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Hands-On Equations Balance Model Enhances Algebraic Equation Solving in Upper Elementary and Middle School Students

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Abstract

The struggle of students to solve algebraic equations has been identified as a major challenge in mathematics education. This study investigated the effectiveness of the Hands-On Equations® early algebra balance model in improving the equation-solving skills of upper elementary and middle school students. Fourth-grade (n = 123) and eighth-grade (n = 105) students from the United States participated in this study. A pretest-to-posttest design was used to evaluate the performance of the students on six algebraic equations, including three equations with the unknown on both sides of the equal sign. The results showed that eighth graders outperformed fourth graders on the benchmark pretest. However, after seven lessons using this balance model, the fourth graders showed a statistically significant gain of three standard deviations, outperforming the eighth-grade pretest scores. The model helped the younger students to make sense of formal algebraic notation, the relational meaning of the equal sign, and the subtraction property of equality, which are essential concepts for future algebraic studies. The study also found a statistically significant gain in the eighth graders' performance with a moderate effect size. Therefore, upper elementary and middle school students should use this algebra balance model to enhance their equation-solving ability.

Keywords

Balance Model, Teaching Linear Equations, Early Algebra, Isomorphism, Math Manipulatives, Hands-On Equations

1. Introduction

As observed by the National Mathematics Panel, "many students are woefully unprepared for algebra" (Department of Education, 2008: p. 208) and struggle to solve algebraic equations. Equations such as $4x + 3 = 3x + 9$ and $2(2x + 1) = 3x + 12$ require to students have an understanding of the symbolic notation, the relational meaning of the equal sign, and the ability to work with the unknown—all of which are areas of deficiency cited by the Math Panel. Some of these deficiencies occur early in the student's career. For example, elementary school students may have difficulty understanding the meaning of a term such as "$4x$". They may wonder whether it represents a two-digit number written in base 10 (Herscovics & Linchevski, 1994), in which case if $x = 3$, $4x$ would be 43. Alternatively, the student may think that $4x$ means four plus x (Kieran, 1985), in which case $4x$ would be 7 when x is 3. Expressions such as $2(2x + 1)$ can also be confusing. Many students who try and work with such expressions by memorizing the distributive property forget to distribute to each term inside the parentheses and will incorrectly expand to $4x + 1$ (Booth, 1988; Ncube, 2016).

The equal sign is also a source of confusion for many students. Whereas almost every elementary school student is familiar with the operational meaning of the equal sign, that is, as an indicator that the result is coming next, such as in $3 + 4 = 7$, many are not familiar with the relational meaning of the equal sign. For example, in the problem $5 + 3 = \_\_ + 4$, many students will enter an 8 since that is the sum of the numbers on the left side (Falkner et al., 1999). This limited understanding of the meaning of the equal sign may persist into middle school, high school, and college (Knuth et al., 2008). Although the operational meaning is a legitimate and essential one (Ginsburg, 1996), students who only know this meaning will not realize that the equal sign in an equation such as $4x + 3 = 3x + 6$ indicates that the *total value* of each side of the equal sign is the same (Kieran, 1981). Since the meaning of a sign is arbitrary (Chandler, 2007), students who do not have a meaningful experience with the use of the equal sign in its relational sense will not acquire that understanding (Borenson, 2013; Hornburg et al., 2021; Sherman & Bisanz, 2009).

To solve equations such as $4x + 3 = 3x + 9$ containing the unknown on both sides of the equal sign, students must be able to work with—or on—the unknown (Filloy & Rojano, 1989; Herscovics & Linchevski, 1994). Two possible approaches to solving this equation are transposition and the subtraction property of equality (Hall, 2002; Otten et al., 2019). According to Hall (2002), many students who attempt to solve equations using symbolic manipulation find algorithmic work to be "daunting" (p. 17). For example, the "change side-change signs" rule is open to "oversimplification and abuse" (p. 57), leading to many errors. Hall found nine errors that many students make when solving simple linear equations with symbolic notation (switching addends, deletion errors, combining terms that cannot be combined, etc.). Indeed, when students do not "construct meaning for the new symbolism, they are reduced to performing meaningless operations on symbols they do not understand" (Herscovics & Linchevski, 1994: p. 60), resulting in many mistakes (De Lima & Tall, 2008; Hall, 2002; Kieran, 1985).

An approach often used by educators to assist students in understanding linear equations is the balance model (Otten et al., 2019). There are physical, virtual, and drawn versions of a two-pan balance scale. In their systematic review of the literature on the balance model, Otten et al. (2019) found that in Grades 3 - 6, it is most often used to teach the relational meaning of the equal sign and help students find the missing number in problems such as 8 = ___ + 3 and 4 + 3 = ___ + 2. In Grades 7 and 8, the balance model is used to introduce students to equations such as $ax + b = cx + d$, having the unknown on both sides of the equal sign, where the coefficients, constants, and solutions are non-negative whole numbers (Araya et al., 2010; Boulton-Lewis et al., 1997; Vlassis, 2002).

Boulton-Lewis et al. (1997) used a cups and discs balance model with above-average eighth-grade students. In this model, the unknown is represented by a cup, and a disc represents a unit. Hence, to represent the equation $2x + 5 = 17$, the student places two cups and five discs on the left side of a line serving as a partition and seventeen discs on the right side. The objective is to determine the number of discs in each cup so that both sides have the same number. To solve the equation, students physically remove discs from each side. The students in the Bolton study preferred solving the above equation mentally rather than use the cups and discs. Although the authors attributed student hesitancy to use the materials to cognitive load, there are other explanations. First, since the students could solve this equation mentally, they saw no need to use these materials; secondly, the cups and discs balance model is very cumbersome. For example, the above equation would require 24 objects to represent the problem. This would be an operational, rather than a cognitive load, issue.

