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Tree of Fermat-Pramanik Series and Solution of $A^M + B^2 = C^2$ with Integers Produces a New Series of $\left(C_1^2 - B_1^2\right) = \left(C_2^2 - B_2^2\right) = \left(C_3^2 - B_3^2\right) = \text{Others}$

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Abstract

The Fermat–Pramanik series are like below: $A_1^2 + A_2^2 + A_3^2 + A_4^2 + A_5^2 + A_6^2 + \cdots + A_{n-1}^2 = A_n^2$. The mathematical principle has been established by factorization principle. The Fermat-Pramanik tree can be grown. It produces branched Fermat-Pramanik series using same principle making Fermat-Pramanik chain. Branched chain can be propagated at any point of the main chain with indefinite length using factorization principle as follows:

$$A_1^2 + A_2^2 + A_3^2 + A_4^2 + A_5^2 + \cdots + A_{n-1}^2 = A_n^2$$
$$+^1$$
$$C_1^2 = C_2^2 \Rightarrow \left(A_4^2 + C_1^2 = C_2^2\right) \Rightarrow 21^2 + 20^2 = 29^2$$

Same principle is applicable for integer solutions of $A^M + B^2 = C^2$ which produces series of the type $\left(C_1^2 - B_1^2\right) = \left(C_2^2 - B_2^2\right) = \left(C_3^2 - B_3^2\right) = \cdots = \left(C_n^2 - B_n^2\right)$. It has been shown that this equation is solvable with $N\{A, B, C, M\}$. $A^M = A^{M_1 + M_2} = A^{M_1} A^{M_2} = (C + B)(C - B) = C^2 - B^2$ where $C + B = A^{M_1}$, $C - B = A^{M_1}$, $M = M_1 + M_2$ and $M_1 > M_2$. Subsequently, it has been shown that $C_1^2 - B_1^2 = C_2^2 - B_2^2 = C_3^2 - B_3^2 = \cdots = C_n^2 - B_n^2 = A^{M_1 + M_2 + M_3 + M_4 + M_5 + M_6 + \cdots + \text{so on}} = A^M$ using $M = M_1 + M_2 + M_3 + \cdots$. The combinations of *M*s should be taken so that the values of both the parts $(C_n + B_n)$ and $(C_n - B_n)$ should be even or odd for obtaining $Z\{B, C\}$. Hence, it has been shown that the Fermat triple can generate a) Fermat-Pramanik multiplate, b) Fermat-Pramanik Branched

[1]+: symbol denotes branching of A_4 "square" to C_1 square.

multiplate and c) Fermat-Pramanik deductive series. All these formalisms are useful for development of new principle of cryptography.

Keywords

Fermat Theorem, Fermat-Pramanik Tree, Solution of $A^M + B^2 = C^2$, Deductive Series, Generation of Fermat's Triode, Generation of Fermat Series

1. Introduction

Theory of number has become crown of mathematics since Pythagoras time.

The Pythagorean equation,

$$A_1^2 + A_2^2 = A_3^2 \tag{1}$$

has an infinite number of positive integer solutions for A_1, A_2 and A_3; these solutions are known as Pythagorean triplets (P.T.) (with the simplest example $3^2 + 4^2 = 5^2$ [1]-[6]. Around 1637, Fermat wrote in the margin of a book that the more generalized equation, $A_1^n + A_2^n = A_3^n$ had no solutions in the positive integers if n is an integer greater than 2. In theory, this statement is known as Fermat's Last Theorem (it is also called as Fermat's conjecture before 1995). The cases $n = 1$ and $n = 2$ have been known from Pythagoras time having infinite solutions [1]. The proposition was first stated as a theorem by Pierre de Fermat around 1637. It was written in the margin of a copy of Arithmetica. Fermat claimed that he had a proof and due to the lengthy calculation, he was unable to fit in the margin of the copy. However, after his death no document was found to substantiate his claim. Consequently, the proposition became as a conjecture rather than a theorem. After 358 years of effort by mathematicians, the first successful proof was completed in 1994 by Andrew Wiles and formally published in 1995. It was described as a "stunning advance in mathematics" in the citation for Wiles's Abel Prize award in 2016. It was also proved many parts of the Taniyama-Shimura conjecture. Afterward, it was defined as the modularity theorem, and opened up new approaches to numerous other problems and developed powerful technique known as modularity lifting in mathematics. It is among the most outstanding out come in mathematical analysis [7] [8] [9]. Very few attempts have been made to extend the Fermat's equation upto the 4th term [10] [11]. Recently Pramanik et.al has shown that Pythagoras triplet can adapt n number of terms in place of three terms [12].

