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A Sandwich Theorem for *m*-Convex Stochastic Processes

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Abstract

In this paper, we present some properties of m-convex stochastic processes. The most important results are: a generalization of the sandwich theorem and a result on Hyers-Ulam stability, given for m-convex functions. The first result allows us to bound an m-convex stochastic process by two convex stochastic processes, and the second allows us to approximate controlled perturbations of an m-convex stochastic process by an m-convex function. As a consequence of these two results, we obtain a Hermite-Hadamard type inequality for m-convex stochastic processes.

Keywords

m-Convex Stochastic Processes, Hermite-Hadamard Inequality, Sandwich Theorem, Hyer-Ulam's Stability

1. Introduction

In 1974, B. Nagy in [1] developed a stochastic processes characterization to solving a generalization of the Cauchy functional equation. Then, in 1980, K. Nikodem in [2] obtained some properties of convex stochastic processes and gave generalizations of several results proved in [1]. Both research works began a line of research on convex stochastic processes; as a result of this line of research, the following papers have been obtained [3]-[9].

The concept of m-convex function was introduced by G. H. Toader in [10]. We can find this type of functions in the articles [11]-[29], in which some algebraic properties for this type of functions were demonstrated, classical integral inequalities of the Hermite-Hadamard type and some stability results and sandwich theorems.

Interesting and important inequalities for m-convex functions were developed by M. K. Bakula in [11] and M. E. Özdemir in [30]. On the other hand, some Venezuelan researchers have developed numerous works on this topic; some examples can be found in the papers [3] [4] [16]-[24]. Additionally, in 1994, K. Baron, J. Matkowski and K. Nikodem in [12] developed a characterization of real functions that can be separated by a convex function and in 1995, K. Nikodem and S. Wasowicz proved a sandwich theorem for affine functions in [28].

For their part, in 2007, K. Nikodem and Z. Páles in [29] studied the classic Kakutani theorem and extended it to the convexity in the sense of Beckenbach, getting as consequences stability results of the Hyers-Ulam type. Subsequently, in 2016, N. Merentes and K. Nikodem in [27] proved that a pair of functions can be separated by functions strongly convex, approximately concave, or c-quadratic-affine functions, obtaining as a consequence, stability results of the Hyers-Ulam type. In [27] it has been shown an analogue result of the sandwich theorem for convex functions that is not true in the class of *m*-convex functions with $m \in (0,1)$. However, T. Lara in 2017 proved a useful sandwich result for the function *m*-convex in [20].

The main objective of this paper is to perform a sandwich-type theorem and a Hyer-Ulam's stability theorem for *m*-convex stochastic processes as a counterpart to those performed for *m*-convex functions.

2 Preliminaries

Definition 2.1. *A function* $f: I \to \mathbb{R}$, *where* $I \subseteq \mathbb{R}$ *is an interval, is convex if*

$$f\left(tx+(1-t)y\right) \le tf\left(x\right)+(1-t)f\left(y\right),$$

para todo $x, y \in I$ y $t \in (0,1)$.

If the inequality is strict (<) *for* $x, y \in I$, $t \in (0,1)$, *then we say that the function* f *is strictly convex. If the inequality holds in the opposite direction* (≥) *we say that* f *is concave and if it is verified in the strict sense* (>) *we say that f is strictly concave.*

Definition 2.2. *Let* $(\Omega, \mathbb{A}, \mathbb{P})$ *be a probability space. A function* $X: \Omega \to \mathbb{R}$ *is a random variable if it is* $\mathbb{A}$*-measurable. A function* $X: I \times \Omega \to \mathbb{R}$, *where* $I \subseteq \mathbb{R}$ *is an interval, is a stochastic process if for each* $t \in I$ *the function* $X(t,\cdot)$, *is a random variable.*

A stochastic process $X: I \times \Omega \to \mathbb{R}$ is:

Definition 2.3. *Jensen-Convex if, for each* $a, b \in I$, *the following inequality is satisfied*:

$$X\left(\frac{a+b}{2},\cdot\right) \le \frac{X(a,\cdot)+X(b,\cdot)}{2}. \quad (a.e.)$$

Definition 2.4. *Convex if, for each* $a, b \in I$, $t \in (0,1)$ *the following inequality is satisfied*:

$$X\left(ta+(1-t)b,\cdot\right) \le tX(a,\cdot)+(1-t)X(b,\cdot). \quad (a.e.)$$

Definition 2.5. *Quasi-Convex if, for each* $a,b\in I$, $t\in(0,1)$ *the following inequality is satisfied*:

$$X\left(ta+(1-t)b,\cdot\right)\le \max\left[X\left(a,\cdot\right),X\left(b,\cdot\right)\right].\quad (a.e.)$$

Also, we say that a stochastic process $X:I\times\Omega\to\mathbb{R}$ is:

Definition 2.6. *Continuous in probability in the interval* I, *if for all* $t_0\in I$ *we have*

$$P-\lim_{t\to t_0}X\left(t,\cdot\right)=X\left(t_0,\cdot\right)\quad (a.e.)$$

where $P-\lim$ denotes the limit in probability.