A study by Araya et al. (2010) with seventh-grade students did not use any physical props. Instead, half of the group saw a 15-minute video demonstration of the cups and discs balance model employed to solve algebraic equations. In this instance, the cups and discs are illustrated as being placed on the bins of a two-pan stationary balance scale, and the cup's weight is assumed to be zero. The objective is to find the weight of each disc. The other half of the group was presented with a video showing the traditional abstract solution. Both groups were given a posttest shown on a computer screen. All calculations were performed mentally. The group presented with the cups and disc balance model video scored significantly higher than the other group. Furthermore, students with a below-average GPA who had been presented with the balance model videos did as well as the above-average GPA students who were presented with the symbolic notation videos. In considering why the analogies model could have such an immediate and significant positive impact, the authors attributed it to a) the ease with which the mapping could be understood and b) the intuitive nature of the two-pan balance and the principles for maintaining equilibrium. They suggested that the latter may be part of our biological primary knowledge. However, the 15-minute video exposure to the model did not result in any improvement for the lowest-achieving mathematics students.

Vlassis (2002) used a drawn balance model in her study with lower-achieving eighth-grade students. Her objective was to have students transfer the procedure used in simplifying the pictorial equation to the traditional written notation for solving equations. For example, she presents the students with a drawing showing a balance scale with two bins containing images representing weights. On one bin, there is a drawing of two squares with an x inside each of them and a circled 14; on the other, there is a drawing of three squares, each one containing an x, and a circled 8. The students used arrows or cross-outs to remove the same weight from each side. Thereafter, they were presented with equations written in the traditional symbolic notation. Her study showed that the students could transfer the procedures learned with the drawing. Vlassis concluded that the isomorphism between the representation and the equation enabled the students to form an operative mental image that they could readily access, even months after instruction. In particular, the model enabled the students to understand the equality between the two sides of the equation and that removing the same value maintains the balance between the two sides. Whereas the model helped the students apply their learning to equations with positive values, it did not do so with those involving negative values, such as $8x - 5 = 2x + 7$ or $-6x = 24$.

In summary, the studies by Araya et al. (2010) and Vlassis (2002) show that the balance model concept can enable seventh- and eighth-grade students to understand equivalence and the subtraction property of equality and apply those concepts to the solution of linear equations with the unknown on both sides of the equal sign. As noted earlier, students in Grades 3 - 6 can use the balance model to understand equivalence and find the missing number in simple addend problems, such as 8 = __ + 3 (Otten et al., 2019). Students in these grades also understand that removing the same weight from each side of a two-pan balanced system maintains the balance of the system (Brizuela & Schliemann, 2004; Mann, 2004). Taylor-Cox (2003) demonstrated that even 5-year-old children understand the concept of maintaining balance using a two-pan seesaw. As one child noted, "If Alex wants to get off the seesaw, then Angela has two get off too since she is the one who weighs the same" (p. 18).

Consequently, it makes sense to inquire whether upper elementary school students can learn to solve linear equations with the unknown on both sides of the equal sign if they are provided with a concrete version of the balance model *that they can easily manipulate*. In solving those equations, the students would be learning essential algebraic concepts. Further, it would be of interest to explore the effect of the concrete model on the achievement of students in Grade eight since, according to the National Math Panel, many of those students have difficulty with such equations.

2. Development of the New Early Algebra Balance Model

As a mathematics supervisor in the 1980s, the author learned that many algebra students had difficulty in solving algebraic linear equations having the unknown on both sides of the equal sign. He wondered if a hands-on approach could be developed to enable elementary school students to experience success with such equations, thereby enhancing their self-perception as learners and introducing them to powerful algebraic concepts that may pay dividends later on. At the time, the author was aware of the existence of algebra tiles. These are manipulatives intended to model operations with polynomials (Howden, 1985). From attending NCTM conferences, however, the author realized that teachers also used them to model linear equations in Grades 6 - 8 and high school. Algebra tiles are an area-based model: A long green rectangular bar is considered to have a length of x and a width of 1, thereby having a surface area of x; a small 1×1 yellow square has a surface area of 1. Using algebra tiles, the equation $3x + 2 = 2x + 3$ is illustrated as shown in **Figure 1**, with the two sides of the equation representation separated by a vertical line.

The student removes two green bars and two yellow blocks from each side to solve. The remaining setup shows one green bar on the left side of the partition and one yellow square on the right. Notwithstanding the difference in areas between the long green bar and the small yellow square, the student is expected to conclude that in this example, the green bar (or x) has the same value as the yellow square—that is, $x = 1$. The author of this paper did not think that upper elementary-grade students would have an easy time grasping and conceptualizing why two entities of obviously different sizes were equal to each other. Hence, he concluded that algebra tiles would be inappropriate for elementary school students.

The author considered the notation proposed by Sawyer (1960) to be more sensible for introducing algebraic notation to elementary school students. Sawyer suggested a method for teaching algebraic number puzzles to fifth-graders. In this approach, the unknown number is represented by a drawing of a sack containing an unknown number of stones; the drawing of a stone represents each additional unit. With this model, the instruction, "Think of a number. Add 3", would be expressed pictorially as shown in **Figure 2**. Although the author did not find this approach appealing for working with equations, it did demonstrate that young students could understand the concept of an unknown.

The author came across an encouraging statement by Barbel Inhelder, a student of Piaget: "Advanced notions of mathematics are perfectly accessible to children of seven to ten years of age, *provided they are divorced from their mathematical expression and studied through materials that the child can handle*

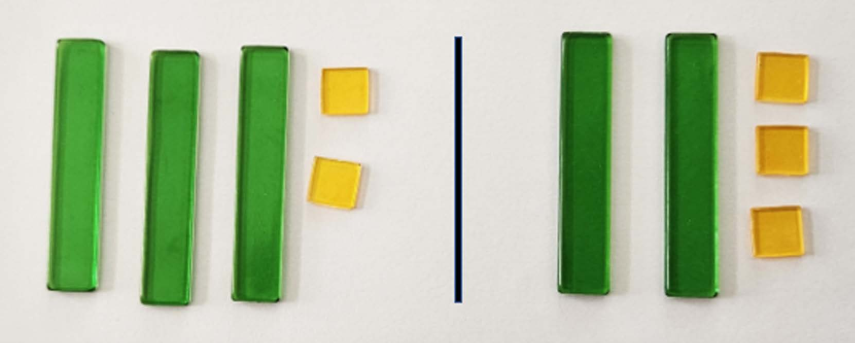

Figure 1. Algebra tiles representation of equation $3x + 2 = 2x + 3$.