$$A_1^2 + A_2^2 + A_3^2 + \cdots + A_{n-1}^2 = A_n^2 \quad \text{(Fermat-Pramanik multiplate)} \tag{2}$$

It is already discussed how to generate Pythagoras triplet by simple method which is illustrated briefly below [3].

$$A_1^2 + A_2^2 = A_3^2$$

$$A_1^2 + A_2^2 = A_3^2 \Rightarrow A_1^2 = A_3^2 - A_2^2 \tag{3}$$

$$\therefore A_1^2 = (A_3 + A_2)(A_3 - A_2) \quad (4)$$

Now let us consider $A_1 = B_1 B_2$ where all are odd or even. If A_1 will be prime then one of the B will be 1.

Henceforth from Equation (4) we can obtain,

$$A_3 + A_2 = B_2^2 \text{ and } A_3 - A_2 = B_1^2 \quad \text{Involving } B_1 \text{ and } B_2 \quad (5)$$

$$\text{Thus, } A_3 = \frac{B_1^2 + B_2^2}{2} \text{ and } A_2 = \frac{B_2^2 - B_1^2}{2} \quad (6)$$

With this principle it has been shown that Fermat-Pramanik multiplate can be generated [3]. It is to be noted that A_3 and any of A_1 and A_2 of Pythagorean triplets should be odd numbers if there is no common factor for A_1, A_2 and A_3.

Now principle of generation of branching of Fermat-Pramanik multiplate will be illustrated by a simple principle. Let A_1 is even and it is related with A_2 and A_3 through Equation (4) which is $A_1^2 = (A_3 + A_2)(A_3 - A_2)$.

A_2 and A_3 can be generated from any combination of B_1, B_2, B_3, B_4 etc. If all Bs are "odd" then the following combinations will be permitted for A_1 as illustrated in Table 1.

Order of values of Bs are $B_1 < B_2 < B_3 < B_4 < \cdots$. For Illustration the following values of B_1, B_2, B_3, B_4 are taken $B_1 = 5$, $B_2 = 11$, $B_3 = 19$, $B_4 = 29$. Thus, $A_3 = (B_1^2 + B_2^2)/2$ and $A_2 = (B_2^2 - B_1^2)/2$. Now sets will be generated are as follows,

$$A_1^2 + A_2^2 = A_3^2 \Rightarrow A_1^2 = A_3^2 - A_2^2$$

If all Bs are even the choice for solution of A_1, A_2, A_3 have no problem. A_3 may have any number of any of Bs and A_2 may have any number of any of Bs. It is to be noted that all A_3 and A_1 are odd (Table 2).

Table 1. Scheme of formation of Fermat triode $A_1^2 + A_2^2 = A_3^2$ where all Bs are odd. $B_1 = 5$, $B_2 = 11$, $B_3 = 19$, $B_4 = 29$.

Serial no	Values for A_3 as per Equation (6)		Values of A_2 As per Equation (6)		$A_3 > A_2$	Value of A_1 as per Equation (6)
1	B_2	73	B_1	48	System of 2 Bs	55
2	B_3	193	B_1	168	System of 2 Bs	95
3	B_3	241	B_2	120	System of 2 Bs	209
4	B_4	433	B_1	408	System of 2 Bs	145
5	B_4	481	B_2	360	System of 2 Bs	319
6	B_4	601	B_3	240	System of 2 Bs	551
7	$(B_1 + B_2 + B_3)$	1033	B_4	192	System of 4 Bs	1015
8	$(B_2 + B_3 + B_4)$	1753	B_1	1728	System of 4 Bs	295
9	$(B_1 + B_2 + B_4)$	1193	B_3	832	System of 4 Bs	855
10	$(B_1 + B_3 + B_4)$	1465	B_2	1344	System of 4 Bs	583
11	$(B_2 + B_3 + B_4)$	1801	B_2	1680	System of 4 Bs	649
12	$(B_1 + B_2 + B_3)$	673	B_2	552	System of 4 Bs	385
13	$(B_3 + B_4 + B_2)$	1433	$(B_1 + B_2 + B_3)$	592	System of 6 Bs	1305
14	$(B_2 + B_2 + B_4)$	1721	$(B_1 + B_1 + B_3)$	880	System of 6 Bs	1479

Table 2. Scheme of formation of Fermat triode $A_1^2 + A_2^2 = A_3^2$ where all Bs are even $B_5 = 10 < B_6 = 16 < B_7 = 18 < B_8 = 24$.

