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A New Method for Solving Algebraic Equations

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Abstract

High degree algebraic equations are an unsolved problem in algebra. The author has been able to solve high-order algebraic equations using elementary algebraic methods through exploration. This article will introduce new methods for solving algebraic equations, including factorization, collocation, and construction, collectively known as the L-algorithm. At the same time, we discuss the proof of Hilbert's 10th problem.

Keywords

Indefinite Equation, High-Order Indefinite Equation, L-Algorithm, Congruences

1. Introduction

Definition 1: An equation in algebra that contains unknowns is called an equation, and the value of the unknown variable that satisfies the equation is called the solution of the equation. If all unknowns and known numbers are integers are called indefinite equations [1]. When the exponent is greater than 2, it is also called a high-order equation.

For general linear to quadratic algebraic equations, we already have mature solutions.

For general quadratic equations $ax^2+bx+c=0$, we have the most familiar formula for solving:

$$x=-b+\frac{\sqrt{a^2\pm 4ab}}{2}$$

For cubic and quartic equations, there are also methods of finding roots to solve the equations, such as the Cardano formula for solving cubic equations

$$x=\sqrt[3]{-\frac{q}{2}+\frac{q^2}{4}+\frac{p^2}{27}}+\sqrt[3]{-\frac{q}{2}-\frac{q^2}{4}+\frac{p^2}{27}}$$

But for high-order algebraic equations, Norwegian mathematician Abel has proven that algebraic equations greater than fourth order (*i.e.* fifth order) cannot be solved by finding roots. French mathematician Galois also proposed Galois theory, which uses group methods to solve algebraic equations, but so far we have not seen an example of solving fifth degree equations [2]. It is an important unresolved problem in algebra and number theory.

We have introduced a new algorithm for high-order indefinite equations [3]. This article will further elaborate on using new algorithm to solve various types of the high-order equations.

2. Algebraic Principles and Congruences Theory

2.1. Algebraic Principles

The following three theorems are familiar and commonly used mathematical principles in algebraic operations, and will not be proven here.

Theorem 2. 1. If *a, b* and *c* are integers, there must be $a+b=c$.

Theorem 2.2: If *a, b* and *c* are integers and $a+b=c$ holds, then $(a+b)d=cd$ also holds.

Theorem 2.3. If *a*, *b* and *c* are integers, $ab=c$, Then *c* must contain factor *a* or *b*.

2.2. Congruences Theory

In this article, we will apply the following congruences theory, for example

If $a\equiv c(\text{mod}\, p)$, $b\equiv d(\text{mod}\, p)$, and $e=c+d$,
then

$$a+b\equiv e(\text{mod}\, p).$$

If $c+d=0$, then $a+b\equiv 0(\text{mod}\, p)$.

Regarding the theory and formulas of congruence, we can refer to some books [4].

3. A New Method Solving Algebraic Equations (L-Algorithms)

3.1. The First Algorithm (Factorization Method)

Factorization method: It is applicable to polynomial equations, characterized in that all equations are in the following form

$$k_1x^i+k_2x^{i-1}+\cdots+k_ix+f=0 \tag{1}$$

where $k_1,k_2,\cdots,k_i$ and f are integers.
So, the equation (1) can be changed to

$$\left(k_1x^{i-1}+k_2x^{i-2}+\cdots+k_i\right)x=-f. \tag{2}$$

It is clear that the condition for equation (2) to hold is $x\mid f$ or $\left(k_1x^{i-1}+k_2x^{i-2}+\cdots+k_i\right)\mid f$ [5].

Example 3.1: Attempt to find an integer solution to the quadratic equation (3) of one variable

$$x^4 + x^3 + x^2 + x - 26520 = 0. \tag{3}$$

The solution can be obtained from equation (4)

$$x^4 + x^3 + x^2 + x = 26520, \tag{4}$$

$$\left(x^3 + x^2 + 1\right)x = 26520. \tag{5}$$

According to the Fundamental Theorem of Arithmetic and Theorem 3, assuming equation (5) has an integer solution, 26520 must contain a factor *x*. Given that 26,520 = 5 × 8 × 13 × 51, *x* may be one of them. Now, substitute each factor into equation (5) and test them one by one. When $x = -13$, substituting it into the equation yields

$$(-13)^4 + (-13)^3 + (-13)^2 + (-13) = 26520,$$

$$28561 - 2197 + 169 - 13 = 26520$$

Obviously, when $x = -13$, both sides of equation (3) are equal, so $x = -13$ is the solution to equation (3).

