Scientific Research Publishing

Creative Education, 2021, 12, 2881-2885
https://www.scirp.org/journal/ce
ISSN Online: 2151-4771
ISSN Print: 2151-4755

The Geometric Meaning of Several Concepts in Linear Algebra

Wenbing Wu, Jinquan Xiong

Nanchang Normal University, Nanchang, China
Email: wwbysq@fjnu.edu.cn

How to cite this paper: Wu, W. B., & Xiong, J. Q. (2021). The Geometric Meaning of Several Concepts in Linear Algebra. *Creative Education, 12,* 2881-2885.
https://doi.org/10.4236/ce.2021.1212215

Received: November 14, 2021
Accepted: December 24, 2021
Published: December 27, 2021

Abstract

Linear algebra is a relatively abstract science. Through the geometric explanation of several algebraic concepts, this article strives to make these concepts easy to understand in teaching. Through the visualization of concepts such as determinants, linear transformations and eigenvalues, this course has changed from abstract equations to concrete geometric shapes in the teaching process, so as to achieve the purpose of improving the teaching effect.

Keywords

Algebra Teaching, Determinant, Linear Transformation, Eigenvalues, Geometric Meaning

1. Introduction

The teaching of algebra class is often boring and monotonous, so how to make the explanation of this class lively and interesting is an important issue that teachers of this class should consider. This article explains geometrically several basic concepts in algebra, which should be helpful to the teaching of this course.

2. The Geometric Meaning of Equations

The linear equation of two variables geometrically represents a straight line, and the system of equations containing two linear equations of two variables geometrically represents the positional relationship of the two straight lines:

Intersect ====> has a unique solution, Parallel ====> no solution.

Coincidence ====> Infinitely many solutions.

A system of equations composed of three ternary linear equations: If there is only one intersection of three planes, that is, the system of equations has a unique solution; If the three planes intersect on a straight line, the equation system has infinitely many solutions; If there is no intersection or line of intersection between the three planes, the equation system has no solution.

3. The Geometric Meaning of the Second and Third Order Determinants

Two-dimensional case: There is a parallelogram OACB on the plane. The coordinates of points A and B are respectively: (a_1, b_1), (a_2, b_2), as shown in the figure below, find the area of the parallelogram OACB. Analysis: Cross point A as the vertical line of the x axis, and cross the x axis at point E; cross point B as a line parallel to the x axis and cross point C as a line parallel to the y axis and cross at point D. Obviously we can get the triangle CDB and triangle AEO congruence, then:

$$S_{\text{OACD}} = S_{\text{OEDB}} + S_{\text{CDB}} - S_{\text{AEO}} - S_{\text{AEDC}} = S_{\text{OEDB}} - S_{\text{AEDC}} = a_1b_2 - a_2b_1,$$

as shown in Figure 1.

According to the definition of the second-order determinant, the area of the parallelogram is just the second-order determinant formed by the coordinates of A and B:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

In general, it can also be proved that the area of a parallelogram formed by two straight lines (vectors) passing through the origin, such as OA and OB, is the absolute value of the second-order determinant formed by the coordinates of A and B.

In three-dimensional situation, three vectors are known
$u = (a_1, a_2, a_3), v = (b_1, b_2, b_3), w = (c_1, c_2, c_3)$, that is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

The volume of the parallelepiped formed by these three vectors is the absolute value of the third-order determinant, as shown in Figure 2.

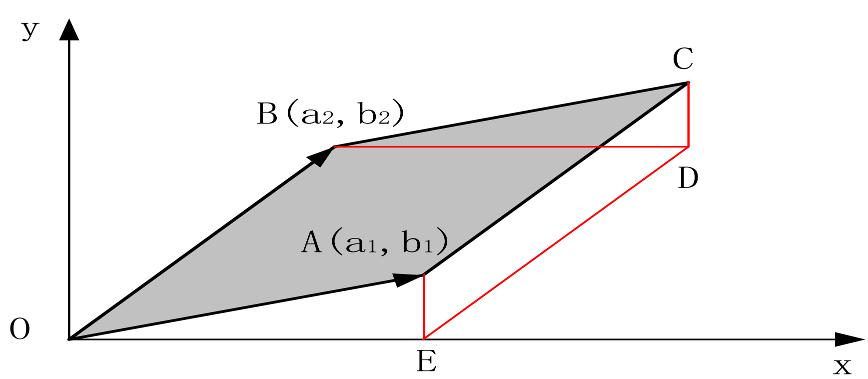

Figure 1. The area of the second determinant.