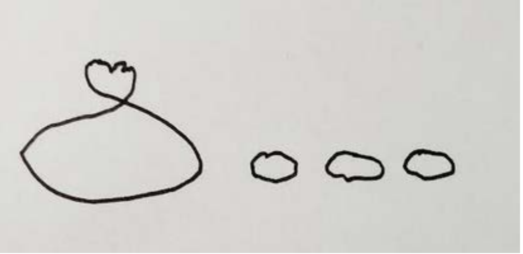

Figure 2. Sawyer's (1960) representation of "Think of a number. Add 3".

himself" (as cited in Bruner, 1960: p. 43, emphasis added). Inhelder was referring to a hands-on isomorphic system that would be the counterpart of the abstract mathematical system. The power of an isomorphic system to enable young students to experience advanced mathematical concepts was further confirmed by Post (1981: p. 112): "An isomorphism is an extremely important concept in mathematics, for if any two systems can be shown to be isomorphic to one another, it becomes possible to work in the simpler and more available system and transfer all conclusions to the less accessible one".

With this perspective in mind, the author undertook a two-year research and experimentation process seeking to develop an isomorphic manipulative system enabling students as early as the third grade to solve equations with the unknown on both sides of the equal sign. He wanted the system to work with equations containing the terms x and/or $-x$ and positive and/or negative constants. Upon completing the instructional system, the author applied for and was granted a patent (Borenson, 1986a). The system, known as Hands-On Equations (Borenson, 1986b), consists of a series of sequential lessons and the accompanying manipulatives, as described below.

In this instructional system, concrete objects and physical actions are the counterparts of abstract symbols and mathematical processes. A game piece, namely a blue pawn, represents the unknown x, and another game piece, namely a white pawn, represents $(-x)$. The system includes eight blue pawns and eight white pawns. Red-numbered cubes represent positive constants, and green-numbered cubes represent negative constants. The system has two red cubes numbered 0 - 5 and two numbered 5 - 10; the green cubes are similarly numbered. This innovation for representing the constants makes it possible, for example, to have the constant of 9 represented by just one game piece, namely the red cube displaying the number 9, thereby simplifying the representation of the constant compared to the Sawyer or algebra tile model. Furthermore, the author wanted to use a balance to model the two sides of the equation, but he did not want students to rely on a moving balance to determine whether their solutions were correct. Hence, he decided to use a flat laminated scale for the student and a three-dimensional stationary balance scale for the teacher. **Figure 3** shows the physical representation of equation $4x + 5 = 2x + 13$.

In this mapping, the "$4x$" is translated to the placement of four blue pawns on the left side of the balance scale. The plus sign followed by the 5-constant is an instruction to place on the same side of the balance a red cube displaying the number 5. The equal sign is an instruction to continue the setup on the other side of the scale. Once the setup is completed, to solve the equation, the student performs "legal moves", that is, moves that maintain the theoretical balance of the system. In this example, the student simultaneously removes one pawn from each side of the balance scale, as shown in Figure 4. The student does so again, and then removes a 5-value from the cubes on each side. This leaves two pawns on the left side of the scale and a value of 8 on the cube(s) on the right. At this point, the student realizes that the value of the pawn is 4, since 4 + 4 = 8. The student writes the solution as $x = 4$. The check value of 21 = 21 is obtained by evaluating the original physical setup shown in Figure 3 when the pawn has the value of 4.

The first six lessons are presented with manipulatives. The seventh lesson transitions to a pictorial representation of the concrete solution using only paper and pencil. Whether using the hands-on or the pictorial solution (see **Appendix**), the isomorphic solution process is the same: the student translates or maps the given abstract equation into its concrete or pictorial representation. Next, legal moves are made to simplify the setup and thereby solve for the value of the pawn or

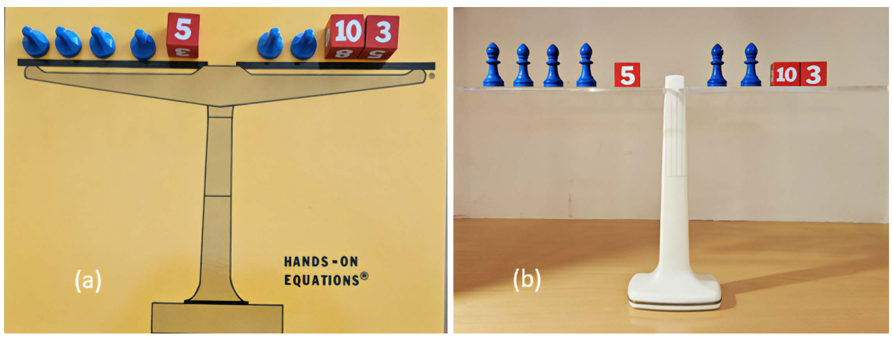

Figure 3. Hands-On Equations representation for $4x + 5 = 2x + 13$ is shown on the (a) student laminated scale and (b) teacher stationary balance scale.

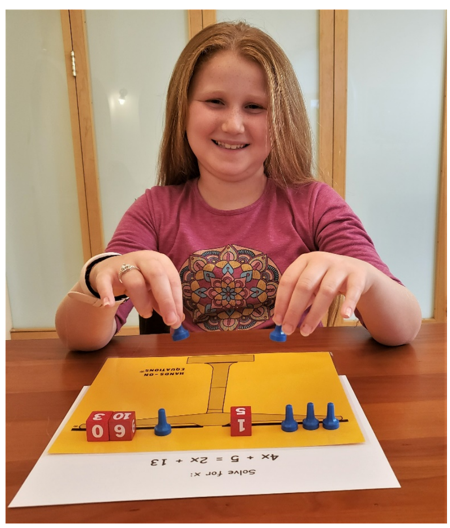

Figure 4. This student demonstrates the legal move of simultaneously removing a pawn from each side of the balance scale.

shaded triangle that will make both sides "balance". That value will be the value of x that solves the given abstract equation. The process is shown schematically in Figure 5.

