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Groupoid Approach to Ergodic Dynamical System of Commutative von Neumann Algebra

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Abstract

Given a compact and regular Hausdorff measure space (X,μ), with μ a Radon measure, it is known that the generalised space $\mathcal{M}(X)$ of all the positive Radon measures on X is isomorphic to the space of essentially bounded functions $L^{\infty}(X,\mu)$ on X. We confirm that the commutative von Neumann algebras $\mathscr{M}\subset B(\mathcal{H})$, with $\mathcal{H}=L^{2}(X,\mu)$, are unitary equivariant to the maximal ideals of the commutative algebra $C(X)$. Subsequenly, we use the measure groupoid to formulate the algebraic and topological structures of the commutative algebra $C(X)$ following its action on $\mathcal{M}(X)$ and define its representation and ergodic dynamical system on the commutative von Neumann algebras $\mathscr{M}$ of $B(\mathcal{H})$.

Keywords

Measure Groupoid, Groupoid Equivalence, Ergodic Action, Convolution Algebra, von Neumann Algebra, Generalized Space

1. Preliminaries

Assuming the preliminary materials of [1] which form the background of this work, we are motivated to explore an alternative approach to the representation of the dynamical system of the commutative algebra $C(X)$ on a commutative von Neumann algebra using groupoid framework.

This work uses the more complex and profound analytic method, according to [2], for operator algebra. In place of the usual tool of polar decomposition of linear forms, we employ the inherent decomposition of measures within the measure groupoid. This enables the use of groupoid equivalence to present the relations existing between the von Neumann algebra and its commutants. Hence, the groupoid representations relate the predual of the algebra to the Hilbert space constituting its domain.

The starting point for the analytical method is the relationship between the closure of a space or subspace with the boundedness of the functions defined on it. Because the measurability of functions imposes a very little restriction on the space, according to Connes [3], which translates to closure of the space or subspace supporting the function; the space must be a closed interval $I=[a,b]$ or a standard Borel space X for measurability to be granted. Hence, measurability of X is a structure defined on X by measurable functions. This measure structure is invariant under $Aut(X)$ the transformations of X. (cf. [3]). The definition of a Borel measure as a positive operator valued set map by [4] connects these structures to operator algebra.

The connection is based on the positivity and completeness of the Hilbert space of square integrable functions $L^2(X,\mu)$, and the fact that the Radon-Nikodym Theorem asserts that the derivatives $\frac{\mathrm{d}\mu}{\mathrm{d}\nu}$ are measurable functions f on X. It follows that while the Borel structure on X gives rise to the Hilbert space $L^2(X,\mu)$ and von Neumann algebra, the topological structure defined by continuous functions $f: X \to I\!R$ gives rise to the Banach algebra $C(X)$ with norm $\|f\| = \sup_{x\in X}|f(x)|$. This good interaction between the two structures is established using the Borel measures on X, whose mutual derivatives define measurable functions, and define linear forms $\varphi(f)$ on $C(X)$ and bilinear/sesquilinear forms or inner product $\varphi(\bar{f}f)$ on $\mathcal{H}=L^2(X,\mu)$, with $\varphi(\bar{f}f)\geq 0$.

Spectral consideration is used to identify the algebra of these measures as von Neumann algebra of operators on $\mathcal{H}$. For the spectrum of a self-adjoint operator $T\in\mathcal{B}(\mathcal{H})$ can be analysed using polynomials $p(x)$ which which are constitutive of the maximal ideals $\{\mathfrak{m}_x : x\in X\}$ and the geometric structure of the algebra $C(X)$. Every polynomial $p(x)$ on X defines an operator as a measurable function, and there is always a sequence of polynomials uniformly approximating a measurable function by Weierstrass approximation theorem. Thus, the continuous extensions of any continuous function vanishing at a point $x\in X$ to a function vanishing in some closed subsets containing x define the spectrum of the resulting partially ordered operators. Cf. [4].

These continuous extensions are captured by the uniformly convergent sequences or nets, such that the sequence $f_n \to f$ means $f_n(x)\to f(x)$ for all $x\in X$; which also gives rise to weak convergence $\langle f_n(T)\varepsilon,\eta\rangle \to \langle f(T)\varepsilon,\eta\rangle$, $\forall\, \varepsilon,\eta\in\mathcal{H}$. A net defines a unique measure class $[\mu]$ on X supported on the closed (compact) interval $I=[-\|T\|,\|T\|]$ such that $f(T)=0 \Leftrightarrow \int|f|\mathrm{d}\mu = 0$. This interval is related to the spectral radius of the operator T in a von Neumann algebra $\mathscr{M}$ which can be defined as follows.

Definition 1.1 (Cf. [3]) A commutative von Neumann algebra $\mathscr{M} \subset B(\mathcal{H})$ is the algebra of operators on $\mathcal{H}$ of the form $f(T)$ for some bounded Borel function *f*. This is called the von Neumann algebra generated by the operator *T*.

Hence, $\mathscr{M}$ is made up of operators with the same symmetries as *T*. This means that they commute with all unitary operators ($U \in B(\mathcal{H})$, commuting with *T*. Thus, given that $S \in \mathscr{M}$, then $UTU^* = T \Rightarrow Uf(T)U^* = USU^* = S$, where $S = f(T)$ for some Borel function *f*. The commutative von Neumann algebra $\mathscr{M}$ is naturally isomorphic to $L^\infty(I,\mu)$-the algebra of bounded measurable functions on $I = [-\|T\|, \|T\|]$ that is equal μ-a.e.

Because proper actions relate directly to slice theorem used in the cohomogeneity-one *G*-space analysis as in [5] [6] [7], some of the main results of the paper also relate to slice theorem.

2. The Algebra and the Generalized Space

According to [8], the time evolution of dynamical systems modelled by measure-preserving actions of integers $\mathbb{Z}$ or real numbers $I\!R$ which represent passage of time are generalized by measure-preserving actions of *lattices* which are usually "subgroup" of Lie groups. The two basic constituents of the commutative algebra $C(X)$: the Borel group of units $G(1)$, and the maximal ideals $\mathfrak{m}_x$, are used to model the above in the action of $C(X)$. The dynamical system defined by the Borel group $G(1)$ of units on the geometric point $\mathcal{X} \subset \mathcal{M}(X)$ is the ergodic action of the (lattice) algebra $C(X)$ on the generalized space $\mathcal{M}(X)$ of nonnegative Radon measures on the space *X*; and the maximal ideals $\mathfrak{m}_x$ are the (projective) modules which characterize and encode the symmetries of measurable functions vanishing on the neighbourhoods of each point of *X*.

These symmetries are represented by *z*-ultrafilters $\mathcal{F}_x$ of zero sets (affine algebraic varieties of $C(X)$) converging to each $x \in X$. The complements of these algebraic sets constitute the open neighbourhood base of points of *X*. The ultrafilter $\mathcal{F}_x$ convergence of closed sets to *x* has associated nets of polynomials or measurable functions converging to a function *f* defined on *x*. Given a net f_α of contractions in the complete metric space *X*, as in [9], it follows that $f_\alpha \to f$ such that $f(x) = x$. All these are represented on the generalized space $\mathcal{M}(X)$ with ergodic action of $C(X)$ in form of ergodic groupoid.

The idea of a generalized space $\mathcal{M}(X)$ of Radon measures on X which is conceived as the *state space* (cf. [8]) is a direct extension of Mackey's conception of a measure class *C* as a *generalized subset*. At the centre of this extension is the focus on i) measure preserving transformations of the compact metric space *X*, and ii) the Dirac measures δ_x as generalized or geometric points embedding the points of X in the generalized space. The role assigned to the ergodic transformations by Mackey, is to translate along time in such a way as to ensure the invariance of measure or state.