Definition 2.7. *Mean-Square contiuous in* I, *if for all* $t_0\in I$ *we have*

$$\lim_{t\to t_0}\mathbb{E}\left[\left(X\left(t,\cdot\right)-X\left(t_0,\cdot\right)\right)^2\right]=0.\quad (a.e.)$$

where $\mathbb{E}\left[X\left(t,\cdot\right)\right]$ denotes the expectation value of the random variable $X\left(t,\cdot\right)$.

Definition 2.8. *Differentiable at a point* $t\in I$, *if there is a random variable* $X'\left(t,\cdot\right):I\times\Omega\to\mathbb{R}$ *defined as follows*:

$$X'\left(t,\cdot\right)=P-\lim_{t\to t_0}\frac{X\left(t,\cdot\right)-X\left(t_0,\cdot\right)}{t-t_0}.\quad (a.e.)$$

Remark 2.9. *Every mean-square continuous stochastic process is a continuous in probability stochastic process*; *however*, *the converse is not true*.

Definition 2.10. *Let* $X:I\times\Omega\to\mathbb{R}$ *be a stochastic process such that* $\mathbb{E}\left[\left(X\left(t,\cdot\right)\right)^2\right]<+\infty$ *for all* $t\in I$. *The random variable* $Y:\Omega\to\mathbb{R}$ *is called the mean-square integral of the stochastic process* X *en* $[a,b]\subseteq I$, *if for any partition* $a=t_0<t_1<\cdots<t_n=b$ *of the interval* $[a,b]$ *y* $\Theta_k\in\left[t_{k-1},t_k\right]$ ($k=1,\cdots,n$), *we have*

$$\lim_{n\to+\infty}\mathbb{E}\left[\left(\sum_{k=1}^{+\infty}X\left(\Theta_k,\cdot\right)\left(t_k-t_{k-1}\right)-Y\left(\cdot\right)\right)^2\right]=0.\quad (a.e.)$$

In this case, the following notation is used:

$$Y\left(\cdot\right)=\int_a^b X\left(s,\cdot\right)\mathrm{d}s.\quad (a.e.)$$

Remark 2.11. *For the existence of the mean-square integral of the stochastic process* X, *it is sufficient that* X *be mean-square continuous. Basic properties of the mean-square integral can be read in* [31].

Definition 2.12. *Let* $m\in[0,1]$ *and* $I=[0,c]$. *A mean-square continuous stochastic process* $X:I\times\Omega\to\mathbb{R}$, *is m-convex*, *if for all* $a,b\in I$ *y* $t\in[0,1]$, *the following inequality is satisfied*:

$$X\left(ta+m(1-t)b,\cdot\right)\le tX\left(a,\cdot\right)+m(1-t)X\left(b,\cdot\right).\quad (a.e.)$$

We denote by $S_m\left(c,\cdot\right)$, the class of stochastic processes *m*-convex in I, such that $X\left(0,\cdot\right)\le 0$.

Remark 2.13. *If in the previous definition*, *we take* $t=0$, *then*

$$X(mb,\cdot) \le mX(b,\cdot), \quad (a.e.)$$

for all $b \in I$.

3. Main Results

First, let's establish some algebraic properties for m-convex stochastic processes.

Lemma 3.1. *Let* $Y:[a,b]\times\Omega\to\mathbb{R}$ *be a mean-square stochastic process,* $m_1, m_2 \in [0,1]$, *such that* $m_1 \le m_2$ *and* $m_2 \ne 0$. *If* Y *is* m_2*-convex almost everywhere, then* Y *is* m_1*-convex almost everywhere.*