3.2. The Second Algorithm (Index Method)

This method is applicable to equations of the following form

$$A^x + B^y + \cdots + U^v = W^z. \tag{6}$$

The condition for using the Index method in equation (6) is that all exponents on the left side of the equation are coprime with *z*. Additionally, the left-hand side of the equation can have coefficients.

We refer to the equation in the form of equation (7) as the Beal equation [6]. Below we will introduce its solution.

$$A^x + B^y = C^z. \tag{7}$$

where A, B, C, x, y, z are integers.

Regardless of whether *a*, *b*, *x*, and *y* are any integers, an integer *c* is can be found to make equation (8) hold (equation (8) is called the initial equation).

$$a^x + b^y = c \tag{8}$$

By Theorem 3.2.1, multiply both sides of the initial equation (8) by c^{xy} to obtain

$$\left(a^x + b^y\right)c^{xy} = cc^{xy},$$

$$a^x c^{xy} + b^y c^{xy} = cc^{xy}. \tag{9}$$

Rewrite equation (9) as

$$\left(ac^y\right)^x + \left(bc^x\right)^y = c^{xy+1}.$$

So, if $z = xy + 1$ or $z \mid (xy+1)$, it is easy to see that the solution to equation (7) is $A = ac^y$, $B = ad^y$, $C = c$..(If $z \mid (xy+1)$, $xy + 1 = kz$, then $C = c^k$).

Example 3.2.1 Solving the Beal equation

$$A^3+B^5=C^4. \tag{10}$$

According to the above method, for equation (10), we need to establish a new equation

$$a^3+b^5=c. \tag{11}$$

Foe equation (11), let a = 3, b = 2, and substituting them into the left side of equation (11) yields a new equation (12)

$$3^3+2^5=59, \tag{12}$$

Multiply $59^{3\times5}$ on both sides of equation (12) to obtain

$$\left(3^3+2^5\right)59^{3\times5}=59\times59^{3\times5},$$

$$3^3\times59^{3\times5}+2^5\times59^{3\times5}=59\times59^{3\times5}, \tag{13}$$

Equation (13) can be rewritten as:

$$\left(3\times59^5\right)^3+\left(2\times59^3\right)^5=\left(59^4\right)^4, \tag{14}$$

Obviously, the solution to equation (10) is $A=3\times59^5=714924299$, $B=2\times59^3=410758$, and $C=59^4=12117361$.

3.3. The Third Algorithm (Generol and Congruences Method)

The third algorithm is applicable to multivariate multiple equations, characterized in that all equations as equation (15) (Simply put, it contains a term with an unknown exponent of 1)

$$k_1A^i+k_2B^{i-1}+\cdots+k_2U^2+k_iV+n=0, \tag{15}$$

where $k_1,k_2,\cdots,k_i$ and n are integers.

Its L-algorithm is to transform equation (15) into

$$k_1A^i+k_2B^{i-1}+\cdots+k_2U^2+n=-k_iV \tag{16}$$

Let

$$w=k_1A^i+k_2B^{i-1}+\cdots+k_2U^2+n,$$

then by equation (16) we obtain

$$V=-w/k_i.$$

1) For the rational number solution of equation (Generol method);

Example 3.3.1. Try to find the rational number solution of the fifth degree equation (18).

$$3a^5+7b^4+11c^3+17d^2+23e+71=0. \tag{18}$$

Let

$$w=3a^5+7b^4+11c^3+17d^2+71. \tag{19}$$

Since only seeking that the solution to the equation is a rational number, the unknowns on the left side of the equation can be any integer (of course, the smaller the better), where w is an integer and V is a rational number.

For example, let a = 3, b = 4, c = 5, d = 2, substitute them into the left part of equation (19) to obtain

$$w = 3\times 3^5 + 7\times 4^4 + 11\times 5^3 + 17\times 2^2 + 71 = 4035,$$

$$e = -\frac{w}{23} = -\frac{4035}{23}.$$

In this way, the rational number solution of equation (13) is $a = 3$, $b = 4$, $c = 5$, $d = 2$. $e = -\frac{4035}{23}$.

2) The solution to the equation are integers, which can be divided into two situations:

a) When $k_1 \mid n$:

$$k_1 A^i + k_2 B^{i-1} + \cdots + k_2 U^2 + k_i V + n = 0 \quad (20)$$

where $k_1, k_2, \cdots, k_i$ and n are integers, and $k_1 \mid n$.