Every measure preserving continuous linear transformation $\varphi : X \to X$ decomposes into ergodic components. This is based on the fact that the "noncommutative spaces" replacing the "phase space" are basically quotient spaces determined by ergodic actions of a Borel group. Hence, X is embedded in the generalized space as its geometric or generalized points; that is, the $G(1)$-space $\mathcal{X}=\{\delta_x : x\in X\}$. Within this quotient setting, it is clear that non-ergodic transformations are constituted by ergodic components which are considered limits of nets of the former. This makes ergodic sense of what is said above on the operator T, and its transforms $S=f(T)$ constituting the von Neumann algebra $\mathscr{M}$, and the commutants $\mathscr{M}'$ made up of unitary operators $\mathscr{U}(\mathcal{H})$ leaving them invariant. This is given as follows.

Proposition 2.1 *The algebra $C(X)$ defines an ergodic and equivariant transformations φ by its Borel group $G(1)$ on X and on the generalized space $\mathcal{M}(X)$.*

Proof. The homothety $G(1): X\to X$ defined by $g: x\mapsto \frac{y-x}{g(x)}$, is a transformation of balls $B(x,g(x))$ centred at x. It is measure preserving since the push forward of a Radon measure μ under the map is given as $\varphi_{(x,g)_*}\mu(A)=\mu(g(x)A+x)$, $A\subset X$. The measure class is preserved because $g(x)>0$ for all $g\in G(1)$, $x\in X$.

According to [8], ergodic theorems express a relationship between averages taken along the orbit of a point under the iteration of a measure-preserving map or transformation. The iteration of the transformations $\varphi: X\to X$ on X which induces $\varphi_*:\mathcal{X}\to\mathcal{X}$ on the generalized points $\mathcal{X}$ represents passage of time, and its invariance in both spaces $\varphi: x\mapsto\varphi(x)\simeq\varphi_*:\delta_x\mapsto\delta_{\varphi(x)}$ constitutes limits of nets of transformations involving the maximal filter convergence $\mathcal{F}_x\to x$ and the convergence of net of tangent measures of $\mu_\alpha\to\nu$. These represent averages over time.

The induced iteration on the generalized space $\mathcal{M}(X)$ with respect to some invariant measure μ (or measure class $\varphi_*:[\mu_f]\to[\mu_{f\circ\varphi}]$) represents invariance over states. The ergodicity represented by average over space or states (averages taken over the classes of measures) is given by nets of invariant non-ergodic measures converging to an ergodic limit. It is also the convergence of operators to an ergodic operator in a von Neumann algebra. Cf. [4]. These two averages are given by the invariance or stability of $\mathcal{X}$ and the measure classes $\{[\mu_f]: f\in\mathfrak{m}_f\}$ under $G(1)$-actions.

Remark 2.2 From the proposition, we see that the restriction of φ_* to the generalized points $\mathcal{X}=\{\delta_x : x\in X\}\subset\mathcal{M}(X)$ gives a transformation of $\mathcal{X}$ defined as $\varphi_*:\delta_x\mapsto\delta_{\varphi(x)}$, such that for any $A\subseteq X$, we have

$$(\varphi_*\delta_x)(A)=\delta_x(\varphi^{-1}A)=\delta_{\varphi(x)}(A).$$

This shows that the set $\mathcal{X}=\{\delta_x : x\in X\}\subset\mathcal{M}(X)$ of the generalized points can be continuously and affinely extended to the generalized space $\mathcal{M}(X)$.

Subsequently, the generalized points $\mathcal{X} := \{\delta_x : x \in X\}$ can generate the generalized space $\mathcal{M}(X)$ and the generalized subspaces $\{C_f : f \in \mathfrak{m}_x\}$ or the measure classes. This is stated in the following result.

Proposition 2.3 *The generalized space* $\mathcal{M}(X)$ *of Radon measures on X is an affine and continuous extension of the geometric points* $\mathcal{X} := \{\delta_x : x \in X\}$.

Proof. The coincidence of zero sets of $C(X)$ with null sets of $\mathcal{M}(X)$ establishes the existence of nets $\{\mu_\alpha\}$ of non-ergodic Radon measures related to $\{\mu_f : f \in \mathfrak{m}_x, x \in X\}$ which converge to the Dirac measures $\{\delta_x : x \in X\}$ as the *z*-ultrafilter $Z[\mathfrak{m}_x]$ converges $\mathcal{F}_x \to x$. Since the elements of $\mathfrak{m}_x$ vanish at *x*, its $G(1)$-action is transferred to fibre of measure classes (or tangent measures to δ_x) via φ_* (see [1]). □

From this result, the dynamism defined by the transformations φ is encoded in the symmetry of the measures classes contributing to the convergent nets. Hence, the connection between ergodic theory and the dynamics defined by continuous transformations on compact metric spaces is encoded by the closure of the resulting convex set of non-ergodic φ-invariant measures with ergodic measures as boundary. Cf. [8].

3. The Principal Groupoid and its Action

In what follows, we present the action of the commutative algebra $C(X)$ using the groupoid equivalence. The $C(X)$-action is determined at each point $x \in X$ by the maximal ideal $\mathfrak{m}_x$ and the Borel group $G(1)$. The maximal ideal is a module of the (lattice) algebra $C(X)$ and a $G(1)$-space at every point $x \in X$. Hence, there is a trivialization of an action groupoid on *X* which we will now explore in order to describe the $C(X)$ dynamical system on *X* and on the generalized space $\mathcal{M}(X)$.

The two algebraic objects $\mathfrak{m}_x$ and $G(1)$ aid in the understanding of the dynamics associated to the commutative algebra $C(X)$ at each $x \in X$. Their employment also associates a *z*-ultrafilter related to a maximal ideal $\mathfrak{m}_x$ to the dynamical system. Thus, given the zero map $Z : C(X) \to X$, the family of closed sets $\{Z(f) : f \in \mathfrak{m}_x\}$ is a closed cover for *X*. Cf. [10]. An open cover for *X* can be constructed from their complements, a countable number of $U_f = X - Z(f)$, such that (U_f, ϕ_f) is an open covering for *X* and each inverse image $\pi^{-1}(U_f)$ is fibrewise homeomorphic to $U \times \mathfrak{m}_x$. These give a system of homeomorphisms $\phi_f : U_f \times \mathfrak{m}_x \to \pi^{-1}(U_f)$ forming the transition functions

$$\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathfrak{m}_x \to (U_\alpha \cap U_\beta) \times \mathfrak{m}_x.$$

These transition functions also form unitary group if the Borel measures defined on the closed subsets are given the following characterisation.

Definition 3.1 [4] A positive operator valued measure is a triple $(X, \mathcal{B}, \mu)$, where X is a set, $\mathcal{B}$ is a ring (or σ-algebra) of subsets of X, and μ is an operator valued set function on $\mathcal{B}$ with the following properties

1) μ is positive, *i.e.* $\mu(M) \geq 0$ for each $M \in \mathcal{B}$.

2) μ is additive, *i.e.* $\mu(M \cup N) = \mu(M) + \mu(N)$ whenever $M \cap N = \varnothing$ in $\mathcal{B}$.

3) μ is continuous in the sense that $\mu(M) = LUB\{\mu(M_n)\}$ if M_n is an increasing sequence of sets in $\mathcal{B}$ whose union M is also in $\mathcal{B}$. So μ is called positive operator-valued measure on X or $\mathcal{B}$. It is monotone on $\mathcal{B}$ if $M \subset N \Rightarrow \mu(M) \leq \mu(N)$.

These conditions are satisfied by the complements of ultra filters of zero sets $Z(\mathfrak{m}_x)$ of the maximal ideals $\mathfrak{m}_x$ of $C(X)$ [10]. Given a maximal filter $\mathfrak{m}_x$, we have $M = \bigcup U_f$, where $U_f = X - Z(f), f \in \mathfrak{m}_x$; it follows that $\mu(U_f)$ is an increasing sequence of Hermitian operators, with $\mu(U_f) \leq \mu(M)$ $\forall f \in \mathfrak{m}_x$.