Proof. Since Y is m_2-convexo and $m_1 \le m_2$, we have

$$\begin{aligned} Y(ta+m_1(1-t)b,\cdot) &= Y\left(ta+m_2(1-t)\left(\frac{m_1}{m_2}b,\cdot\right)\right) \\ &\le tY(a,\cdot)+m_2(1-t)Y\left(\frac{m_1}{m_2}b,\cdot\right) \\ &\le tY(a,\cdot)+m_2(1-t)\left(\frac{m_1}{m_2}\right)Y(b,\cdot) \\ &= tY(a,\cdot)+m_1(1-t)Y(b,\cdot). \quad (a.e.) \end{aligned}$$

Therefore, Y is m_1-convex almost everywhere. □

Proposition 3.2. *Let* $a \ge 0$ *and* $X, Y:[a,b]\times\Omega\to\mathbb{R}$ *be mean-square stochastic processes, If* X *is* m_1*-convex and* Y *is* m_2*-convex almost everywhere, with* $m_1 \le m_2$ *and* $m_2 \ne 0$, *then* $X+Y$ *y* αX, $\alpha \ge 0$ *are* m_1*-convex almost everywhere.*

Proof. By the previous lemma, we have that Y es m_1-convex almost everywhere.

For $\lambda_1, \lambda_2 \in [a,b]$ and $t \in [0,1]$, we obtain

$$\begin{aligned} &(X+Y)(t\lambda_1+m_1(1-t)\lambda_2,\cdot) \\ &= X(t\lambda_1+m_1(1-t)\lambda_2,\cdot)+Y(t\lambda_1+m_1(1-t)\lambda_2,\cdot) \\ &\le tX(\lambda_1,\cdot)+m_1(1-t)X(\lambda_2,\cdot)+tY(\lambda_1,\cdot)+m_1(1-t)Y(\lambda_2,\cdot) \\ &= tX(\lambda_1,\cdot)+tY(\lambda_1,\cdot)+m_1(1-t)X(\lambda_2,\cdot)+m_1(1-t)Y(\lambda_2,\cdot) \\ &= t(X+Y)(\lambda_1,\cdot)+m_1(1-t)(X+Y)(\lambda_2,\cdot). \quad (a.e.) \end{aligned}$$

From where, $X+Y$ is m_1-convex almost everywhere.

Besides,

$$\begin{aligned} (\alpha X)(t\lambda_1+m_1(1-t)\lambda_2,\cdot) &= \alpha\left(X(t\lambda_1+m_1(1-t)\lambda_2,\cdot)\right) \\ &\le \alpha\left(tX(\lambda_1,\cdot)+m_1(1-t)X(\lambda_2,\cdot)\right) \\ &= \alpha tX(\lambda_1,\cdot)+\alpha m_1(1-t)X(\lambda_2,\cdot) \\ &= t(\alpha X)(\lambda_1,\cdot)+m_1(1-t)(\alpha X)(\lambda_2,\cdot). \quad (a.e.) \end{aligned}$$

From where, αX is m_1-convex almost everywhere. □

Proposition 3.3. *Let* $X, Y:[0,b]\times\Omega\to\mathbb{R}$ *be nonnegative stochastic processes such that*

$$\left(X(\lambda_1,\cdot)-X(\lambda_2,\cdot)\right)\left(Y(\lambda_1,\cdot)-Y(\lambda_2,\cdot)\right) \ge 0 \quad (a.e.)$$

for all $\lambda_1,\lambda_2 \in [0,b]$. *If* X,Y *are m-convex stochastic processes, then* XY *is m-convex almost everywhere.*

Proof. Let $\lambda_1,\lambda_2 \in [0,b]$ and $t \in (0,1)$.

Since X and Y are *m*-convex stochastic processes, we have

$$\begin{aligned}
&XY\left(t\lambda_1 + m(1-t)\lambda_2,\cdot\right)\\
&= X\left(t\lambda_1 + m(1-t)\lambda_2,\cdot\right)Y\left(t\lambda_1 + m(1-t)\lambda_2,\cdot\right)\\
&\le \left(tX(\lambda_1,\cdot) + m(1-t)X(\lambda_2,\cdot)\right)\left(tY(\lambda_1,\cdot) + m(1-t)Y(\lambda_2,\cdot)\right)\\
&= t^2 X(\lambda_1,\cdot)Y(\lambda_1,\cdot) + tX(\lambda_1,\cdot)m(1-t)Y(\lambda_2,\cdot)\\
&\quad + m(1-t)X(\lambda_2,\cdot)tY(\lambda_1,\cdot) + m^2(1-t)^2 X(\lambda_2,\cdot)Y(\lambda_2,\cdot)\\
&= t^2 X(\lambda_1,\cdot)Y(\lambda_1,\cdot) + mt(1-t)\left[X(\lambda_1,\cdot)Y(\lambda_2,\cdot) + X(\lambda_2,\cdot)Y(\lambda_1,\cdot)\right]\\
&\quad + m^2(1-t)^2 X(\lambda_2,\cdot)Y(\lambda_2,\cdot).
\end{aligned}$$