Since the solution to the equation is an integer and $k_1 \mid n$,the unknowns $A, B, \cdots, U$ on the left side of the equation can be chosen as any multiple of k_i (of course, the smaller the better), then

$$w = k_1 A^i + k_2 B^{i-1} + \cdots + k_2 U^2 + n, \quad w \equiv 0 \pmod{k_i};$$

and $V = -w/k_i$ (V is an integer).

In this way, the equation (20) can obtain a set of integers of a solution.

Example 3.3.2. Try to find the integer solution of the fifth degree equation (21).

$$3a^3 + 7b^2 + 9c + 81 = 0. \quad (21)$$

Since 9|81, set $a = 9$, $b = 18$, substitute them into the equation (21) to obtain

$$3\times 9^3 + 7\times 18^2 + 9c + 81 = 0$$

$$4455 + 9c = 0,$$

$$c = -495.$$

So, the solution of the equation (20) are $a = 9$, $b = 18$, $c = -495$.

Or, set $a = 3$, $b = 3$, substitute them into the equation (21) to obtain

$$3\times 3^3 + 7\times 3^2 + 9c + 81 = 0$$

$$225 + 9c = 0,$$

$$c = -25.$$

So, the other solution of the equation (20) are $a = 3$, $b = 3$, $c = -25$.

b) When here is not $k_1 \mid n$ (need congruence algorithm) [6];

Example 3.3.3. Try to solving equation (22)

$$4a^5 + 7b^4 + 11c^3 + 17d^2 + 19e + 791 = 0. \quad (22)$$

Since

$$w = 4a^5 + 7b^4 + 11c^3 + 17d^2 + 791, \quad (23)$$

in order to make

$$w \equiv 0 \pmod{19},$$

we can use the theory of congruence to obtain the values of a, b, c and d, for example:

Let $a=19$, $4a^5=4\times19^5=9904396$, $9904396\equiv0(\bmod 19)$;

Let $b=1$, $7b^4=7\times1^4=7$, $7\equiv7(\bmod 19)$,

Let $c=3$, $11c^3=11\times3^3=297$, $297\equiv12(\bmod 19)$,

Let $d=5$, $17d^2=17\times5^2=425$, $425\equiv7(\bmod 19)$,

and

$$791\equiv12(\bmod 19);$$

so

$$0+7+12+7+12\equiv0(\bmod 19).$$

Now, we have found that are a = 19, b = 1, c = 3, and d = 5, substitute them into the equation (23) to obtain

$$\begin{aligned}w&=4a^5+7\times b^4+11\times c^3+17\times d^2+791\\&=4\times19^5+7\times1^4+11\times3^3+17\times5^2+791\\&=9905916\end{aligned}$$

Since by the equation (22) we obtain

$$w+19e=0,$$

then

$$19e=-w=-9905916,,$$

$$e=-9905916/19=-521364.$$

In this way, the integer solution of equation (23) is a = 19, b = 1, c = 3, d = 5, e = −521364.

Definition 2: The above new algorithms for equations are referred to as L-algorithms.

4. Analysis and Discussion

4.1. Advantages of the L-Algorithm

1) The L-algorithm only uses multiplication and addition/subtraction (summing up some power terms), and finally uses division to solve. Because multiplication is easier than finding square roots and can be manually operated, this algorithm has significant superiority and operability. In theory, it is not limited by pluralism and large power exponents.

2) By utilizing the principle of L-algorithm, we can also solve more similar types of algebraic equations. for example equation (24)

Example 4.1. Try to solving equation (24)

$$A^7+7B^2+17C+89=0 \tag{24}$$

Because equation (24) has an unknown whose exponent is one, we can use the Congruent complementarity method, as let

$$w=A^7+7B^2+89,$$

We use the Congruent complementarity method to find that when A = 1, B = 3, which

$$1^7\equiv1(\bmod 17),\ \ 7\times3^2\equiv12(\bmod 17),\ \ 89\equiv4(\bmod 17),$$

then $1+12+4\equiv 0(\mathrm{mod}17)$ and $w\equiv 0(\mathrm{mod}17)$ Finally, find $C=-w/17=-153/17=-9$, which is the solution to equation (24) are A = 1, B = 3, C = −9.