Proposition 3.2 *The group of automorphisms or transformations* $G(1) \subseteq Aut(X, \mu)$ *of X constitutes the unitary group* $\mathscr{U}(\mathcal{H})$ *of the space of operators* $B(\mathcal{H})$.

Proof. That μ is positive operator-valued implies the map $\mu : \mathcal{B} \to B(\mathcal{H})$. Then Borel measures on X define positive operators on $\mathcal{H}$ since $\mu(\overline{f}f) \geq 0$ for any pair $\overline{f}, f \in L^2(X, \mu)$. As already noted, they are also linear forms φ on $C(X)$ by the map $\varphi(f) = \int f \mathrm{d}\mu$.

$\phi \in Aut(X, \mu)$ is identified with the unitary operator on $L^2(X, \mu)$ given by $U_\phi(f) = f \circ \phi^{-1}$ and $Aut(X, \mu)$ with a closed subgroup of the unitary $\mathcal{U}(L^2(X, \mu))$, where the map $Aut(X, \mu) \to \mathcal{U}(L^2(X, \mu)), \phi \mapsto U_\phi$ is the Koopman representation of $Aut(X, \mu)$ by [11]. Thus, $G(1)$ is a subgroup of the unitary group in view of the positive operators defined by the measures associated with $\mathfrak{m}_x$. □

Proposition 3.3 *The transformations* $\phi \in Aut(X, \mu)$ *define the system of homeomorphisms which are the transition functions of the fibre bundle structure.*

Proof. Given the definition of operator valued measures and with the preceding formulations, the group of automorphisms or transformations $G(1) \subseteq Aut(X, \mu)$ of X then constitutes the structure group of the fibre bundle since it defines an action on the fibres $\mathfrak{m}_x$ given as

$$G(1) \times \mathfrak{m}_x \to \mathfrak{m}_x, (\phi, f) \mapsto \phi(f) = f \circ \phi^{-1}.$$

The action is fibrewise since $Z(f) = Z(\phi(f))$, where Z is the zero map. □

We now use symmetry groupoid to capture these bundle symmetries.

Theorem 3.4 *The symmetries of the commutative algebra* $C(X)$ *give rise to* *a (Lie) symmetry groupoid.*

Proof. We use the trivialized action of the group of units $G(1)$ on each maximal ideal $\mathfrak{m}_x$ to formulate the bundle structure. The indexed family $\mathcal{Z} = \{\mathfrak{m}_x\}_{x \in X}$ of geometries constitute a bundle $\mathcal{Z}$ over *X*, with the projection $p : \mathcal{Z} \to X$, such that $\mathfrak{m}_x = p^{-1}(x)$. The "symmetries" of these geometric (closed) points of the commutative algebra $C(X)$ is expressed by the groupoid $\mathcal{G}(\mathcal{Z}) \rightrightarrows X$. Hence, with $(\mathcal{Z}, p, X)$ a vector bundle, and $\mathcal{G}(\mathcal{Z})$ the set of all vector space isomorphims $\xi : \mathfrak{m}_x \to \mathfrak{m}_y$ for $x, y \in X$. The Borel group $G(1) = G(\mathfrak{m}_x)$ of automorphisms of $\mathfrak{m}_x$ expresses the particular "symmetry" of $\mathfrak{m}_x$; and the groupoid $\mathcal{G}(\mathcal{Z})$ expresses the smoothly "varying symmetries" of the bundle.

The smooth bundle symmetry $\mathcal{G}(\mathcal{Z})$ is a Lie groupoid on *X* with respect to the following structure. For $\xi : G(\mathfrak{m}_x) \to G(\mathfrak{m}_y), s(\xi) = x, t(\xi) = y$; the objection map is $x \mapsto 1_x = Id_{\mathfrak{m}_x}$, the partial multiplication is the composition of maps; the inverse of $\xi \in G(\mathcal{Z})$ is its inverse as an isomorphism. The isotropy groups are the general linear groups $G(\mathfrak{m}_x)$ of the fibres which are all isomorphic [12]. □

Remark 3.5 *The general linear groups* $G(\mathfrak{m}_x)$ *coincide with the unitary group when the bundle is considered a Hilbert bundle. They define the* (*partial*) *symmetries of the system, which the Lie groupoid* $\mathcal{G}(\mathcal{Z}) \rightrightarrows X$ *represents.*

Given the Lie groupoid $\mathcal{G}(\mathcal{Z}) \rightrightarrows X$, its symmetries are modelled on the generalized space $\mathcal{M}(X)$. This will be achieved through the formulation of the action of the Lie groupoid $\mathcal{G}(\mathcal{Z}) \rightrightarrows X$ on the space of the generalized points $\mathcal{X} = \{\delta_x : x \in X\}$ homeomorphic to $\mathcal{Z}$. This will present the generalized space $\mathcal{M}(X)$ as a measure groupoid giving a generalized measure-theoretic approach to the dynamical system defined by the action of $C(X)$ [13].

From Mackey's definition of generalized subset, there is a correspondence between closed subsets of *X* and Radon measures in $\mathcal{M}(X)$; such that the points of *X* coincide with the Dirac measures $\delta_x \in \mathcal{M}(X)$ which are invariant ergodic measures [8]. The Dirac measures define the *point functionals* (cf. [14]). Because $\delta_x(X) = 1$, $\forall x \in X$, δ_x a probability measure. Hence, $\delta_x(f) = f(x)$ for any $x \in X$, a Borel subset $A \subset X$, and $f \in C(X)$. The action of the Lie groupoid on the set of generalized points $\mathcal{X} = \{\delta_x : x \in X\}$ is now considered.

Proposition 3.6 *Given the Lie groupoid* $\mathcal{G}(\mathcal{Z}) \rightrightarrows X$*, the set of generalized points* $\mathcal{X} = \{\delta_x : x \in X\}$ *is a* $\mathcal{G}(\mathcal{Z})$*-space.*

Proof. The homeomorphism $\rho : \mathcal{X} \to X$, which is a continuous open map from the space $\mathcal{X}$ onto the unit space *X*, defines a left action of $\mathcal{G}(\mathcal{Z})$ on $\mathcal{X}$, where $\mathcal{G}(\mathcal{Z}) \star \mathcal{X}$ is the set of composable pair (ξ, δ_x). This means $(\xi, \delta_x) \mapsto \xi x$ with $s(\xi) = \rho(\delta_x)$. In other words, $\mathcal{X}$ is a left $\mathcal{G}(\mathcal{Z})$-space if

$\rho : \mathcal{X} \ni \delta_x \mapsto s(\xi) \in X$. The action defines a groupoid equivalence on $\mathcal{X}$. Given any pair $\delta_x, \delta_y \in \mathcal{X}$, we say that $\delta_x \sim \delta_y$ if $\rho(\delta_x) = \rho(\delta_y)$ which implies $s(\xi) = \rho(\delta_x) = \rho(\delta_y)$. Since ξ in $\mathcal{G}(\mathcal{Z})$ are isomorphisms, $(\xi, \delta_x), (\xi, \delta_y)$ are composable pairs in $\mathcal{G}(\mathcal{Z}) \star \mathcal{X}$ [15].