Research Objectives

This study with fourth- and eighth-graders aimed to achieve three research objectives: first, it aimed to determine the extent to which the seven-lesson instructional treatment improved student performance from the pretest to the post-tests with and without manipulatives; secondly, it aimed to compare the performance of each grade group at the three test points; and lastly, it aimed to assess the pre- and posttest performance of the fourth-grade students using the eighth-grade pretest as a point of reference or benchmark. These objectives were accomplished by addressing the research questions outlined in Table 1.

Since the author of this paper is the author/inventor of this early algebra

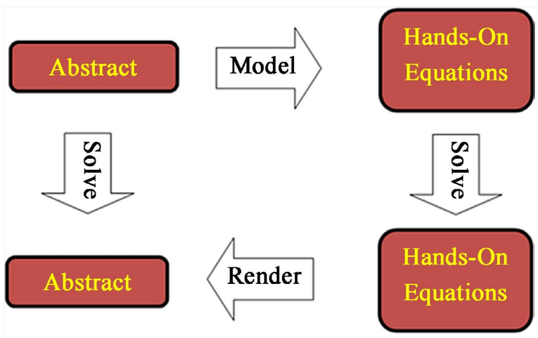

Figure 5. Isomorphic modeling. An abstract algebraic linear equation is mapped into and solved physically or pictorially within the isomorphic world of Hands-On Equations. The value obtained for the pawn will be the value of x that solves the given abstract equation.

Table 1. List of the research questions.

Research Questions
RQ1) Does the mean value of the algebra test change significantly between any of these pairwise steps of lessons? • Fourth grade students: L1 vs. L6; L1 vs. L7; L6 vs. L7 • Eighth grade students: L1 vs. L6; L1 vs. L7; L6 vs. L7
RQ2) Is there a significant difference on the algebra test at any of these paired test points? • Fourth grade pretest (L1) vs. Eighth grade pretest (L1) • Fourth grade posttest (L6) vs. Eighth grade posttest (L6) • Fourth grade posttest (L7) vs. Eighth grade posttest (L7)
RQ3) Is there a significant difference on the algebra test at any of these paired test points? • Fourth grade posttest (L6) vs. Eighth grade pretest (L1) • Fourth grade posttest (L7) vs. Eighth grade pretest (L1)

program, and obtains compensation from its sale, he needed to take steps to minimize the possibility of bias entering the study, as noted in Conflicts of Interest section of this paper.

3. Materials and Methods

3.1. Participants

A total of 228 students from eleven classes participated in this study. **Grade 4:** 123 students from six fourth-grade classes from a large district in the south-eastern United States. Three classes were from urban schools, and three were from suburban schools. **Grade 8:** 105 students in total. Thirty-four students from two rural classes in Kentucky were taught by the same teacher and 71 from three suburban classes, one in each of the following states: Illinois, Maryland, and Missouri. The instructional program was delivered to the students by their regular classroom teacher. All the teachers in this study had at least three years of teaching experience; however, this was their first time teaching this algebra program.

The six elementary teachers had responded to a call from their district seeking teachers to participate in the study. The teachers received a one-day onsite training session on using the program. Each teacher received a class set of materials. The five eighth-grade teachers were sent to a public workshop and were also provided with a class set of materials. The teachers were informed of the teaching and testing protocol and provided with a summary data sheet on which to enter student test results, coded to hide student identity. It was also made clear that no teacher would be individually identified in the study report. At the end of the testing, the math coordinator collected and sent the completed elementary data sheets to the researcher for statistical analysis, as did each eighth-grade teacher.

3.2. Teaching Procedure

As the research protocol specified, the teachers followed the instructional manual for Level 1 in presenting their lessons. The first six lessons involved the use of teacher and student manipulatives. The teacher employed the stationary demonstration scale and game pieces for the first six lessons with the specific teaching example specified in the manual. Next, students were asked to apply the learned strategy to a practice problem. This was followed by a class discussion or student presentation of the solution. This process was repeated a second time, after which the students were given a worksheet containing ten examples: four on the new work and six on prior lessons. A seventh lesson without manipulatives followed the six manipulative lessons. That lesson taught the students a pictorial solution method involving only paper and pencil. Each of the seven lessons, including the time needed to work on the worksheet, was completed within a class period of 50 minutes.

3.3. Tests and Testing Procedure

Each fourth- and eighth-grade class took a total of three tests. A pretest (X1) was taken before instruction on the program. After completing the first six lessons using the manipulatives, the students took posttest Lesson 6 (X6) with the manipulatives. After learning the pictorial notation in Lesson 7, they took Lesson 7 posttest (X7) without using the manipulatives. The first two test items were purposely selected to be relatively easy so that the fourth-grade students would experience some level of success on the pretest and thereby not be unduly intimidated by the algebraic notation. The remaining four test items were progressively more difficult. Five versions of this test were designed, with each differing only in the value of one or more constants. Three of these versions were selected by tossing a numbered cube as the testing instrument, as shown in **Table 2**.

The students were given 15 minutes to complete each test. Although the test form had a field for students to enter the value of the check, the rating of each test paper was based exclusively on the value provided for x. Each of the six test items was assigned one point regardless of its level of difficulty.

3.4. Statistical Analysis

The data were analyzed using SPSS 28.01. Starting with the assumption of normality, the rules of thumb for medium-sized samples ($50 < n < 300$) provided by Kim (2013) were followed. This means that skewness and kurtosis z-values > |3.29| were considered to reflect significant deviations from normality. Because deviations from normality were found, it was decided to use non-parametric tests. Moreover, given the medium sample size, it was decided to use a conservative significance level of .01 to avoid type I errors.