Example 4.2. Try to solving equation (25)

$$64x^5+48x^4-32x^3+68x^2-10x-13=0 \tag{25}$$

By the equation (25), we obtain

$$64x^5+48x^4-32x^3+68x^2-10x=13 \tag{26}$$

According to the Fundamental Theorem of Arithmetic, if equation (26) has an integer solution, 13 must contain an x factor; But 13 is a prime number, substituting x into the equation does not make the equation hold. So the equation may have a solution or it could be a real number. After observation, the coefficients on the left side of the equation are all even, so there may be 2 hidden factors on the right side of the equation, that is, 13 = 2 × 13 × 0.5. Perhaps $x=0.5$, substituting it into equation (5) actually satisfies the requirements of the equation.

$$64\times 0.5^5+48\times 0.5^4-32\times 0.5^3+68\times 0.5^2-10\times 0.5=2+3-4+17-5=13.$$

Obviously, when x = 0.5, both sides of equation (26) are equal, so x = 0.5 is a rational of the solution to equation (25).

Example 4.3. Try to solving equation (27)

$$5A^7+7B^5+17C^3-5D^2+93=0 \tag{27}$$

According to the principle of L-algorithm,

$$w=5A^7+7B^5+3C^3+93, \tag{28}$$

and it is best to consider D^2 = 25 and w = 125, then we obtain

$$w=5A^7+7B^5+3C^3+93=125,$$

Substituting A = 1, B = 1, C = 2 into equation (28) yields

$$w=5A^7+7B^5+3C^3+93=5\times 1^7+7\times 1^5+3\times 2^3+93=125,$$

By equation (27) we obtain

$$w-5D^2=0,$$

$$5D^2=w,$$

$$D^2=w/5=125/5,$$

$$D^2=25,$$

$$D^2=\sqrt{25}=5.$$

In this way, the solution to equation (27) are A = 1, B = 1, C = 2, D = 5.

4.2. Limitations of L-Algorithms

In section 3, the typical standards and calculation principles of the L-algorithm have been clearly explained (i.e. their scope of use). The current limitations we know are:

a) In the first algorithm, for a univariate high-order equation, if x is not a known integer term factor, the equation may have no real number solution.

b) When the power exponent on the left side of the equation is congruent with the power exponent of a term on the right side, it cannot be used the index method.

c) In the equation (20), sometimes V can be the power of the lowest exponent.

5. Hilbert's 10th Problem

In 1900, the German genius mathematician Hilbert at the World Congress of Mathematicians held in Paris, France, 23 mathematical problems were listed in one go. Among them, the 10th problem is called Hilbert's 10th problem, which is actually a problem related to computation. Using the language of modern mathematics and computation, this problem can be summarized as follows: "If given any multivariate algebraic equation with integer coefficients, $F(x_1, x_2, \cdots, x_n) = 0$, do we have an algorithm to solve the equation?" From this algorithm, we can determine whether the equation has an integer solution. Note that only the given algebraic equation has an integer solution, and we do not care about its specific solution. To put it more formally, Hilbert's 10th problem can be defined as a "decision problem" with only positive (yes) or negative (no) answers.

From the perspective of modern computational theory. Hilbert is actually an "algorithm" for determining the existence or non existence of any indefinite equation. This algorithm can tell us whether the indefinite equation has a solution or not [7].

Impossible Previously, we were helpless in solving high-order indeterminate equations, and it was to determine whether the equation had integer solutions. now

Thanks to the L-algorithm, it is possible to solve integer solutions of high-order equations and determine whether algebraic equations have integer solutions.

Below, we will delve into the solutions of various types of indefinite equations.

1) A linear equation is a type that we are very familiar with, and determining whether an equation has an integer solution is a thatched hut problem.

2) For quadratic equations, we can use the L-algorithm to determine whether the equation has integer solutions. For example:

a) If there is an unknown variable in the equation, such as equation (29)

$$k_1 A_1^x + k_2 A_2^y + \cdots + k_{i-1} A_{i-1}^z + k_i A_i + n = 0. \tag{29}$$

We will write equation (29) as

$$k_1 A_1^x + k_2 A_2^y + \cdots + k_{i-1} A_{i-1}^z + n = -k_i A_i. \tag{30}$$

$$k_1 A_1^x + k_2 A_2^y + \cdots + k_{i-1} A_{i-1}^z = w.$$

Using the previous knowledge, we know that through the theory of congruence, we can determine whether we can find the values of $A_1, A_2, \cdots, A_{i-1}$, such that

$$k_1A_1^x + k_2A_2^y + \cdots + k_{i-1}A_{i-1}^z \equiv 0 \pmod{k_i},$$

you can determine whether the equation has an integer solution.

b) Actually, if there has not an unknown variable (the exponent of power is 1), we can operate in this way.