This action of the Lie groupoid $\mathcal{G}(\mathcal{Z})$ on $\mathcal{X}$ is free and proper. Because $\xi \cdot \delta_x = \delta_x$ implies that ξ is a unit. It is *proper* also because the map $\mathcal{G}(\mathcal{Z}) \star \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ given by $(\xi, \delta_x) \mapsto (\xi \cdot \delta_x, \delta_x)$ is a proper map; that is, the inverse image of a compact set is compact. The two make $\mathcal{X}$ a principal $\mathcal{G}(\mathcal{Z})$-space. Hence, the natural projection $\pi : \mathcal{X} \to \mathcal{G}(\mathcal{Z}) \backslash \mathcal{X}$ onto the locally compact and Hausdorff orbit space $\mathcal{G}(\mathcal{Z}) \backslash \mathcal{X}$ is an open map, where $\mathcal{G}(\mathcal{Z}) \backslash \mathcal{X}$ means that the groupoid $\mathcal{G}$ has a left action on $\mathcal{X}$ [15]. □

Given that $\mathcal{X}$ is a left principal $\mathcal{G}(\mathcal{Z})$-space, then $\mathcal{X} \star \mathcal{X} = \{(\delta_x, \delta_y) \in \mathcal{X} \times \mathcal{X} : \rho(\delta_x) = \rho(\delta_y)\} \subset \mathcal{X} \times \mathcal{X}$ is the equivalence relation defined by the open map ρ (or $\mathcal{G}(\mathcal{Z})$-action) on $\mathcal{X}$. The equivalence classes are defined by having the same image in *X*. Since $\mathcal{G}(\mathcal{Z})$ acts by composition on $\mathcal{X}$, we have $\xi \cdot \delta_x \Rightarrow s(\xi) = \rho(\delta_x)$; that is, $\xi \circ \rho$ is defined on $\mathcal{X}$. Thus $\mathcal{X} \star \mathcal{X} \subset \mathcal{X} \times \mathcal{X}$ is a space of equivalence classes or pairs in $\mathcal{X}$ on which a diagonal action of $\mathcal{G}(\mathcal{Z})$ is defined as follows:

$$\mathcal{G}(\mathcal{Z}) \star (\mathcal{X} \star \mathcal{X}) \to \mathcal{X} \star \mathcal{X}, \xi \cdot (\delta_x, \delta_y) \mapsto (\xi \cdot \delta_x, \xi \cdot \delta_y).$$

Let $H = \mathcal{G}(\mathcal{Z}) \backslash \mathcal{X} \star \mathcal{X}$ be the orbit space of the diagonal action. Then H has a natural groupoid structure with multiplication defined as $[\delta_x, \delta_y] \cdot [\delta_y, \delta_z] = [\delta_x, \delta_z]$ with $H^o = \mathcal{G}(\mathcal{Z}) \backslash \mathcal{X}$ as the unit space. Thus, $\mathcal{G}(\mathcal{Z}) \backslash \mathcal{X} \star \mathcal{X} \rightrightarrows \mathcal{G}(\mathcal{Z}) \backslash \mathcal{X}$ is a groupoid which is denoted $H \rightrightarrows H^o$, where $t([\delta_x, \delta_y]) = [\delta_x]$ and $s([\delta_x, \delta_y]) = [\delta_y]$, for $[\delta_x], [\delta_y] \in \mathcal{G}(\mathcal{Z}) \backslash \mathcal{X}$.

Proposition 3.7 *The groupoid of equivalence* $H \rightrightarrows H^o$ *defined by* $\mathcal{G}(\mathcal{Z})$*-action on* $\mathcal{X}$ *defines a right action on the space* $\mathcal{X}$.

Proof. Given the derived groupoid $H \rightrightarrows H^o$, where $\sigma : \mathcal{X} \to H^o$ is a continuous open map from the (locally) compact space $\mathcal{X}$ onto the unit space $H^o = G(\mathcal{Z}) \backslash \mathcal{X}$, given as $\delta_x \mapsto [\delta_x] \Rightarrow \sigma(\delta_x) = t([\delta_x, \delta_y]) = t(\xi(\delta_x), \xi(\delta_y))$. Thus, the quotient groupoid H defines a right action on $\mathcal{X}$. We therefore have:

$$\mathcal{X} \star H = \{(\delta_z, h) = (\delta_z, [\delta_x, \delta_y]) \in \mathcal{X} \times H : \sigma(\delta_z) = [\delta_z] = [\delta_x] = t([\delta_x, \delta_y])\}.$$

Thus, the action is given by composition $\delta_z \cdot [\delta_x, \delta_y] = \xi(\delta_y)$, where ξ is unique in $G(\mathcal{Z})$ and satisfies $\delta_z = \xi \delta_x$.

The action is well defined for given $[\delta_{x'}, \delta_{y'}] = [\delta_x, \delta_y]$, then there exists a unique $h \in H$ such that $\delta_x h = \delta_{x'}$ and $\delta_y h = \delta_{y'}$. Hence, by definition $[\delta_{x'}] = [\delta_z] \Rightarrow [\delta_x] = [\delta_z]$, and if the three $[\delta_{x'}], [\delta_x], [\delta_z]$ are same orbit then there must be a unique element of $\mathcal{G}(\mathcal{Z})$ such that $\delta_{x'} \mapsto \delta_z$. This is given by

$\delta_{x'} \overset{h^{-1}}{\mapsto} \delta_x \overset{\xi}{\mapsto} \delta_z$. It therefore follows that

$$\delta_z \cdot \left[\delta_{x'}, \delta_{y'}\right] = \left(\xi h^{-1}\right) \delta_{y'} = \xi\left(\delta_y\right) = \delta_z \cdot \left[\delta_x, \delta_y\right].$$

□

Corollary 3.8 *The left* $\mathcal{G}(\mathcal{Z})$*-action* ρ *and right* H*-action* σ *commute on* $\mathcal{X}$.

Proof. Given the right action σ of the equivalence groupoid H on $\mathcal{X}$, it follows that $\mathcal{X}$ is a right principal H-space. The left action ρ of $\mathcal{G}(\mathcal{Z})$ and the right action σ of H commute $\sigma \circ \bar{\rho} = \rho \circ \bar{\sigma}$. The following diagram illustrates this commutativity of left $\mathcal{G}(\mathcal{Z})$-action ρ and right H-action σ on $\mathcal{X}$.

$$\begin{tikzcd}
\mathcal{G}(\mathcal{Z}) \arrow[dr, "\bar{\rho}"] \arrow[dd, "s"'] & & H \arrow[dl, "\bar{\sigma}"'] \arrow[dd, "t"] \\
& \mathcal{X} \arrow[dl, "\rho"] \arrow[dr, "\sigma"'] & \\
X & & \mathcal{G}(\mathcal{Z}) \setminus \mathcal{X}
\end{tikzcd}$$

So, the action ρ induces a homeomorphism of $\rho \circ \bar{\sigma} : \mathcal{X}/H \to X$ given as $\rho \circ \bar{\sigma}\left(\delta_z \cdot \left[\delta_x, \delta_y\right]\right) = \rho\left(\delta_y\right) = s(\xi)$. □

Theorem 3.9 *The space of generalized or geometric points* $\mathcal{X}$ *is a* $\left(\mathcal{G}(\mathcal{Z}), H\right)$*-equivalence.*

Proof. The proof follows from the above. As we have seen, $\mathcal{G}(\mathcal{Z})$ and H are locally compact groupoids, and $\mathcal{X}$ is a (locally) compact space that is i) a left principal $\mathcal{G}(\mathcal{Z})$-space, ii) a right principal H-space; and iii) the two actions commute; iv) the map $\rho : \mathcal{G}(\mathcal{Z}) \star \mathcal{X} \to \mathcal{X}$ induces a bijection of $\mathcal{X}/H$ onto X, and v) the map $\sigma : \mathcal{X} \star H \to \mathcal{X}$ induces a bijection of $\mathcal{G}(\mathcal{Z}) \setminus \mathcal{X}$ onto H^o.