To meet the first research objective, Friedman tests were carried out to verify whether the test results within each grade changed from pretest to posttest (with or without manipulatives) (Field, 2018). Kendall's W coefficient was used to measure effect size for the overall test, where values closer to 1 reflect stronger effects (Tomczak & Tomczak, 2014). For the pairwise comparisons, effect sizes were calculated by point-biserial correlations (Field, 2018). These were calculated by dividing the standardized test statistic by the square root of the total observations (Field, 2018; Tomczak & Tomczak, 2014). For interpretation, $r < .10$ is considered a small effect, values around .30 a moderate effect, and

Table 2. The six equations of each test.

Test Label Test Name	X1 Benchmark Pretest	X6 Lesson 6 Posttest	X7 Lesson 7 Posttest
Item #1	$2x = 8$	$2x = 10$	$2x = 6$
Item #2	$x + 3 = 8$	$x + 3 = 8$	$x + 3 = 10$
Item #3	$2x + 1 = 13$	$2x + 2 = 12$	$2x + 1 = 7$
Item #4	$3x = x + 12$	$3x = x + 4$	$3x = x + 2$
Item #5	$4x + 3 = 3x + 6$	$4x + 3 = 3x + 9$	$4x + 3 = 3x + 7$
Item #6	$2(2x + 1) = 2x + 6$	$2(2x + 1) = 2x + 8$	$2(2x + 1) = 2x + 10$

$r > .50$ a large effect (Field, 2018).

To answer the second research objective, the fourth- and eighth-grade scores were compared at each moment of testing. For this purpose, Kruskal Wallis tests were used, followed by post hoc pairwise comparisons (Field, 2018). Point-biserial correlations were used for these latter tests to measure effect size.

To answer the third research objective, the posttest scores of the fourth graders were compared against the eighth-grade pretest score. For this purpose, Mann-Whitney U tests were used with point-biserial correlations to measure effect size.

4. Results

Table 3 shows the overall descriptive statistics. There was negative skewness (i.e., more values in the higher ranges) and positive kurtosis (i.e., a leptokurtic or heavy-tailed distribution), particularly on the post-tests. For this reason, non- parametric testing was used.

4.1. Within Grade Changes in Test Scores (Research Objective 1)

The average test results are separately visualized in Figure 6 for Grades 4 and 8. Rank-based Friedman tests were used to verify the effect of the manipulative

Table 3. Descriptive statistics (N = 228).

	Min	*Max*	*M*	*SD*	*Skewness*	*Kurtosis*
Pretest	0.00	6.00	2.77	1.73	0.36	−0.66
Posttest with manipulatives	0.00	6.00	5.14	1.27	−1.75	2.76
Posttest without manipulatives	1.00	6.00	5.33	1.05	−1.87	3.35

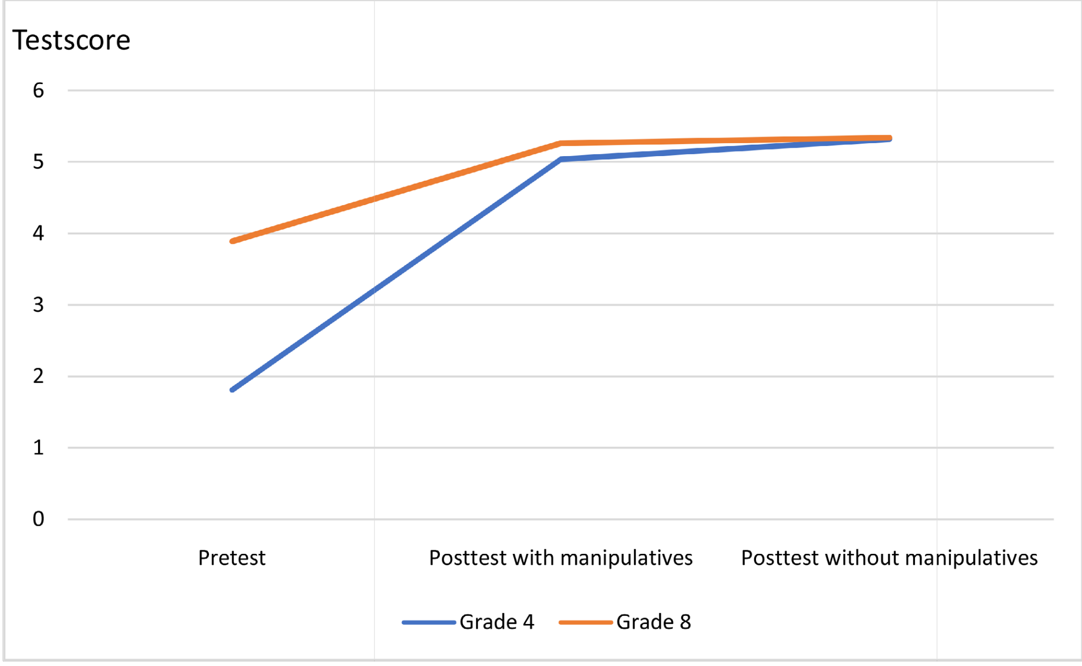

Figure 6. Average test scores on the pretest and post-tests by grade.

instruction in the two grades separately. In both grades, a significant main effect was found, with χ^2 (2, N = 123) = 197.63, $p < .001$ for the fourth grade and χ^2 (2, N = 105) = 89.15, $p < .001$ for the eighth grade. Especially in Grade 4, there was a large effect size of $W = .80$ versus a medium effect size of $W = .38$ in Grade 8.

Post hoc Bonferroni corrected pairwise comparisons revealed that in Grade 4, the students had significantly lower pretest scores ($M = 1.81$, $SD = 1.10$, *median* = 2) compared to their posttest scores ($p < .001$ for both comparisons), while the posttest scores with manipulatives ($M = 5.04$, $SD = 1.32$ *median* = 5) and without manipulatives ($M = 5.32$, $SD = 0.97$, *median* = 6) were similar, $p = .214$. The effect size for the pre-posttest comparison was $r = .68$ for the posttest with manipulatives and $r = .76$ for the posttest without manipulatives, indicating large effect sizes, equivalent to more than three standard deviations. Similarly, in Grade 8, there was no difference between the two post-tests with ($M = 5.26$, $SD = 1.20$, *median* = 6) and without manipulatives ($M = 5.34$, $SD = 1.15$, *median* = 6), $p = .225$. Both posttest scores were significantly higher than the pretest score ($M = 3.89$, $SD = 1.67$, *median* = 4), $p < .001$. The pre-post effects found among the eighth graders were more moderate, with r = .41 and .49 for the post-tests with and without manipulatives, respectively, showing an almost one standard deviation increase.