Furthermore, the inequality:

$$\left(X(\lambda_1,\cdot) - X(\lambda_2,\cdot)\right)\left(Y(\lambda_1,\cdot) - Y(\lambda_2,\cdot)\right) \ge 0.$$

Which implies

$$X(\lambda_1,\cdot)Y(\lambda_1,\cdot) + X(\lambda_2,\cdot)Y(\lambda_2,\cdot) \ge X(\lambda_1,\cdot)Y(\lambda_2,\cdot) + X(\lambda_2,\cdot)Y(\lambda_1,\cdot).$$

Hence,

$$\begin{aligned}
&XY\left(t\lambda_1 + m(1-t)\lambda_2,\cdot\right)\\
&\le t^2 X(\lambda_1,\cdot)Y(\lambda_1,\cdot) + mt(1-t)\left[X(\lambda_1,\cdot)Y(\lambda_2,\cdot) + X(\lambda_2,\cdot)Y(\lambda_1,\cdot)\right]\\
&\quad + m^2(1-t)^2 X(\lambda_2,\cdot)Y(\lambda_2,\cdot)\\
&\le t^2 X(\lambda_1,\cdot)Y(\lambda_1,\cdot) + mt(1-t)\left[X(\lambda_1,\cdot)Y(\lambda_1,\cdot) + X(\lambda_2,\cdot)Y(\lambda_2,\cdot)\right]\\
&\quad + m^2(1-t)^2 X(\lambda_2,\cdot)Y(\lambda_2,\cdot)\\
&= \left[t^2 + mt(1-t)\right]X(\lambda_1,\cdot)Y(\lambda_1,\cdot) + \left[mt(1-t) + m^2(1-t)^2\right]X(\lambda_2,\cdot)Y(\lambda_2,\cdot)\\
&= t\left[t + m(1-t)\right]X(\lambda_1,\cdot)Y(\lambda_1,\cdot) + m(1-t)\left[t + m(1-t)\right]X(\lambda_2,\cdot)Y(\lambda_2,\cdot)\\
&= \left[t + m(1-t)\right]\left[tX(\lambda_1,\cdot)Y(\lambda_1,\cdot) + m(1-t)X(\lambda_2,\cdot)Y(\lambda_2,\cdot)\right].
\end{aligned}$$

On the other hand,

$$t + m(1-t) \le t + (1-t) = 1,$$

since, $0 \le m \le 1$ and $t \in (0,1)$.

Therefore

$$\begin{aligned}
&XY\left(t\lambda_1 + m_1(1-t)\lambda_2,\cdot\right)\\
&\le \left[t + m(1-t)\right]\left[tX(\lambda_1,\cdot)Y(\lambda_1,\cdot) + m(1-t)X(\lambda_2,\cdot)Y(\lambda_2,\cdot)\right]\\
&\le tX(\lambda_1,\cdot)Y(\lambda_1,\cdot) + m(1-t)X(\lambda_2,\cdot)Y(\lambda_2,\cdot)\\
&= tXY(\lambda_1,\cdot) + m(1-t)XY(\lambda_2,\cdot) \quad (a.e.).
\end{aligned}$$

From this last inequality, we conclude that XY is *m*-convex almost everywhere. □

Proposition 3.4. *Let* $X:[0,b]\times\Omega \to \mathbb{R}$ *and* $Y:[0,r]\times\Omega \to \mathbb{R}$ *be m-convex* *stochastic processes such that* $rang\left[X(\lambda,\cdot)\right]\subset[0,r]$ *for all* $\lambda\in[0,b]$. *If* Y *is increasing, then the composition function* $Y\circ X$ *is m-convex in* $[0,b]$ *almost everywhere.*

Proof. Let $\lambda_1,\lambda_2\in[0,b]$ y $t\in[0,1]$, then we have

$$\begin{aligned}(Y\circ X)(t\lambda_1+m(1-t)\lambda_2,\cdot)&=Y\left(X\left(t\lambda_1+m(1-t)\lambda_2,\cdot\right),\cdot\right)\\&\le Y\left(tX(\lambda_1,\cdot)+m(1-t)X(\lambda_2,\cdot),\cdot\right)\\&\le tY\left(X(\lambda_1,\cdot),\cdot\right)+m(1-t)Y\left(X(\lambda_2,\cdot),\cdot\right)\\&=t(Y\circ X)(\lambda_1,\cdot)+m(1-t)(Y\circ X)(\lambda_2,\cdot)\quad(a.e.).\end{aligned}$$

Therefore, the function $Y\circ X$ is *m*-convex in $[0,b]$ almost everywhere. □

The following result gives necessary conditions under which a pair of stochastic processes may be separated by a stochastic process *m*-convex. We shall prove a sandwich type theorem inspired in [20].