From the construction, (iv) and (v) follow from the fact that if we have $\left[\delta_x, \delta_y\right] \in \mathcal{G}(\mathcal{Z}) \setminus \mathcal{X} \star \mathcal{X}$, where $\rho\left(\delta_x\right) = \rho\left(\delta_y\right)$, then there exists a unique $h \in H$ such that $\delta_x h = \delta_y$; the correspondence $\left[\delta_x, \delta_y\right] \mapsto h$ is the desired isomorphism between $\mathcal{G}(\mathcal{Z}) \setminus \mathcal{X} \star \mathcal{X}$ and H. Thus, the $\left(\mathcal{G}(\mathcal{Z}), H\right)$-equivalence of $\mathcal{X}$ implies H is naturally isomorphic to $\mathcal{G}(\mathcal{Z}) \setminus \mathcal{X} \star \mathcal{X}$ and $\mathcal{G}(\mathcal{Z})$ is naturally isomorphic to $\mathcal{X} \star \mathcal{X}/H$. □

Remark 3.10 In [16] it was shown that every action of a Lie groupoid $\mathcal{G}$ on the arrows induces an action on the space of objects. So, the partial multiplication defined by $\mathcal{G}$ defines a self-action of the arrows which is reflected on the space of objects X and corresponds to elements in $Aut(X)$ of the compact set X by homomorphisms. The composition of elements of $Aut(X)$ form a unitary group which preserves the nets of Radon measures converging to ergodic measures or operators. Another formulation of the above as a gauge groupoid of a principal G-bundle is given in [12]. We will consider the Haar system of measures for the

Lie groupoid next.

4. The Measure Groupoid and Measure Classes

Ergodic or measure groupoids act ergodically (or metric transitively) through the closed or invariant measure classes. The class $[\mu]$ of a measure μ is the set of all equivalent measures to μ having the same null set. Every measure class contains a probability since any measure can be normalized on its support (cf. [13]). Deitmar [17] showed the existence of Haar system of measures given the groupoid equivalence on X. Seda [18] showed that with a suitable separability condition on a groupoid $\mathcal{G}$, each probability measure μ_x uniquely determine a class of measure $[\mu]$ on X for which it serves as integral of 'translates' of the Haar measure ν on the structure or isotropy group $\mathcal{G}_x^x$. These translates constitute a system of Haar measures and a measure class $[\mu]=C$ defined on the fibres of $\mathcal{G}$. Theorem 2.1 in [13] and the associated definitions form the background for treating the metric transitive nature of the measure classes.

The principal Lie groupoid $\mathcal{G}(\mathcal{Z})$ is analytic given that its Borel structure is analytic and the space $(X,\mathcal{B})$ is countably separated. Given a probability measure ν on the t-fibre $t^{-1}(x)=\mathcal{G}(\mathcal{Z})(x,-), x\in X$, an arrow $\xi\in\mathcal{G}(\mathcal{Z})$ with $s(\xi)=x$, and $B\subset\mathcal{G}(\mathcal{Z})$, the map

$$B\mapsto\int\chi_B(\xi\eta)\mathrm{d}\nu(\eta)$$

defines a probability $\xi\cdot\nu$ on $\mathcal{G}(\mathcal{Z})(t(\xi),-)$. Since the product $\xi\eta$ is defined for ν-almost all η, the support of ν is $\mathcal{G}(\mathcal{Z})(s(\xi),-)$. Thus, $supp\left(\nu^{s(\xi)}\right)=\mathcal{G}(\mathcal{Z})(s(\xi),-)\Rightarrow supp\left(\xi\cdot\nu^{x}\right)=\mathcal{G}(\mathcal{Z})(t(\xi),-)$, which gives $\xi\cdot\nu^{s(\xi)}=\nu^{t(\xi)}$.

Theorem 4.1 *The principal groupoid* $H\rightrightarrows H^o$ *is the groupoid equivalence established on the normalized generalized space* $\mathcal{M}_1(X)\simeq\mathcal{P}(X)$ *by the action of the Lie groupoid* $\mathcal{G}(\mathcal{Z})$.

Proof. Two probabilities $(\mu,\nu)\in\mathcal{P}(X)\times\mathcal{P}(X)$ for which there exists $\xi\in\mathcal{G}(\mathcal{Z}):t(\xi)=\mu,s(\xi)=\nu$, are said to be equivalent. This is the equivalence on geometric space $\mathcal{X}$ defined by the Lie groupoid $\mathcal{G}(\mathcal{Z})$ as the image $\xi\mapsto(t(\xi),s(\xi))$. Therefore, if $\mu,\nu\in\mathcal{X}/\mathcal{G}(\mathcal{Z})$, then $\mu\sim\nu$ if on the fibre $\mu_f\sim\nu_f\Rightarrow\xi\mapsto(\mu_f,\nu_f)$. This agrees with the induced action of the Borel group $G(1)\subset C(X)$ on each measure class in $\mathcal{M}(X)$. Thus, isomorphisms on the fibres are constituted by invariant measure classes given as follows

$$\xi\cdot\mu_f^{s(\xi)}\to\nu_{fg}^{t(\xi)},\ \text{where } \mu_f^{s(\xi)}\sim\nu_{fg}^{t(\xi)}.$$

Hence, the $(H,\mathcal{G}(\mathcal{Z}))$-equivalence of the generalized points $\mathcal{X}$ is related to the action of the commutative algebra (or lattice) $C(X)$ on the generalized space $\mathcal{M}(X)$ and represented by the measure groupoid $(\mathcal{G}(\mathcal{Z}),C)$. □

There is always a symmetric quasi-invariant probability in a measure class. For according to Hahn in Theorem 2.1 of [13], every probability measure in an invariant measure class C is quasi-invariant. Using his conditions, we strengthened the quasi-invariant condition for a Haar measure by modifying it to agree with a maximal ideal $\mathfrak{m}_x$ or the corresponding zero-sets $Z[\mathfrak{m}_x]$ decomposition of measures on X as follows.

Lemma 4.2 *Let* $(\mathcal{G}(\mathcal{Z}),C)$ *be a measure groupoid,* $\nu \in C$ *a probability with t-decomposition* $\nu = \int \nu^x \mathrm{d}\bar{\nu}(x)$. *There is a* $\bar{\mu}$ *-conull Borel set* $U_f \subset X$ *such that*

1) $\nu^x(\mathcal{G}(\mathcal{Z})) = 1$ if $x \in U_f$.

2) $\nu^x\left(\mathcal{G}(\mathcal{Z}) - \mathcal{G}(\mathcal{Z})|_{U_f}\right) = 0$ if $x \in U_f$.

3) $x \in U_f \Rightarrow \nu^x\left(t^{-1}(x)\right) = 1$.

4) if $\xi \in \mathcal{G}(\mathcal{Z})|_{U_f}$, then $\xi \cdot \nu^{s(\xi)} \sim \nu^{t(\xi)}$.

Given this modification, we now have that for every co-null Borel set $U_f \subset X$, $\left(\mathcal{G}(\mathcal{Z})|_{U_f}, C\right)$ is a measure groupoid called an *inessential reduction (i.r)* of $(\mathcal{G}(\mathcal{Z}),C)$ in [13], where the inessential reduction for an open conull subset $U_f \subset X$ is also denoted as $\mathcal{G}(\mathcal{Z})_o$.

From this, we see that each invariant measure class $C \in \mathcal{C}$ form a system of Haar measures $\{\mu_f^x\}_{x\in X}$ for each $f \in \mathfrak{m}_x$ on the *inessential reduction (i.r)* $\left(\mathcal{G}(\mathcal{Z})|_{U_f}, C\right)$. Since each system is defined on the *t*-fibre $t^{-1}(x) = \mathcal{G}(\mathcal{Z})(x,-)$, it follows that the system of Haar measures is not unique. Hence, any invariant measure class C determines the measure groupoid $(\mathcal{G}(\mathcal{Z}),C)$, with a Haar measure defined as follows.