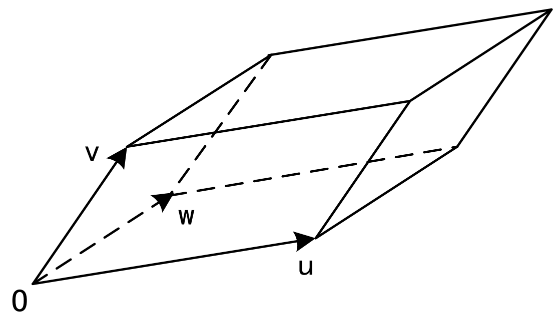

Figure 2. The volume of the third determinant.

4. The Geometric Meaning of Linear Transformation ($y = Ax$) on the Plane

Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In the A matrix, let the first row (1, 0) represent the X axis and (0, 1) represent the Y axis, then AX obtains the coordinate (2, 1) of x in the $x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ XOY plane.

$$A_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then the result of A_1x is (−2, 1). The first line of A_1 is (−1, 0), which is equivalent to the X axis in the new coordinate system, which can be considered as the opposite of the X axis in the previous coordinate system.

$$A_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0.5 & 0 \\ 0 & 2 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \left(\alpha = \frac{\pi}{3} \right)$$

Observe A_1x, the result of its transformation is that the coordinate system is changed (X-axis is reversed), but the relative position of the vector X in the new coordinate system remains unchanged. The result of A_1x is the coordinates of the transformed vector X in the old coordinate system. Other A_2 to A_4 can be understood by reference (the new coordinate system can be shrunk and rotated).

The result of the transformation from A_1 to A_4 is shown in Figure 3.

5. The Geometric Meaning of Eigenvalues

Suppose A is a square matrix of order n. If the number λ and the n-dimensional non-zero column vector x make the relationship $Ax = \lambda x$ hold, then such a number λ is called the eigenvalue of matrix A, and the non-zero vector x is called the corresponding feature of A The eigenvector of the value λ. The formula $Ax = \lambda x$ can also be written as $(A - \lambda E)X = 0$. This is a homogeneous system of linear equations with n unknowns and n equations. The necessary and sufficient condition for it to have a non-zero solution is that the coefficient determinant $|A - \lambda E| = 0$.

When the vector x is dragged with the mouse to rotate clockwise, Ax also starts to rotate. The trajectory of the vector x is a circle, and the trajectory of the vector Ax is generally an ellipse. Draw graphics as shown in Figure 4.

When the vector x is rotating, if the vector x and the vector Ax are collinear (including the same direction and the reverse direction), then there is an equation

$$A_1 x = \lambda x$$

λ is a real multiplier, λ is positive means that the two vectors are in the same

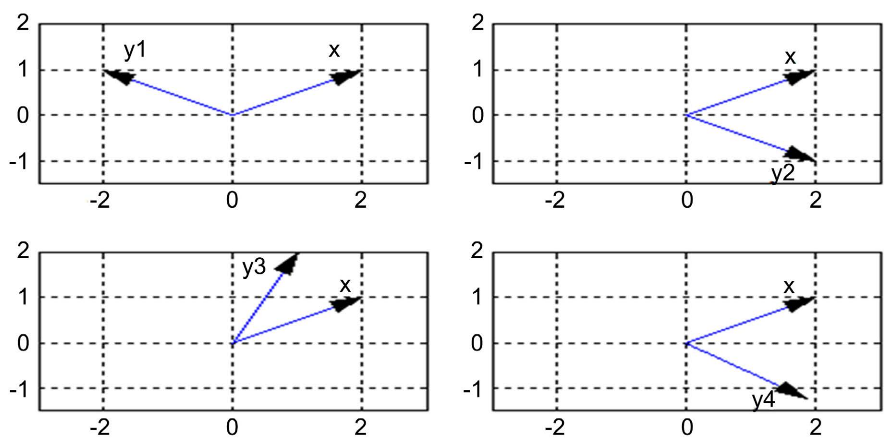

Figure 3. The volume of the third determinant.