Note that:

Remark 3.5. *If* $I=(0,+\infty)$ *or* $I=[0,+\infty)$ *and* $X:I\times\Omega\to\mathbb{R}$ *is a m-convex stochastic process, then*

$$X(ma,\cdot)\le mX(a,\cdot),\quad a\in I.\quad(a.e.)\tag{1}$$

Theorem 3.6. *Let* $I=(0,+\infty)$ *or* $I=[0,+\infty)$ *and* $X:I\times\Omega\to\mathbb{R}$ *be a mean-square integrable stochastic process, no negative and m-convex, then there exist a convex stochastic process* $Z:I\times\Omega\to\mathbb{R}$ *such that*

$$X(a,\cdot)\le Z(a,\cdot)\le mX\left(\frac{a}{m},\cdot\right),\quad \text{for all } a\in I.\quad(a.e.)$$

or equivalent

$$\frac{1}{m}Z(ma,\cdot)\le X(a,\cdot)\le Z(a,\cdot),\quad \text{for all } a\in I.\quad(a.e.)$$

Proof. Let $a,b\in I$. Since $X:I\times\Omega\to\mathbb{R}$ is a *m*-convex stochastic process, we have

$$X\left(ta+m(1-t)b,\cdot\right)\le tX(a,\cdot)+m(1-t)X(b,\cdot),$$

for all $t\in[0,1]$.

Replacing b with $\frac{b}{m}$ in the previous inequality, we obtain:

$$X\left(ta+m(1-t)\frac{b}{m},\cdot\right)\le tX(a,\cdot)+m(1-t)X\left(\frac{b}{m},\cdot\right).$$

where from,

$$X\left(ta+(1-t)b,\cdot\right)\le tX(a,\cdot)+m(1-t)X\left(\frac{b}{m},\cdot\right).$$

On the other hand, from inequality (1), it follows that:

$$\begin{aligned}X\left(ta+(1-t)b,\cdot\right)&\le tX(a,\cdot)+m(1-t)X\left(\frac{b}{m},\cdot\right)\\&\le tmX\left(\frac{a}{m},\cdot\right)+(1-t)mX\left(\frac{b}{m},\cdot\right).\end{aligned}$$

Let $Y : I \times \Omega \to \mathbb{R}$ be the convex stochastic process, defined as follows:

$$Y(a,\cdot) := mX\left(\frac{a}{m},\cdot\right),$$

we get

$$X(ta+(1-t)b,\cdot) \le tY(a,\cdot) + m(1-t)Y(b,\cdot).$$

Applying the sandwich theorem for convex functions, we conclude that there exists a convex stochastic process $Z : I \times \Omega \to \mathbb{R}$, such that

$$X(a,\cdot) \le Z(a,\cdot) \le Y(a,\cdot), \quad a \in I.$$

Hence,

$$X(a,\cdot) \le Z(a,\cdot) \le mX\left(\frac{a}{m},\cdot\right), \quad a \in I, \quad (a.e.)$$

or equivalent

$$\frac{1}{m}Z(ma,\cdot) \le X(a,\cdot) \le Z(a,\cdot), \quad a \in I. \quad (a.e.)$$

□

Theorem 3.7. *Let* $b > 0$, $m \in [0,1]$ *and* $X, Y : [0,b] \times \Omega \to \mathbb{R}$ *be stochastic processes, where* Y *is non-negative. There exists a m-convex stochastic process* $Z : [0,b] \times \Omega \to \mathbb{R}$, *such that* $X \le Z$ *in* $[0,mb]$ *and* $Z \le Y$ *in* $[0,b]$, *if and only if, for any* $\lambda_1, \lambda_2 \in [0,b]$ *and* $t \in [0,1]$, *the following inequality holds almost everywhere*

$$X(t\lambda_1 + m(1-t)\lambda_2,\cdot) \le tY(\lambda_1,\cdot) + m(1-t)Y(\lambda_2,\cdot).$$

Proof. ($\Rightarrow$) Suppose there exists a *m*-convex stochastic process $Z : [0,b] \times \Omega \to \mathbb{R}$, such that $X \le Z$ in $[0,mb]$ and $Z \le Y$ in $[0,b]$.