In these analyses, all of the fourth graders were considered as one group, as were all of the eighth graders, not considering potential differences between the urban, rural, and suburban classes. Concerning the fourth graders, either urban or suburban students, there was no significant difference on the pretest, $U = 1659$, $p = .229$, $r = .09$; therefore, they were considered as one homogeneous group. However, for the eighth graders who were either from rural classes or suburban classes, there was a significant difference, $U = 491.5$, $p < .001$, $r = .49$. The suburban students had pretest scores ($M = 4.44$, $SD = 1.56$, *Median* = 5.00) that were substantially higher than those of the rural students ($M = 2.74$, $SD = 1.26$, *Median* = 3.00). The average and median test scores of each of the three groups are presented in **Table 4**.

Even though significant effects for both the suburban and rural eighth graders were found when comparing their posttest scores to the pretest scores, it is noted that the effect size was larger for the rural students, with χ^2(2, N = 34) = 36.56,

Table 4. Test scores of the Grade 4, Grade 8 rural, and Grade 8 suburban students.

Group	Pretest	Posttest with Manipulatives	Posttest without Manipulatives
Grade 4 (123)	$M = 1.81$, $SD = 1.10$, *Median* = 2	$M = 5.04$, $SD = 1.32$ *Median* = 5	$M = 5.32$, $SD = 0.97$, *Median* = 6
Grade 8 rural (34)	$M = 2.74$, $SD = 1.26$, *Median* = 3.00	$M = 5.15$, $SD = 1.10$, *Median* = 5.00	$M = 5.03$, $SD = 1.38$, *Median* = 6.00
Grade 8 suburban (71)	$M = 4.44$, $SD = 1.56$, *Median* = 5.00	$M = 5.31$, $SD = 1.25$, *Median* = 6.00	$M = 5.49$, $SD = 1.00$, *Median* = 6.00

$p < .001$, $W = .54$ versus $\chi^2(2, N = 71) = 53.63$, $p < .001$, $W = .38$ for the suburban students. More specifically, for the suburban eighth graders, it was found that the pretest scores increased by about half a standard deviation, $p < .001$, $r = .33$ and $p < .001$, $r = .44$ to the posttest with and without manipulatives, respectively. For the rural eighth-graders, the pretest scores increased by almost two standard deviations, with $p < .001$, $r = .57$, and $p < .001$, $r = .58$ to the posttest with and without manipulatives, respectively.

4.2. Comparison of Grade 4 and Grade 8 Pretest and Posttest Scores (Research Objective 2)

Comparing the test scores of the Grade 4 students, Grade 8 suburban students, and Grade 8 rural students, it was found that on the pretest, the three groups showed a significant difference, $H(2) = 96.41$, $p < .001$. All three pairwise comparisons were significant at $p < .001$. The rural eighth graders' lower scores than the suburban eighth graders have already been discussed. In line with this difference, it was further found that while the fourth graders had lower pretest scores than the rural eighth graders, this was a moderate effect ($r = .26$). In contrast, the difference with the eighth-grade suburban students was huge ($r = .70$). On the post-tests, on the other hand, the scores of the three groups were more similar, $H(2) = 5.36$, $p = .068$ and $H(2) = 6.27$, $p = .043$ for the posttest with and without manipulatives respectively.

4.3. Comparison of Grade 4 Posttest Scores to Grade 8 Pretest Scores (Research Objective 3)

Research Objective 2 already noted that the pretest scores of the eighth-grade group were significantly larger than the fourth-grade pretest scores. Research Objective 1 noted that the instructional program led to a gain of more than three standard deviations for the fourth-grade group. In consideration of this large gain, a test was conducted to determine whether the posttest scores of the fourth-graders were significantly greater than the eighth-grade pretest scores. This was indeed the case, with $U = 3767$, $p < .001$, $r = .37$ for the test with manipulatives, and $U = 3193.5$, $p < .001$, $r = .45$ for the posttest without manipulatives. These significant differences were also found when comparing the fourth-grade posttest scores to the initial scores of the suburban eighth-grade scores only, with $U = 3346.5$, $p = .004$, $r = .21$ for the test with manipulatives and $U = 2959.5$, $p < .001$, $r = .29$ for the posttest without manipulatives Although the eighth-grade rural group increased their pretest scores by two standard deviations, their post-test scores were not significantly larger than the eighth-grade suburban pretest scores, with $U = 1514.5$, $p = .028$, $r = .21$ for the test with manipulatives and $U = 1496$, $p = .038$, $r = .20$ for the posttest without manipulatives.

In summary, the instructional program proved effective for all three groups, as evidenced by significant gains from the pretest to the post-tests, with and without manipulatives. Additionally, while the eighth-grade suburban groups outperformed both the fourth-grade group and, to a lesser extent, the eighth-grade rural group on the benchmark pretest, the posttest scores of all three groups became more similar after receiving the instructional treatment. Furthermore, the fourth-grade group demonstrated an increase in performance that enabled them to achieve posttest scores, with or without manipulatives, surpassing the pretest scores of the much stronger eighth-grade suburban group.