Definition 4.3 Let $(\mathcal{G}(\mathcal{Z}),C)$ be a measure groupoid. Let $\nu \in C$ and let $\mu \in \bar{C} = (t,s)(C)$ be a probability on the base space. The pair (ν,μ) is called a Haar measure for $(\mathcal{G}(\mathcal{Z}),C)$ if ν has a *t*-decomposition $\nu = \int \nu^x \mathrm{d}\mu(x)$ with respect to μ such that for some inessential reduction $\mathcal{G}(\mathcal{Z})_o$ of $\mathcal{G}(\mathcal{Z})$, for all $\xi \in \mathcal{G}(\mathcal{Z})_o$ and $F \geq 0$ a Borel function on $\mathcal{G}(\mathcal{Z})$ we have

$$\int F(\gamma)\mathrm{d}\nu^{t(\xi)}(\gamma) = \int F(\xi\gamma)\mathrm{d}\nu^{s(\xi)}(\gamma). \tag{1}$$

Thus, for a Borel function F on the groupoid $\mathcal{G}(\mathcal{Z})$, given a symmetric probability $\nu \in C$ with *t*-decomposition $\nu = \int \nu^x \mathrm{d}\bar{\nu}(x)$, where U_f is conull as in the above; the quasi-invariance of ν implies

$$\left(F \mapsto \int F(\gamma)\mathrm{d}\nu^{t(\xi)}(\gamma)\right) \sim \left(F \mapsto \int F(\xi\gamma)\mathrm{d}\nu^{s(\xi)}(\gamma)\right).$$

With these constructions, Hahn showed that every measure groupoid has a measure ν satisfying (1) for ξ in an inessential reduction.

Furthermore, the symmetricity of the probability ν implies $t_*\nu = s_*\nu$.

Hence, the quasi-invariance for right translation uses the s-decomposition $\nu = \int \nu_x \mathrm{d}\overline{\nu}(x)$. The result is that the left and right invariance of C are equivalent. Using these, a (t,s)-decomposable measure for the composable space $\mathcal{G}(\mathcal{Z})^{(2)}$ is defined as

$$\nu^{(2)} = \int \nu_x \times \nu^x \mathrm{d}\overline{\nu}(x), \text{ with measure class } \left[\nu^{(2)}\right],$$

which is usually written $C^{(2)}$ and is dependent on C. Since the space of composables of any groupoid has a goupoid structure, the conclusion is that $\left(\mathcal{G}(\mathcal{Z})^{(2)}, C^{(2)}\right)$ is a measure groupoid. The proofs of the following results follow from 3.3 and 3.4 of [13].

Lemma 4.4 *If* $\mathcal{G}(\mathcal{Z})$ *is an analytic (standard) Borel groupoid,* $\mathcal{G}(\mathcal{Z})^{(2)} \subset \mathcal{G}(\mathcal{Z}) \times \mathcal{G}(\mathcal{Z})$ *is an analytic (standard) Borel groupoid.*

Proposition 4.5 *If* $(\mathcal{G}(\mathcal{Z}), C)$ *is an analytic groupoid with invariant measure class, so is* $\left(\mathcal{G}(\mathcal{Z})^{(2)}, C^{(2)}\right)$.

The concept of ergodicity will be delineated next and related to the dynamical system of the measure groupoid.

5. Convolution Algebra and Dynamical System

According to Hahn [13], the measure groupoid is ergodic if and only if there is a single point $x_o \in X$ such that $X \setminus \{x_o\}$ is null. In other words, ergodicity implies the existence of Dirac probability measures $\{\delta_x : x \in X\}$ defined at each point of X. Thus, the existence of a $\mathcal{G}(\mathcal{Z})$-action on $\mathcal{X}$ makes it ergodic groupoid. This also implies that every Borel function ϕ on the base X can be expressed in the form of a positive Borel function F on the arrows $\mathcal{G}(\mathcal{Z})$ given as $\phi \circ t(\xi) = F(\xi, s(\xi))$, where F satisfies $F \circ t^{-1} = F \circ s^{-1}$. This means that the Borel functions on the arrows preserve the equivalence the groupoid $\mathcal{G}(\mathcal{Z})$ defines on the base space X. Alternatively, as stated above, the $\mathcal{G}(\mathcal{Z})$-action preserves the Borel structure of the generalized space.

Subsequently, a real-valued Borel function F on the measure groupoid $\mathcal{G}(\mathcal{Z})$ satisfying $F(\xi^{-1}\gamma) = F(\gamma)$ for $\mu^{t(\xi)}$-a.e and for μ-almost all ξ, corresponds to a Borel function ϕ on X such that $F = \phi \circ s$ a.e. Thus, the invariant functions on the equivalence space $X \star X$ (or on the space X with $\mathcal{G}(\mathcal{Z})$-action) are of the form $\phi \circ t$ or $\phi \circ s$. This shows the Borel functions are $\mathcal{G}(\mathcal{Z})$-invariant.

Therefore, the dynamical system is related to the convergence of the ultrafilters $\mathcal{F} \to x$ associated to each maximal ideal $\mathfrak{m}_x$. We can therefore define a net of such positive Borel function F on the arrows $\mathcal{G}(\mathcal{Z})$ given as $F_\alpha(\xi, s(\xi)) \to F(\xi, s(\xi))$; or in the form $\phi_\alpha \circ t(\xi) \to \phi \circ t(\xi)$ which can be considered local bisections. Since such a net $F_\alpha \to F$ (or in terms of local bisections $\phi_\alpha \to \phi$) corresponds to the ultrafilter, it represents the dynamical system of the commutative algebra $C(X)$. Following from 2.6 of [13], the ergodic measure groupoid is therefore defined as follows.

Definition 5.1 The measure groupoid $(\mathcal{G}(\mathcal{Z}), C)$ is called ergodic if the only Borel functions $\phi: X \to I\!R$ satisfying $\int |\phi \circ t - \phi \circ s| \mathrm{d}\mu = 0$ are such that $\phi = \text{constant}$ $\bar{\mu}$-a.e. Alternatively, $(\mathcal{G}(\mathcal{Z}), C)$ is ergodic if and only if for all $A \in \mathcal{B}(X)$, $\int |I_A \circ t - I_A \circ s| \mathrm{d}\bar{\mu} = 0 \Rightarrow \bar{\mu}(A) = 0$ or $\bar{\mu}(X - A) = 0$.

If then we denote the space of all the Borel function on the measure groupoid $\mathcal{G}(\mathcal{Z})$ with $\mathscr{B}(\mathcal{G}(\mathcal{Z}))$, the convolution product of two Borel function $f, g \in \mathscr{B}(\mathcal{G}(\mathcal{Z}))$ on the space is defined as follows.

$$f * g(\xi) = \int_{\xi\gamma=\tau} f(\xi) g(\gamma) \mathrm{d}\mu^{s(\xi)}(\gamma) = \int_{\mathcal{G}(\mathcal{Z})(t(\xi),-)} f(\xi) g(\xi^{-1}\gamma) \mathrm{d}\mu^{t(\xi)}(\gamma).$$

This follows from the involutive map on any Borel $f \in \mathscr{B}(\mathcal{G}(\mathcal{Z}))$ which is defined as $f^*(\xi) = f(\xi^{-1}) = \overline{f(\xi)}$. Thus, the space $\mathscr{B}(\mathcal{G}(\mathcal{Z}))$ is made into a normed $*$-algebra, with the norm of f given as the supremum norm

$$\|f\| = \sup_{x\in X} |\phi \circ t| = \sup_{x\in X} |\phi \circ s| = \sup_{x\in X} |\phi(x)|.$$

The representation of this convolution algebra $\mathscr{B}(\mathcal{G}(\mathcal{Z}))$ of the measure groupoid makes use of the modular function Δ which Peter Hahn defined and employed in Theorem 3.8 of [13] as $\mu^{(2)}$-a.e. homomorphism $\Delta := \gamma \mapsto P(\gamma) / P(\gamma^{-1})$.

