm{d}t \\
&\quad+\frac{\tau}{2}\int_0^1 p(t)|h'(t)|^2+q(t)|h(t)|^2\,\mathrm{d}t \\
&\quad-\frac{\tau}{2}\sum_{i=1}^k \alpha_i h^2(t_i).
\end{aligned} \tag{A.1}$$

For any $h \in H$, by (2.1), we have

$$\begin{aligned}
(x-Bx,h) &= \left(x-\int_0^1 G(t,s)(f(s,x(s))+Mx(s))\mathrm{d}s+\sum_{i=1}^k \alpha_i G(t,t_i)x(t_i),h\right) \\
&= \int_0^1 p(t)x'(t)h'(t)+[q(t)+M]x(t)h(t)\mathrm{d}t-\sum_{i=1}^k \alpha_i x(t_i)h(t_i) \\
&\quad-\int_0^1 p(t)\left(\int_0^1 G_t'(t,s)(f(s,x(s))+Mx(s))\mathrm{d}s\right)h'(t)\mathrm{d}t \\
&\quad-\int_0^1 [q(t)+M]\left(\int_0^1 G(t,s)(f(s,x(s))+Mx(s))\mathrm{d}s\right)h(t)\mathrm{d}t \\
&\quad-\int_0^1 p(t)\sum_{i=1}^k \alpha_i G_t'(t,t_i)x(t_i)h'(t)\mathrm{d}t \\
&\quad-\int_0^1 [q(t)+M]\sum_{i=1}^k \alpha_i G(t,t_i)x(t_i)h(t)\mathrm{d}t \\
&\quad+\sum_{i=1}^k \alpha_i\left(\int_0^1 G(t_i,s)(f(s,x(s))+Mx(s))\mathrm{d}s\right)h(t_i) \\
&\quad+\sum_{i=1}^k \alpha_i \sum_{j=1}^k \alpha_j G(t_i,t_j)x(t_j)h(t_i).
\end{aligned} \tag{A.2}$$

By integrating by parts and Lemma 2.2, we can obtain immediately

$$\begin{aligned}
&\int_0^1 p(t)\left(\int_0^1 G_t'(t,s)(f(s,x(s))+Mx(s))\mathrm{d}s\right)h'(t)\mathrm{d}t \\
&=\sum_{i=1}^k \alpha_i\left(\int_0^1 G(t_i,s)(f(s,x(s))+Mx(s))\mathrm{d}s\right)h(t_i) \\
&\quad-\int_0^1\left(p(t)\left(\int_0^1 G_t'(t,s)(f(s,x(s))+Mx(s))\mathrm{d}s\right)\right)'h(t)\mathrm{d}t
\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}&\int_0^1 p(t)\sum_{i=1}^{k}\alpha_i G_t'(t,t_i)x(t_i)h'(t)\mathrm{d}t\\&=\sum_{i=1}^{k}\alpha_i\sum_{j=1}^{k}\alpha_j G(t_i,t_j)x(t_j)h(t_i)-\int_0^1\left(p(t)\sum_{i=1}^{k}\alpha_i G_t'(t,t_i)x(t_i)\right)'h(t)\mathrm{d}t.\end{aligned}\tag{A.4}$$

By Lemma 2.2

$$\begin{aligned}&-\int_0^1\left(p(t)\left(\int_0^1 G_t'(t,s)\left(f(s,x(s))+Mx(s)\right)\mathrm{d}s\right)\right)'h(t)\mathrm{d}t\\&-\int_0^1\left(p(t)\sum_{i=1}^{k}\alpha_i G_t'(t,t_i)x(t_i)\right)'h(t)\mathrm{d}t\\&+\int_0^1\left[q(t)+M\right]\left(\int_0^1 G(t,s)\left(f(s,x(s))+Mx(s)\right)\mathrm{d}s+\sum_{i=1}^{k}\alpha_i G(t,t_i)x(t_i)\right)h(t)\mathrm{d}t\\&=\int_0^1 h(t)L\left(\int_0^1 G(t,s)\left(f(s,x(s))+Mx(s)\right)\mathrm{d}s+\sum_{i=1}^{k}\alpha_i G(t,t_i)x(t_i)\right)\mathrm{d}t\\&=\int_0^1\left(f(t,x(t))+Mx(t)\right)h(t)\mathrm{d}t.\end{aligned}\tag{A.5}$$

Substituting (A.3), (A.4) into (A.2), and using (A.5), one has

$$\begin{aligned}(x-Bx,h)=&\int_0^1 p(t)x'(t)h'(t)+q(t)x(t)h(t)\mathrm{d}t\\&-\sum_{i=1}^{k}\alpha_i x(t_i)h(t_i)-\int_0^1 f(t,x(t))h(t)\mathrm{d}t\end{aligned}\tag{A.6}$$

From (A.1) and (A.6)

$$\left|\frac{\varphi(x+\tau h)-\varphi(x)}{\tau}-(x-Bx,h)\to 0 \text{ as } \tau\to 0\right|$$

From Definition 1.1 and Remarks 1.2 in [20], we conclude that $grad\varphi(x)=x-Bx$.