If $\lambda_1, \lambda_2 \in [0,b]$ and $t \in [0,1]$, then

$$\begin{aligned} X(t\lambda_1 + m(1-t)\lambda_2,\cdot) &\le Z(t\lambda_1 + m(1-t)\lambda_2,\cdot) \\ &\le tZ(\lambda_1,\cdot) + m(1-t)Z(\lambda_2,\cdot) \\ &\le tY(\lambda_1,\cdot) + m(1-t)Y(\lambda_2,\cdot). \quad (a.e.) \end{aligned}$$

($\Leftarrow$) Suppose for all $\lambda_1, \lambda_2 \in [0,b]$ y $t \in [0,1]$, the following inequality holds

$$X(t\lambda_1 + m(1-t)\lambda_2,\cdot) \le tY(\lambda_1,\cdot) + m(1-t)Y(\lambda_2,\cdot).$$

Let us consider the following set

$$\mathbb{A} = EnvConv_Y\left\{(p,q) \in [0,b] \times \mathbb{R} : Y(p,\cdot) \le q\right\}.$$

That is, $\mathbb{A}$ is the set of the convex hull of the epigraphs of Y.

If $(p,q) \in \mathbb{A}$, then by Caratheodory's theorem, (p,q) belongs to the interior of $S \subset \mathbb{A}$, where S is the affine convex set of the form $t(p_1,q_1) + (1-t)(p_2,q_2)$ for $t \in [0,1]$ and $(p_1,q_1),(p_2,q_2)$ vertices of S.

Let $q_0 = inf\left\{r \in \mathbb{R} : (p,r) \in S\right\}$.

Since Y is non-negative, we have S is bounded set. Therefore, $q \ge q_0$ and

(p,q_0) is a limit point of S, where from

$$(p,q_0)=t(p_1,q_1)+(1-t)(p_2,q_2),$$

for some, $t\in[0,1]$ and $(p_1,q_1),(p_2,q_2)$ vertices of S.

Hence

$$\begin{aligned} q_0 &\geq tY(p_1,\cdot)+(1-t)Y(p_2,\cdot) \\ &\geq tY(p_1,\cdot)+(1-t)mY(p_2,\cdot) \\ &\geq X(tp_1+m(1-t)p_2,\cdot) \\ &= X(p,\cdot). \end{aligned}$$

Let's define

$$Z(p,\cdot):=inf\{q\in\mathbb{R}:(p,q)\in\mathbb{A}\},$$

this infimum exists because Y is non-negative.

It is clear that, $X\leq Z$ in $[0,mb]$, given that $p\in[p_1,mp_2]$.

Besides, $(p,Y(p,\cdot))\in\mathbb{A}$ for any $p\in[0,b]$ and $Z\leq Y$ by the definition of infimum.

It remains to be shown that, Z es *m*-convex.

Let $p_1,p_2\in[0,b]$ and $t\in[0,1]$. If q_1,q_2 are such that

$$(p_1,q_1),(mp_2,mq_2)\in\mathbb{A},$$

then

$$(tp_1+(1-t)mp_2,tq_1+(1-t)mq_2)\in\mathbb{A}.$$

Consequently,

$$Z(tp_1+(1-t)mp_2,\cdot)\leq tq_1+(1-t)mq_2,$$

for any q_1,q_2, in particular for the infimum.

Therefore

$$Z(tp_1+(1-t)mp_2,\cdot)\leq tZ(p_1,\cdot)+(1-t)mZ(p_2,\cdot).\quad (a.e.)$$

□

Corollary 3.8. *Let* $b>0$, $m\in[0,1)$ *and* $X,Y:[0,b]\times\Omega\to\mathbb{R}$ *be stochastic processes, with* Y *non-negative, such that*

$$X(t\lambda_1+(1-t)m\lambda_2,\cdot)\leq tY(\lambda_1,\cdot)+(1-t)mY(\lambda_2,\cdot),\quad (a.e.)$$

for all $t\in[0,1)$, then $X(0,\cdot)\leq 0$.

Proof. By the previous theorem, there exists a *m*-convex stochastic process, $Z:[0,b]\times\Omega\to\mathbb{R}$, such that $X\leq Z$ in $[0,mb]$ y $Z\leq Y$ in $[0,b]$.