Notice that $\bar{\mu}$ is a system of Haar measures supported on the fibres $\{\mathfrak{m}_x, x \in X\}$; but given simply as $\bar{\mu}$ because they are same or (groupoid) equivalent measures. Thus, the Hilbert space $\mathcal{H} = L^2(X, \bar{\mu})$ can be considered a bundle space made up of the fibres $\mathcal{G}(\mathcal{Z})(x,-)$. But because the Borel functions F defined on the arrows are equal to Borel functions $\phi \circ t$ defined on X, having the net convergence $\phi_\alpha \circ t \to \phi \circ t$ we described above, we put the Hilbert space simply as $\mathcal{H} = L^2(X, \bar{\mu})$.

6. Unitary Representation of $\mathscr{B}(\mathcal{G}(\mathcal{Z}))$

Given the convolution algebra $\mathscr{B}(\mathcal{G}(\mathcal{Z}))$ of Borel functions defined on the measure groupoid $(\mathcal{G}(\mathcal{Z}), C)$, the formulation of the unitary representation of the convolution algebra $\mathscr{B}(\mathcal{G}(\mathcal{Z}))$ on the space $B(\mathcal{H})$ of bounded operators on the Hilbert space $\mathcal{H} = L^2(X, \bar{\mu})$ is patterned on [13], which is a simplified definition of von Neumann algebra arising from the maximal ideals $\mathfrak{m}_x$ and ergodic action of the Borel group $G(1)$ on a compact measure space X. From the foregoing, the simplification is achieved by considering the system of Haar measures on the principal Lie groupoid $\mathcal{G}(\mathcal{Z}) \rightrightarrows X$, and using them in the definition of the convolution algebras of Borel functions on measure groupoid $(\mathcal{G}(\mathcal{Z}), C)$.

The definition of the convolution algebra of the principal groupoid $\mathscr{B}(\mathcal{G}(\mathcal{Z}))$ gives a $*$-algebra that coincides with the von Neumann algebra $B(\mathcal{H})$, where $\mathcal{H} = L^2(X, \bar{\mu})$ and $\bar{\mu}$ a probability measure on X. The following result on $*$-representation of the resulting algebra is the focus of the paper.

Theorem 6.1 *The map* $T : \mathscr{B}(\mathcal{G}(\mathcal{Z})) \to B(\mathcal{H})$ *is a unitary* $*$*-representation.*

Proof. As in [13], a representation is defined as follow. Given $f \in \mathscr{B}(\mathcal{G}(\mathcal{Z}))$ and $u, v \in L^2(X, \bar{\mu}) = \mathcal{H}$; define a homomorphism

$$T : \mathscr{B}(\mathcal{G}(\mathcal{Z})) \to B(\mathcal{H}), f \mapsto T_f = \int f(\gamma) P(\gamma) \mathrm{d}\mu^{t(\gamma)}(\gamma).$$

This defines an operator $T_f : \mathcal{H} \to \mathcal{H}$ by $u \mapsto T_f(u)$ such that

$$\begin{aligned}\langle T_f(u), v \rangle &= \int f(\gamma) u(s(\gamma)) \overline{v(t(\gamma))} P(\gamma) \mathrm{d}\mu(\gamma) \\ &= \int \left(\int f(\gamma) u(s(\gamma)) P(\gamma) \mathrm{d}\mu^x(\gamma) \right) \overline{v(x)} \mathrm{d}\bar{\mu}(x) \\ &= \int T_f(u(x)) \overline{v(x)} \mathrm{d}\bar{\mu}(x)\end{aligned}$$

Thus, the map $T_f u = \left(x \mapsto \int f(\gamma) u(s(\gamma)) P(\gamma) \mathrm{d}\mu^x(\gamma) \right)$ is also in $\mathcal{H}$, which makes $T_f \in B(\mathcal{H})$.

Likewise, we have the map $T : f^* \mapsto \int f(\gamma^{-1}) P(\gamma^{-1}) \mathrm{d}\mu^{t(\gamma)}(\gamma)$ such that

$$\begin{aligned}\langle T_{f^*}(u), v \rangle &= \int f^*(\gamma) u(s(\gamma)) \overline{v(t(\gamma))} P(\gamma) \mathrm{d}\mu(\gamma) \\ &= \int f(\gamma^{-1}) u(s(\gamma)) \overline{v(t(\gamma))} P(\gamma^{-1}) \mathrm{d}\mu(\gamma) \\ &= \int \overline{f(\gamma) v(s(\gamma))} u(t(\gamma)) P(\gamma) \mathrm{d}\mu(\gamma) \\ &= \overline{\langle T_f v, u \rangle} = \langle u, T_f v \rangle = \langle T_f^* u, v \rangle. \text{ Thus, } T_{f^*} = T_f^*.\end{aligned}$$

Given the convolution product $f \star g(\xi) = \int f(\xi) g(\gamma) \mathrm{d}\mu^{t(\xi)}(\gamma)$; its image under T is given as follows.

$$\begin{aligned}T_{f \star g} &= \iint f(\xi\gamma) g(\gamma^{-1}) P(\gamma) \mathrm{d}\mu^{s(\xi)}(\gamma) P(\xi) \mathrm{d}\mu(\xi) \\ &= \iiint f(\gamma\xi) g(\xi^{-1}) P(\xi) P(\gamma) \mathrm{d}\mu^{s(\gamma)}(\xi) \mathrm{d}\mu^x(\gamma) \mathrm{d}\bar{\mu}(x); \text{ by t-decomposition of } \mu \\ &= \iiint f(\xi) g(\xi^{-1}\gamma) P(\xi) P(\gamma) \mathrm{d}\mu^{t(\gamma)}(\xi) \mathrm{d}\mu^x(\gamma) \mathrm{d}\bar{\mu}(x); \text{ by convolution property} \\ &= \iint f(\xi) \left(\int g(\xi^{-1}\gamma) P(\gamma) \mu^x(\gamma) \right) P(\xi) \mathrm{d}\mu^x(\xi) \mathrm{d}\bar{\mu}(x) \\ &= \iint f(\xi) \left(\int g(\xi^{-1}\gamma) P(\gamma) \mathrm{d}\mu^{t(\xi)}(\gamma) \right) P(\xi) \mathrm{d}\mu^x(\xi) \mathrm{d}\bar{\mu}(x) \\ &= \int f(\xi) \left(\int g(\gamma) P(\gamma) \mathrm{d}\mu^{s(\xi)}(\gamma) \right) P(\xi) \mathrm{d}\mu(\xi); \text{ by reversal of t-decomposition} \\ &= T_f \circ T_g.\end{aligned}$$

Finally, from $T_{f^*} = \int f^*(\gamma) P(\gamma) \mathrm{d}\mu(\gamma)$ we have

$$
\begin{aligned}
T_{g^*\star f^*} &= \iint g^*(\xi\gamma) f^*(\gamma^{-1}) P(\gamma)\,\mathrm{d}\mu^{s(\xi)}(\gamma) P(\xi)\,\mathrm{d}\mu(\xi) \\
&= \iint f(\gamma) g(\gamma^{-1}\xi^{-1}) P(\gamma)\,\mathrm{d}\mu^{s(\xi)}(\gamma)\Delta(\xi)^{-1} P(\xi)\,\mathrm{d}\mu(\gamma). \\
&= \iint f(\xi\gamma) g(\gamma^{-1}) P(\gamma)\,\mathrm{d}\mu^{t(\xi)}(\gamma)\Delta(\xi)^{-1} P(\xi)\,\mathrm{d}\mu(\xi); \left(\text{using } \xi \mapsto \xi^{-1}\right) \\
&= \int \overline{g(\gamma)}\left(\int \overline{f(\gamma^{-1}\xi^{-1})}\Delta(\xi)^{-1} P(\xi)\,\mathrm{d}\mu(\xi)\right) P(\gamma)\,\mathrm{d}\mu^{s(\xi)}(\gamma) \\
&= T_g^* \circ T_f^*.
\end{aligned}
$$

□

The operator is shown to be an isometry as follows.