5. Discussion

This study sought to determine if the Hands-On Equations balance model could enhance the ability of fourth- and eighth-grade students to solve linear equations, including equations such as $4x + 3 = 3x + 9$ and $2(2x + 1) = 3x + 12$, where the coefficients, constants, and solutions are non-negative whole numbers, and where the unknown occurs on both sides of the equal sign. The statistical analysis showed that this was indeed the case. An item analysis can provide insight into the extent to which the instructional program improved group performance on each equation. As the eighth-grade suburban group performed significantly better than the eighth-grade rural group on the pretest, comparing the fourth-grade group to the eighth-grade suburban group on the pretest and the posttest without manipulatives will provide a better measure of the effectiveness of the instructional treatment on the younger students. **Table 5** provides an item analysis showing how the fourth-graders and eighth-grade suburban students performed on each equation, pre- and post-instruction.

The fourth-grade group made large gains on each of the six test items. This discussion will begin with the first three, as these did not require knowledge of the relational meaning of the equal sign nor the ability to work with the unknown. Since 38% of the fourth-grade group did not provide a correct response to $2x = 8$, it is evident that these students were confused about the meaning of the concatenated term "$2x$". The isomorphic mapping attends to this issue by instructing students to read a term such as $2x$ as "two x's" and to represent it by two blue pawns. Since $2x = x + x$, this approach is consistent with the definition of multiplication by a whole number. Even though the third test item, $2x + 1 = 13$, only required students to implement in two steps the knowledge needed to solve the

Table 5. Percentage of the fourth-grade group and eighth-grade suburban group that had the specified item correct on the pretest or the comparable item on the Lesson 7 posttest taken without manipulatives.

Pretest Test Item	Pretest: Grade 4	Pretest: Grade 8 Suburban	Posttest: Grade 4	Posttest: Grade 8 Suburban
1. $2x = 8$	62%	94%	94%	96%
2. $x + 3 = 8$	80%	94%	97%	100%
3. $2x + 1 = 13$	22%	89%	91%	94%
4. $3x = x + 12$	10%	55%	91%	93%
5. $4x + 3 = 3x + 6$	2%	61%	89%	87%
6. $2(2x + 1) = 2x + 6$	5%	51%	69%	79%

first two test items, they had much more difficulty with it on the pretest. The visualization provided by the model of having two pawns and a 1-cube on one side of the balance and a 7-cube on the other side enabled the large majority of the fourth-graders to make sense of the comparable posttest item and thereby provide a correct response, as shown in **Table 5**.

On the benchmark pretest, fewer than 10% of the fourth graders could solve any of the last three equations. These contained more than one term on the right side of the equal sign, and the unknown on both. Solving these equations involves understanding the relational meaning of the equal sign. For example, the equation $3x = x + 12$, requires students to recognize that both sides of the equal sign have the same value. In the Hands-On Equations balance model, the relational meaning of the equal sign is implicit to the students: as both sides of the balance scale have the same value, so do both sides of the equal sign (Lehtonen & Joutsenlahti, 2017; Suh & Moyer-Packenham, 2007; Vlassis, 2002). Indeed, in no sense can one side of the balance scale be perceived as the result of the operations on the other side (Pirie & Martin, 1997). Furthermore, when students conduct their check after finding the value for the unknown, they physically reset the problem and evaluate both sides to see whether they have the same value, once again reinforcing the relational meaning of the equal sign (Hall, 2002).

Solving the equation $3x = x + 12$ also requires the ability to work with the unknown. Over half of the eighth-grade suburban group provided a correct response to this item on the pretest, showing that they had been instructed in methods for solving such equations. Nonetheless, 45% of them did not experience success. On the other hand, with this algebra balance model, once the student has represented the equation using the game pieces on both sides of the balance scale, the student understands that physically removing a pawn from each side of the balance scale maintains the balance of the system (Lehtonen & Joutsenlahti, 2017; Suh & Moyer-Packenham, 2007; Vlassis, 2002). After doing so, the student is left with two pawns on the left side and cubes displaying the numbers 10 and 2 on the right side. Consequently, the student sees that removing x from each side of the equation $3x = x + 12$ leaves the simplified equation $2x = 12$. Hence, physical actions automatically yield results that would otherwise need to be determined mentally or by using symbolic manipulation (Araya et al., 2010). Also, errors typically made in combining unlike terms when working symbolically are avoided by this algebra model since two non-commensurate objects are used, namely a blue pawn representing the unknown x and numbered cubes representing the constants. It is self-evident to students that these cannot be combined into one entity (Hall, 2002; Lehtonen & Joutsenlahti, 2017). Of the eighth-grade rural students, the percentage of students who correctly responded to $3x = x + 12$ increased from 21% on the pretest to 91% on the Lesson 7 posttest for a similar item. The corresponding percentages for this group on test item $4x + 3 = 3x + 6$ were 18% and 85%. Hence, the seven-lesson instructional treatment enabled practically the same percentage of each of the three groups to answer these two items correctly on the posttest without manipulatives, notwithstanding the considerable advantage exhibited by the eighth-grade suburban group over the fourth-grade group and the eighth-grade rural group, as reflected by the pretest scores of the three groups.

The last item on the pretest involved the equation $2(2x + 1) = 2x + 6$. This was the most challenging test item as it also required knowledge of the distributive property or an understanding of the meaning of the left side of the equation. This algebra model enabled students to solve this equation by providing *meaning* for the expression on the left: They learn that the "2" outside and next to the parentheses in an expression such as "$2(2x + 1)$" is understood as an instruction to set up what is inside the parentheses two times. In this example, they set up two pawns and one 1-cube and then do so a second time. The net result will be to set up four pawns and two 1-cubes, which is the equivalent of the representation for $4x + 2$. Indeed, without mentioning the distributive property, some students will mentally combine the two 1-cubes and represent the expression with six pawns and a 2-cube, thereby learning this property independently (Borenson, 1987). The percentage of the eighth-grade rural group that answered this item correctly increased from 18% on the pretest to 65% on the posttest.

The limited ability of fourth graders to solve equations is reflected in the pretest scores of the fourth graders in this study. Only 22% of them were successful in providing the correct answer to the third test item, $2x + 1 = 13$, which only required a basic understanding of symbolic notation and simple mental arithmetic. However, following the seven instructional lessons, each of the first five test items was answered correctly by at least 89% of the fourth graders. It is interesting to note that the percentage of these students providing a correct item response on the pretest ranged from 2% to 80%. Furthermore, there was a negligible difference of no more than 3% in the percentage of the fourth graders and eighth-grade suburban students successfully answering the first five items on the posttest taken without manipulatives. However, on the last test item which involved parentheses, there was a noticeable but still small difference of 10% between the two groups, with the eighth-grade suburban group having the larger percentage.