Therefore

$$X(t\lambda_1+(1-t)m\lambda_2,\cdot)\leq Z(t\lambda_1+(1-t)m\lambda_2,\cdot)\leq tZ(\lambda_1,\cdot)+(1-t)mZ(\lambda_2,\cdot).$$

If $\lambda_1=\lambda_2=0$, then

$$\begin{aligned} X(0,\cdot) &= X\left(t0+(1-t)m0,\cdot\right) \\ &\le Z\left(t0+(1-t)m0,\cdot\right) \\ &= Z(0,\cdot) \\ &\le tZ(0,\cdot)+(1-t)mZ(0,\cdot) \\ &\le \left(t+m(1-t)\right)Z(0,\cdot) \\ &\le 0, \quad (a.e.) \end{aligned}$$

given that, $t+m(1-t)<1$. □

Next definition is the counterpart to the given for *m*-convex functions in [19].

Definition 3.9. *Let* $\varepsilon \ge 0$ *and* $m \in [0,1]$. *A stochastic process* $X : I \times \Omega \to \mathbb{R}$ *is* ε*-m-convex, if for any* $\lambda_1, \lambda_2 \in [0,b]$ *and* $t \in [0,1]$, *we have*

$$X\left(t\lambda_1 + m(1-t)\lambda_2,\cdot\right) \le tX(\lambda_1,\cdot) + m(1-t)X(\lambda_2,\cdot) + \varepsilon. \quad (a.e.)$$

An important consequence of the previous theorem, is the following Hyers-Ula*m*-type stability result for *m*-convex stochastic processes. More in detail.

Corollary 3.10. *Let* $\varepsilon > 0$ *and* $m \in [0,1]$. *If* $X : [0,b] \times \Omega \to \left[-\frac{\varepsilon}{m}, +\infty\right)$ *is a* ε*-m-convexo stochastic process, ther exixts a function* $Z : [0,b] \times \Omega \to \mathbb{R}$ *m-convex*, *such almost everywhere that*

$$\left|X(\lambda,\cdot) - Z(\lambda,\cdot)\right| \le \frac{\varepsilon}{2m}, \quad \lambda \in [0, mb].$$

Proof. Let $Y := X + \frac{\varepsilon}{m}$.

We have, Y is a non-negative stochastic process.

On the other hand,

$$\begin{aligned} X\left(t\lambda_1+(1-t)m\lambda_2,\cdot\right) &\le tX(\lambda_1,\cdot)+(1-t)mX(\lambda_2,\cdot)+\varepsilon \\ &= tX(\lambda_1,\cdot)+t\frac{\varepsilon}{m}+(1-t)mX(\lambda_2,\cdot)+(1-t)\varepsilon+\varepsilon \\ &\quad -t\frac{\varepsilon}{m}-(1-t)\varepsilon \\ &= t\left(X(\lambda_1,\cdot)+\frac{\varepsilon}{m}\right)+(1-t)m\left(X(\lambda_2,\cdot)+\frac{\varepsilon}{m}\right)+t\varepsilon-t\frac{\varepsilon}{m} \\ &= tY(\lambda_1,\cdot)+(1-t)mY(\lambda_2,\cdot)+\varepsilon t\left(1-\frac{1}{m}\right) \\ &\le tY(\lambda_1,\cdot)+(1-t)mY(\lambda_2,\cdot), \quad \text{since } \varepsilon t\left(1-\frac{1}{m}\right)\le 0. \end{aligned}$$

By the previous theorem, there exists a *m*-convex stochastic process $H : [0,b] \times \Omega \to \mathbb{R}$, such that $X \le H$ in $[0, mb]$ and $H \le Y$ in $[0,b]$. where from,

$$X \le H \le Y = X + \frac{\varepsilon}{m} \quad \text{in } [0, mb]$$

Defining, $Z := H - \frac{\varepsilon}{2m}$, we have to Z is *m*-convex.

Besides,

$$\left|X(\lambda,\cdot)-Z(\lambda,\cdot)\right|\le\frac{\varepsilon}{2m},\quad \lambda\in[0,mb].\quad (a.e.)$$

□

Theorem 3.11. *Let* $X:[0,b]\times\Omega\to\mathbb{R}$ *be a twice differentiable mean square stochastic process and* $k_1,k_2\in\mathbb{R}$, *such that*

$$k_1\le X''\le k_2.$$

Then, for $m\in[0,1]$ *fixed,* $a\in[0,b]$ *and* $t\in[0,1]$ *arbitrary, we have almost everywhere*

$$k_1\frac{(mb-a)^2 t(1-t)}{2}\le tX(a,\cdot)+(1-t)X(mb,\cdot)-X(ta+m(1-t)b,\cdot)$$
$$\le k_2\frac{(mb-a)^2 t(1-t)}{2}.$$

Proof. We define,

$$Y(t,\cdot):=tX(a,\cdot)+(1-t)X(mb,\cdot)-X(ta+m(1-t)b,\cdot)-k_1\frac{(mb-a)^2 t(1-t)}{2},$$

with $t\in[0,1]$.