$$
\begin{aligned}
\left|\left\langle T_f u, v\right\rangle\right| &= \left|\int_E f(\gamma) u(s(\gamma)) v(t(\gamma))\,\mathrm{d}\mu(\gamma)\right| \\
&\le \int_E \left|u(s(\gamma))\right|\left|v(t(\gamma))\right|\mathrm{d}\mu(\gamma);\ \text{since } \|f\| \le 1 \\
&\le \left(\iint \left|u(s(\gamma))\right|^2 \mathrm{d}\mu(\gamma)\left|v(t(\gamma))\right|^2 \mathrm{d}\mu(\gamma)\right)^{1/2} \\
&= \left(\int |u(x)|^2 \mathrm{d}\bar{\mu}(x)\right)^{1/2}\left(\int |v(x)|^2 \mathrm{d}\bar{v}(x)\right)^{1/2} \\
&= \|u\|_2 \|v\|_2 .
\end{aligned}
$$

Thus, the $*$-representation is a unitary representation since $\left\langle T_f u, T_g v\right\rangle \le \langle u, v\rangle$. This is in conformity with our understanding of the Borel functions on the measure groupoid as probability measures on X or Haar system of measures on the groupoid.

Proposition 6.2 *The convolution algebra* $\mathscr{B}(\mathcal{G}(\mathcal{Z}))$ *is a commutative von Neuman algebra by representation.*

Proof. The Borel functions defined on the arrows of $\mathcal{G}(\mathcal{Z})$ are defined on the fibres $\mathfrak{m}_x$ which contain the polynomials on *X*. So they are all defined on the operators on $\mathcal{H}$ as the representation showed. Hence, they are all of the form $f(T)$, which makes them von Neumann algebra as defined in the opening section.

Alternatively, using Connes' characterization of commutative von Neumann algebra in 1.3 of [3] as the algebras of operators on Hilbert space that are invariant under a group (or subgroup) of unitary operators, it follows that the convolution algebra $\mathscr{B}(\mathcal{G}(\mathcal{Z}))$ of the Lie groupoid $\mathcal{G}(\mathcal{Z})$ is a commutative von Neumann algebra since it is invariant under $G(1) \subset \mathcal{U}(\mathcal{H})$ as given by the condition of ergodicity $F \circ t^{-1} = F \circ s^{-1}$ on *F*, which implies invariance under transformations of *X*.

A third characterization of a commutative von Neumann algebra by Connes [3] as an involutive algebra of operators that is closed under weak limits still reinforces the result. The presence of an ultra-filter $\mathcal{F} \to x$ associated to every fibre $\mathfrak{m}_x$ of the principal Lie groupoid $\mathcal{G}(\mathcal{Z})$ implies the convergence of $G(1)$-invariant nets of Borel functions $f_\alpha \to f$ in the normed $*$-algebra $\mathscr{B}(\mathcal{G}(\mathcal{Z}))$ (or equivalently the convergence of nets of local bisections $\phi_\alpha \to \phi$ of the principal Lie groupoid $\mathcal{G}(\mathcal{Z})$.) □

We have shown this to be related to the ergodicity of the measure groupoid

$(\mathcal{G}(\mathcal{Z}),C)$ and central to the dynamical system of the algebra $\mathscr{B}(\mathcal{G}(\mathcal{Z}))$ which is now given as a corollary.

Corollary 6.3 *The dynamical system of the commutative von Neumann algebra $\mathscr{B}(\mathcal{G}(\mathcal{Z}))$ is defined by the convergence of nets of operators T_α defined by the nets of Borel functions $f_\alpha \to f$ or local bisections $\phi_\alpha \to \phi$.*

Proof. This follows from the definition of the operators above as

$$f \mapsto T_f = \int f(\gamma) P(\gamma) \mathrm{d}\mu^{t(\gamma)}(\gamma).$$

That these nets define the dynamical system of the von Neumann algebra follows from their relationship to the convergence of nets of non-ergodic measures to ergodic limits which, as given in [8], represents the dynamical system of ergodic actions. □

The action of the (lattice) commutative algebra $C(X)$ on the generalized space $\mathcal{M}(X)$ also involves a decomposition. Thus, the resulting dynamical system converges to an ergodic limit given by $G(1) \times \mathcal{X} \to \mathcal{X}$ which is represented on the generalized space by $G(1)$-invariant convergent nets of measures $\mu_\alpha \to \delta_x$, generalized by the $(H, \mathcal{G}(\mathcal{Z}))$-equivalence of the geometric points $\mathcal{X}$. This is given as a corollary.

Corollary 6.4 *By the the polarity of the $Aut(X)$-action there exists a canonical form Σ, such that $\mathcal{G}(\mathcal{Z}) \times \mathcal{M}(X) \simeq \mathcal{G}(x,x) \times \Sigma$. So, the canonical form $\mathcal{G}(x,x) \times \Sigma \to \Sigma$ converges to ergodic form $\mathcal{G}(x,x) \times \delta_x \to \delta_x$ implies the action $\mathcal{G}(\mathcal{Z}) \times \mathcal{M}(X) \to \mathcal{M}(X)$ converges to the ergodic limit $\mathcal{G}(\mathcal{Z}) \times \mathcal{X} \to \mathcal{X}$.*

Proof. Given that the above convergent nets of measures can be constituted to be transversal to the orbits of $G(1)$ or the measure classes, then we have $\Sigma = \{\mu_\alpha : \mu_\alpha \in C_\alpha\}$, where $C_\alpha \in \mathcal{C}$ are the measure classes. Then $\mathcal{G}(\mathcal{Z})(x,x) \times \Sigma \simeq \mathcal{G}(\mathcal{Z}) \times \mathcal{M}(X)$ as stated. Hence, the steady ergodic state follows from the convergence of the transversal net(s) Σ, as constructed, to ergodic limits. □

The section Σ of measures is constituted from the measure classes. The existence of many measure classes for the $*$-representation points to the fact that the $*$-representation of the convolution algebra $\mathscr{B}(\mathcal{G}(\mathcal{Z}))$ of the measure groupoid is not uniquely tied to any measure class. In other words, the left Haar system of measures is not unique. The homomorphism of the $*$-representation implies that the convergence of a net of Borel function on the convolution algebra $\mathscr{B}(\mathcal{G}(\mathcal{Z}))$ implies a net of bounded unitary operators in the von Neumann algebra $\mathscr{M}$.

7. Conclusions

We have presented the commutative algebra $C(X)$-action on the generalized space $\mathcal{M}(X)$ as constituted by the decomposition action of its maximal ideals $\mathfrak{m}_x$ and the action of its group of units $G(1)$ on X given as a polar action of

$Aut(X,\mu)$ on $\mathcal{M}(X)$, expressed in form of a principal Lie groupoid $G(\mathcal{Z})$ action on the space $\mathcal{X}$ of geometric (or closed) points of the generalized space $\mathcal{M}(X)$. Ergodic requirements made it into the dynamical system defined by Borel functions on the ergodic or measure groupoid $(\mathcal{G}(\mathcal{Z}),C)$.

The convolution algebra $\mathscr{B}(\mathcal{G}(\mathcal{Z}))$ of these Borel functions has a representation on the commutative von Neumann algebra $\mathscr{M}$ of operators on the Hilbert space $\mathcal{H}=L^2(X,\bar{\mu})$. Hence, the presentation of the geometric space $\mathcal{X}=\{\delta_x : x\in X\}$ as a $(G(\mathcal{Z}),H)$-equivalence was helpful for the $*$-representation of the convolution algebra $\mathscr{B}(\mathcal{G}(\mathcal{Z}))$ of the principal Lie groupoid $\mathcal{G}(\mathcal{Z})$ or the measure groupoid $(G(\mathcal{Z}),\mathcal{C})$ on the von Neumann algebra $B(\mathcal{H})$ of bounded operators on $\mathcal{H}$.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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