6. Conclusion

The Hands-On Equations isomorphic mapping of the abstract algebraic equation into its concrete or pictorial representation of weights on an image of a balance scale enabled the students to work in a concrete or visual system and transfer all of their conclusions to the abstract equations. The use of numbered cubes enabled the students to have a compact and simplified representation of equations. For example, only seven blue pawns and two numbered cubes are needed to represent the equations $4x + 3 = 3x + 9$. The two sides of the balance scale conveyed the notion of equality between the two sides of the equal sign. The legal move of removing a pawn from each side of the balance scale enabled the students to physically or pictorially implement the subtraction property of equality. These factors led to an *efficacious* use of manipulatives for representing and solving linear equations.

The success of the seven instructional lessons in enabling the fourth graders to gain more than three standard deviations from the pretest to the posttest taken without manipulatives is a strong endorsement for introducing algebraic equations with the unknown on both sides of the equal sign to upper elementary students. The fact that six regular elementary school teachers with no particular mathematics background could successfully teach these lessons adds to its value. This balance model helps students understand the symbolic notation, the relational meaning of the equal sign, and the subtraction property of equality, which are essential concepts for future algebraic studies. This instructional strategy also helped the eighth graders, especially those from the rural schools, who gained more than two standard deviations. Thus, the Hands-On Equations balance model has the potential to enhance the equation-solving skills of upper elementary and middle school students, including those who may be struggling with these fundamental concepts.

Limitations

A limitation of this study is that the participating teachers volunteered to do so and therefore may have been more interested in implementing an approach involving manipulatives. Another limitation is that they all attended a workshop at which the instructor modeled each lesson and the teachers played the role of the students. This hands-on experience with the program likely led to the teachers having a high level of confidence in their ability to teach the program, which may have affected program outcome. Hence, any replication of this study should include a hands-on introduction of Level I to the participating teachers.

Postscript

De Lima and Tall (2008) assert that the balance model embodiment, although helpful with positive values, can impede working with subtraction or negative values (p. 6). Other educators have expressed similar concerns (Boulton-Lewis et al., 1997; Pirie & Martin, 1997; Vlassis, 2002). The Hands-On Equations balance model, however, enables students to solve algebraic linear equations containing subtraction and negative values through the progression of lessons. Initially, students are taught to consider the game pieces as having theoretical weights. Once students have learned the need to maintain the balance between both sides, they begin to consider the game pieces as having *values* rather than weights. At that point, the balance scale serves only as a *mnemonic* of the need to maintain the balance with each legal move. For example, just as a student may add a blue pawn to each side of the balance, they may add a white pawn, representing ($-x$), to each side; just as they may remove a red 5-cube from each side, they may remove a green 5-cube, representing a negative 5, from each side. Also, students may place on (or remove from) either side of the balance scale a blue and a white pawn, since together they have a value of zero. Hence, to represent the subtraction of 10, for example, students first place a red 10-cube and a green 10-cube on the same side of the scale—since together they have a value of zero—and then remove the red 10-cube. At the following link, the reader will find a video solution of a 3rd-grade gifted student solving the equation $4x - 2(-x) + (-2) = 2(-x) - 10$ using this early algebra balance model: https://youtu.be/Cvs_hYlTdo4.

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Data Availability Statement

The data set for this study is available in the Dryad repository at this link: https://doi.org/10.5061/dryad.s7h44j13w.

Conflicts of Interest

The author of this article is the author/inventor of the Hands-On Equations instructional system. He is also the President of Borenson and Associates, Inc., the company that markets the program to school districts in the United States. Therefore, in conducting this study, he needed to minimize the possibility of bias entering the study. Moore and McGoff (2019) recommend that a researcher with a conflict of interest (COI) can minimize bias by reducing their roles in the research, such as minimizing their interaction with the subjects, which in this case were students and teachers. They also recommended that an outside third party conduct the data analysis, "so the researcher with the COI would not be able to introduce bias into the analysis". The author implemented these recommendations, determining that the onsite teachers would conduct classroom instruction, testing, and data collection. A third party would conduct the statistical analyses, as noted in the Acknowledgments section of this paper.

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Appendix: The Pictorial Notation

Lesson 7 of the Hands-On Equations program transitions students from the concrete solution using pawns and cubes to a pictorial solution using shaded triangles and boxed numbers to resemble the concrete model. **Figure A1** illustrates this process with the equation $4x + 3 = 3x + 9$.

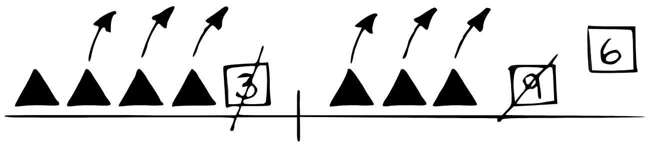

Figure A1. Hands-On Equations pictorial solution of $4x + 3 = 3x + 9$ (Borenson, 1986b). This image is taken from the 2008 printing of the publication.

In this pictorial solution, a line drawing of a two-pan scale replaces the laminated scale. Instead of using a blue pawn to represent the unknown x, a shaded triangle is drawn; instead of using a red numbered cube to represent a positive constant, a boxed number is drawn. The legal moves are indicated using arrows to remove pawns or cross-outs and replacements for the cubes. After performing the legal moves in the above illustration, a shaded triangle remains on the left side of the balance and a boxed-6 on the right side. Hence, $x = 6$. The check conducted in the original pictorial representation gives 27 = 27. Vlassis (2002) observes that the use of arrows to simplify the pictorial equation "is characterized by a less rigid syntax than the formal syntax and has the advantage of providing a friendly environment for non-expert users of symbolic language" (p. 357).