Serial no	Values for A_3 as per Equation (6)		Values of A_2 as per Equation (6)-all		$A_3 > A_2$	Value of A_1
1	B_8	338	B_8	238	System of $2B$s	240
2	B_8	416	B_6	160	System of $2B$s	384
3	$B_8 + B_5$	740	B_7	416	System of $2B$s	612
4	$B_8 + B_6$	850	B_5	750	System of $2B$s	400
5	$B_8 + B_7$	1220	$B_5 + B_6$	544	System of $2B$s	1092
6	$B_5 + B_6$	730	$B_5 + B_7$	54	System of $2B$s	728
7	$(B_5 + B_6 + B_7)$	772	$B_6 + B_7$	672	System of $2B$s	380
8	$(B_6 + B_7 + B_8)$	2020	$B_5 + B_6$	1344	System of $2B$s	1508
9	$(B_5 + B_7 + B_8)$	1690	$B_5 + B_7$	1014	System of $2B$s	1352
10	$(B_5 + B_6 + B_7)$	1360	$B_5 + B_7$	576	System of $2B$s	1232

2. Branching of Fermat-Pramanik Series

Now principle of branching will be illustrated.

If the Fermat-Pramanik series are like below [12],

$$A_1^2 + A_2^2 + A_3^2 + A_4^2 + A_5^2 + A_6^2 + \cdots + A_{n-1}^2 = A_n^2 \tag{7}$$

Branching can be done at any A_x for $x = 1, 2, 3, 4, \cdots$ so on and at any number. Then first it to be checked at A_x for its odd or even character. Let A_4 is taken for illustration. If A_4 is odd and branching is to be done at A_4^2, then A_4 should be the product of two different odd numbers. If A_4 is prime number then one number may be 1.

If A_4 is even and it is the product of two even numbers then it can be used for branching.

$$A_4^2 + C_1^2 = C_2^2 \Rightarrow A_4^2 = C_2^2 - C_1^2 \Rightarrow A_4^2 = (C_2 + C_1)(C_2 - C_1) \tag{8}$$

Now it may be assumed $A_4^2 = X^2 Y^2$ where X and Y are even and $X > Y$.

Therefore, $C_2 + C_1 = X^2$ and $C_2 - C_1 = Y^2$.

$$C_1 = \frac{X^2 + Y^2}{2} \text{ and } C_2 = \frac{X^2 - Y^2}{2} \tag{9}$$

Hence $A_1^2 + A_2^2 + A_3^2 + A_4^2 + A_5^2 + \cdots + A_{n-1}^2 = A_n^2$

$$+^2$$

$$C_1^2(20^2) = C_2^2(29^2) \Rightarrow (A_4^2 + C_1^2 = C_2^2) \Rightarrow 21^2 + 20^2 = 29^2$$

Let it to be illustrated with the numbers. To expand it further, the prime number 29 has been considered which can thus be splitted as the product of 1 × 29. Thus, $C_3 = (29^2 - 1)/2 = 420$ and $C_4 = (29^2 + 1)/2 = 421$. Henceforth, $A_4^2 + C_1^2 + C_2^2 + C_3^2 = C_4^2 \Rightarrow 21^2 + 20^2 + 420^2 = 421^2$.

[2]+: symbol denotes branching of A_4 "square" to C_1 square.

C_1 can also be expanded with the same principle. So any number of branches of any length can be fabricated after proper scrutiny of A_x finding *X* and *Y*, hence *C*s.

3. Solution of $A^M + B^2 = C^2$

This equation is solvable with $N\{A, B, C, M\}$. Even then the combination should be taken so that the values of both the parts will be even or odd.

$$A^M = C^2 - B^2 \tag{10}$$

$$A^M = (C + B)(C - B) \tag{11}$$

Now, if $M = M_1 + M_2$ and $M_1 > M_2$ then,

$$A^M = A^{M_1+M_2} = A^{M_1} A^{M_2} = (C + B)(C - B) \tag{12}$$

Now if A is even, then both A^{M_1} and A^{M_2} are even and $A^{M_1} > A^{M_2}$

Henceforth from Equation (12) we can obtain,

$$B + C = A^{M_1} \text{ and } C - B = A^{M_2} \tag{13}$$

Therefore,

$$B = \frac{A^{M_1} - A^{M_2}}{2} \text{ and } C = \frac{A^{M_1} + A^{M_2}}{2} \tag{14}$$

$$\therefore C^2 - B^2 = A^{M_1+M_2} = A^M \tag{15}$$

Let various combinations of *M*s may be taken (here are $3M_x$) as $M = M_1 + M_2 + M_3$ and values are as follows $M_1 < M_2 < M_3$.