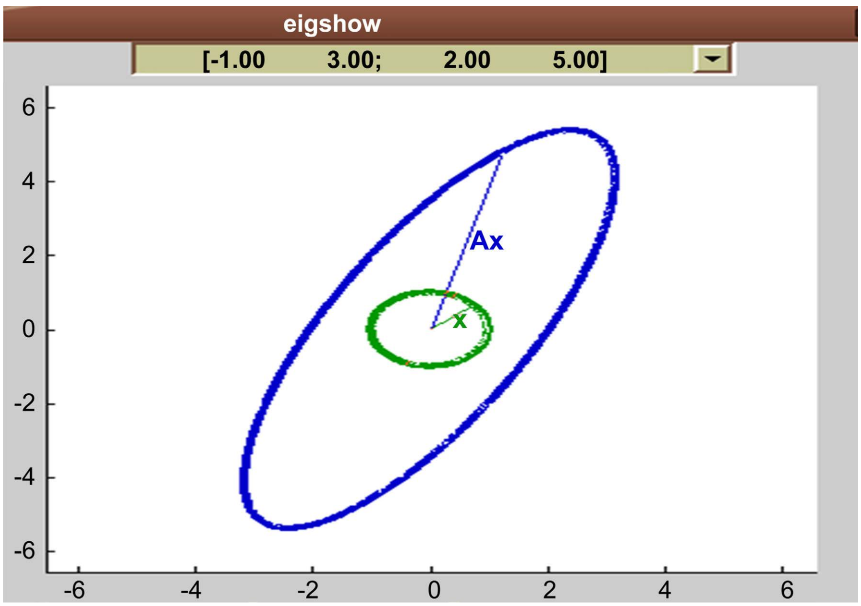

Figure 4. The volume of the third determinant.

direction, and λ is negative means that the two vectors are reversed. People call the position where the vector Ax and the vector x are collinear as the characteristic position, where the real number λ is called the eigenvalue of the matrix, and at this time x is the eigenvector of the matrix A belonging to λ. The eigenvalue represents the amount of enlargement (reduction) of the linear transformation Ax in the direction of the eigenvector x.

6. Conclusion

Linear algebra is a relatively boring course, and there are greater difficulties in the teaching process. In order to solve this difficulty, various attempts have been made in the literature (White et al., 2021; Yildiz & Senel, 2017; Guo et al., 2016; Ding & Rhee, 2011; Shakir, Rao, & Alouini, 2011). This article is also an attempt to improve the specificity and interest of this course. By displaying various equations in geometric form, I hope this method can provide some help to the teaching of this course.

Funding

This paper is supported by Research Foundation of the Nanchang Normal University for Doctors (NSBSJJ2018014). Research project of major education and teaching reform in Fujian Province (FBJG20190284).

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

Ding, J., & Rhee, N. H. (2011). On the Equality of Algebraic and Geometric Multiplicities of Matrix Eigenvalues. *Applied Mathematics Letters, 24,* 2211-2215. https://doi.org/10.1016/j.aml.2011.06.030

Guo, M., Hou, X., Ma, Y. et al. (2016). Facial Expression Recognition Using ELBP Based on Covariance Matrix Transform in KLT. *Multimedia Tools & Applications, 76,* 1-16. https://doi.org/10.1007/s11042-016-3282-9

Shakir, M. Z., Rao, A., & Alouini, M. S. (2011). Collaborative Spectrum Sensing Based on the Ratio between Largest Eigenvalue and Geometric Mean of Eigenvalues. *GLOBECOM Workshops (GC Wkshps),* Houston, 5-9 December 2011. https://doi.org/10.1109/GLOCOMW.2011.6162590

White, J., Costilow, K., Dotson, J. et al. (2021). Guided-Inquiry Experiment for Teaching the Calibration Method of Standard Addition in the Analysis of Lead with Graphite Furnace Atomic Absorption Spectroscopy. *Journal of Chemical Education, 98,* 620-625. https://doi.org/10.1021/acs.jchemed.0c00433

Yildiz, M., & Senel, M. (2017). Teaching Grammar through Task-Based Language Teaching to Young EFL Learners. *The Reading Matrix, 17,* 196-209.