Then,

$$Y''(t,\cdot)=-(a-mb)^2X''(ta+m(1-t)b,\cdot)+k_1(mb-a)^2$$
$$=(mb-a)^2\left[k_1-X''(ta+m(1-t)b,\cdot)\right]$$
$$\le 0.$$

where from, Y ia a concave stochastic process on $[0,1]\times\Omega$, moreover $Y(0,\cdot)=Y(1,\cdot)=0$, therefore, $Y(t,\cdot)\ge 0$, for all $t\in[0,1]$, so left hand side of inequality holds.

On the other hand, we define

$$Z(0,\cdot):=tX(a,\cdot)+(1-t)X(mb,\cdot)-X(ta+m(1-t)b,\cdot)$$
$$-k_2\frac{(mb-a)^2 t(1-t)}{2},\quad (a.e.)$$

with $t\in[0,1]$.

Using a procedure analogous to the previous one, it is shown that $Z(0,\cdot)$ is a convex stochastic process on $[0,1]\times\Omega$, moreover $Z(0,\cdot)=Z(1,\cdot)=0$.

Therefore, $Y(t,\cdot)\le 0$, for all $t\in[0,1]$, so right hand side of inequality holds. □

As a consequence of the previous theorem, we obtain an integral inequality of Hermite-Hadamard type for *m*-convex stochastic processes.

In more detail, the following result is obtained.

Corollary 3.12. *Let* $X:[0,b]\times\Omega\to\mathbb{R}$ *be a twice differentiable mean square stochastic process and* $k_1,k_2\in\mathbb{R}$, *such that*

$$k_1\le X''\le k_2.$$

Then, for $m\in[0,1]$ *fixed,* $a\in[0,b]$ *and* $t\in[0,1]$ *arbitrary, we have almost* *everywhere*

$$k_1\frac{(mb-a)^2}{12}\le\frac{X(a,\cdot)+X(mb,\cdot)}{2}-\frac{1}{mb-a}\int_a^{mb}X(x,\cdot)\mathrm{d}x$$
$$\le k_2\frac{(mb-a)^2}{12}.$$

Moreover, if X *is a m-convexo stochastic process, then the following inequalities of Hermite-Hadamard type take place*:

$$\frac{X(a,\cdot)+X(mb,\cdot)}{2}-k_2\frac{(mb-a)^2}{12}\le\frac{1}{mb-a}\int_a^{mb}X(x,\cdot)\mathrm{d}x$$
$$\le\frac{X(a,\cdot)+mX(b,\cdot)}{2}-k_1\frac{(mb-a)^2}{12}.$$

Proof. By the previous Theorem, for $m\in[0,1]$ fixed, $a\in[0,b]$ and $t\in[0,1]$ arbitrary, we have

$$k_1\frac{(mb-a)^2t(1-t)}{2}\le tX(a,\cdot)+(1-t)X(mb,\cdot)-X(ta+m(1-t)b,\cdot)$$
$$\le k_2\frac{(mb-a)^2t(1-t)}{2}.\quad(a.e.)$$

Integrating each term of the previous inequalities, with respect to $t\in[0,1]$, and by the change of variable $\alpha=ta+m(1-t)b$, we get the first inequalities.

With a similar procedure, the inequalities of Hermite-Hadamard type are obtained, but considering now that $X(mb,\cdot)\le mX(b,\cdot)$, since X is a m-convexo stochastic process.

□

4. Conclusions

This paper establishes fundamental advances in the theory of m-convex stochastic processes. The central results of this research: The generalization of the sandwich theorem and Hyers-Ulam stability for m-convex functions, provide new theoretical tools for bounding such m-convex stochastic processes by classical convex processes and approximating perturbations by m-convex functions.

As a significant corollary, a Hermite-Hadamard-type inequality is obtained, which deepens the analytical structure of these processes and extends their applicability to statistics and applied mathematics.

These contributions not only consolidate a solid theoretical framework for stochastic convexity but also open new avenues of research in the study of approximations in the theory of convex analysis, in stochastic optimization, and in stability analysis in nonlinear contexts.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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