Here two sets of *M*s are taken: (a) $M_1 + M_3$ and M_2 and (b) $M_2 + M_3$ and M_1.

Therefore,

$$C_1 + B_1 = A^{M_1+M_2} \text{ and } C_1 - B_1 = A^{M_2} \tag{16}$$

Thus the product of $(C_1 + B_1)$ and $(C_1 - B_1)$ will result in,

$$C_1^2 - B_1^2 = A^{M_1+M_2+M_3} = A^M \tag{17}$$

Similarly,

$$C_2 + B_2 = A^{M_2+M_3} \text{ and } C_2 - B_2 = A^{M_1} \tag{18}$$

Therefore, the product of $(C_1 + B_1)$ and $(C_1 - B_1)$ will yield in

$$C_2^2 - B_2^2 = A^{M_1+M_2+M_3} = A^M \tag{19}$$

Thus, from Equations (17) and (19) we can obtain,

$$C_1^2 - B_1^2 = C_2^2 - B_2^2 \tag{20}$$

For more elaboration N_1 upto N_8 are accepted and values of *N*s are in this order of $N_1 > N_2 > N_3 > N_4 > N_5 > N_6 > N_7 > N_8$. Some of the combinations are illustrated in **Table 3**.

Table 3. Various combinations of M_x for evaluation of B_x and C_x.

Combination of $C_x + B_x$	Combination of M_xs for $C_x + B_x$	Combination of $C_x - B_x$	Combination of M_xs for $C_x - B_x$
$C_1 + B_1$	$A^{M_1+M_2+M_3+M_4}$	$C_1 - B_1$	$A^{M_5+M_6+M_7+M_8}$
$C_2 + B_2$	$A^{M_1+M_2+M_3+M_4+M_5}$	$C_2 - B_2$	$A^{M_6+M_7+M_8}$
$C_3 + B_3$	$A^{M_1+M_2+M_3+M_4+M_5+M_6}$	$C_3 - B_3$	$A^{M_7+M_8}$
$C_4 + B_4$	$A^{M_1+M_2+M_3+M_4+M_5+M_6+M_7}$	$C_4 - B_4$	A^{M_8}
$\vdots$	$\vdots$	$\vdots$	$\vdots$
so on	so on	so on	so on

Here is a small numerical example. For $A = 2$ and $M = 1+2+3+4 = 10$,

$$C_5 + B_5 = A^{1+2+3} \text{ and } C_5 - B_5 = A^4 \quad (21)$$

Thus, evaluated values of B_5 and C_5 are 40 and 24 respectively and therefore, $40^2 - 24^2 = 2^{1+2+3+4} = 2^{10}$.

Similarly, For $A = 2$ and $M = 1+2+3+4 = 10$,

$$C_6 + B_6 = A^{4+3} \text{ and } C_6 - B_6 = A^{1+2} \quad (22)$$

Therefore, evaluated values of C_6 and B_6 are 68 and 60 respectively and henceforth, $68^2 - 60^2 = 2^{1+2+3+4} = 2^{10}$.

Thus it may be concluded that a new deductive series from Fermat–Pramanik principle can be generated as,

$$\begin{aligned} C_1^2 - B_1^2 &= C_2^2 - B_2^2 = C_3^2 - B_3^2 = \cdots = C_n^2 - B_n^2 \\ &= A^{M_1+M_2+M_3+M_4+M_5+M_6+\cdots+\text{so on}} = A^M \end{aligned} \quad (23)$$

where, $M = M_1 + M_2 + M_3 + M_4 + M_5 + M_6 + \cdots + \text{so on}$.

Thus we have shown that the Fermat triple can generate a) Fermat-Pramanik multiplate [12], b) Fermat-Pramanik Branched multiplate and c) Fermat-Pramanik deductive series. All these formalisms are useful for cryptography and those studies are in progress [13].

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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