Example 5.1. Try to solving equation (31)

$$k_1A_1^x + k_2A_2^y + \cdots + k_{i-1}A_{i-1}^3 - k_iA_i^2 + n = 0 \tag{31}$$

By equation (31), we obtain

$$k_1A_1^x + k_2A_2^y + \cdots + k_{i-1}A_{i-1}^3 + n = k_iA_i^2 \tag{32}$$

Let

$$w = k_1A_1^x + k_2A_2^y + \cdots + k_{i-1}A_{i-1}^3 + n.$$

We can apply the theory of congruence used in the L-algorithm to determine whether we can find the values of $A_1, A_2, \cdots A_{i-1}$, so that

$$k_1A_1^x + k_2A_2^x + \cdots + k_{i-1}A_{i-1}^3 + n \equiv 0 \pmod{k_i},$$

So

$$w = k_iA_i^2,$$

$$A_i = \sqrt{\frac{w}{k_i}}.$$

Now, you can determine whether the equation has an integer solution.

For the equation (22) in the example 3.3.3.

$$4a^5 + 7b^4 + 11c^3 + 17d^2 + 19e + 791 = 0. \tag{22}$$

By equation (22) we obtain

$$4a^5 + 7b^4 + 11c^3 + 17d^2 + 791 = -19e.$$

Let

$$w = 4a^5 + 7b^4 + 11c^3 + 17d^2 + 791,$$

Using the principle of congruence, we can find $a = 19$, $b = 1$, $c = 3$, $d = 5$, such that

$$w \equiv 0 \pmod{19}$$

then

$$e = -521364,$$

the equation (22) has an integer solution.

For the equation (25) in the example 4.2.

$$64x^5 + 48x^4 - 32x^3 + 68x^2 - 10x - 13 = 0 \tag{25}$$

By the equation (25), we obtain

$$64x^5 + 48x^4 - 32x^3 + 68x^2 - 10x = 13,$$

$$\left(64x^4 + 48x^3 - 32x^2 + 68x^2 - 10\right)x = 13$$

Because 13 is a prime number and cannot be decomposed into the product of integers, equation (32) has no integer solution, which means equation (32) has no integer solution.

For the equation (27) in the example 4.3.

$$5A^7+7B^5+17C^3-5D^2+93=0 \tag{27}$$

By the equation (27), we obtain

$$5A^7+7B^5+17C^3+93=5D^2 \tag{33}$$

Let

$$w=5A^7+7B^5+17C^3+93,$$

then $D^2=w/5$.

Therefore, first of all, w must be a multiple of 5 and also an exponent that is an even power.

In example 4.3, we used congruence theory to find $A=1$, $B=1$, $C=2$, $D=5$.

When considering whether the equation has an integer solution, when using (33) to place $5D^2$ separately on the right side of the equation and no integer solution can be found, $5A^7$, $7B^5$, $17C^3$ should also be placed separately on the right side of the equation to test whether an integer solution can be found, and finally determine whether the equation has an integer solution.

We can see the combination of L-algorithm and congruence algorithm from the above examples. It can solve the problem of determining whether an indefinite equation has an integer solution

Example 5.2. Try to solving equation (34)

$$7A^7+7B^4+39=0. \tag{34}$$

From equation (34):

$$7\left(A^7+B^4\right)=39 \tag{35}$$

From equation (35), it is known that $7\left(A^7+B^4\right)$ contains a factor of 7, while 39 does not contain a factor of 7. According to the Fundamental Theorem of Arithmetic, it can be concluded that equation (34) has no integer solution.

In this way, we have solved Hilbert's 10th problem.

6. Conclusion

From the above introduction, we can see that high-order algebraic equations can be solved using the L-algorithm. The several types of high-order equations cited in this article already include most of the commonly used high-order equations. By utilizing its algorithmic principles, we can also solve more different types of high-order equations. The exact solutions of indeterminate equations and rational number solutions of algebraic equations in this article are convincing. This is a very promising and effective new method for solving high-order equations, opening up a practical and promising new path for solving mathematical problems of multivariate high-order equations in number theory. Of course, this only opens another door in the field of high-order algebraic equations in the kingdom of mathematics, and further research and exploration are needed in the future.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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