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On the Proof of the Contradiction of Set Theory

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Abstract

The article is devoted to proving the inconsistency of set theory arising from the existence of strange trees. All steps of the proof rely on common informal set-theoretic reasoning, but they take into account the prohibitions that were introduced into axiomatic set theories in order to overcome the difficulties encountered by the naive Cantor set theory. Therefore, in fact, the article is about proving the inconsistency of existing axiomatic set theories, in particular, the ZFC theory.

Keywords

Set Theory, Inconsistency, Tree, Strange Tree, Through Way, Almost through Way, Isomorphism, Almost Isomorphism, Isomorphism Tree, Place Plane, Superposition of Trees on the Place Plane, Disposition of Trees on the Place Plane

1. Introduction

In the twentieth century, there were crises in mathematics, which led first to its complete axiomatization (in particular, axiomatic set theories appeared), and then to Gödel's famous theorems about the incompleteness and impossibility of proving the consistency of an axiomatic theory by means formalized within the theory itself (the preface in [1] reminds the reader about this).

Now the general opinion of mathematicians is that Peano arithmetic is certainly consistent (the rarest exception is the paper [2]: it makes an assumption about the possible inconsistency of arithmetic), and set theory is almost certainly. This point of view, expressed for example by Kolmogorov and Dragalin in their book [3], means the rejection of the principle of scientific knowability, the validity of which Hilbert always insisted on.

This work is devoted to proving that existing axiomatic set theories (in particular, the ZFC theory) contain a contradiction. Thus, contrary to the generally accepted opinion, the restrictions on the formation of sets made by Zermelo and Frenkel do not save set theory from inconsistency.

The article is a continuation of article [1] and assumes that the reader is familiar with it. In particular, all notations used in the article are borrowed from [1] (and concentrated mainly in Section 2 of [1]), and the numbering of lemmas and theorems continues the numbering adopted in [1].

2. Proof Strategy

To make the reader's work easier, we will describe first in general terms the proof strategy used in the article. A class-set of almost through almost homogeneous trees of height $\propto = \omega_1$ (notations: $T_w = T_w^{\propto}$, $T_t = T_t^{\propto}$, etc.) almost isomorphic to each other is introduced, including along with trees T_w, T_t their cuts at any level $m \le \propto$: $T_w^m = cut(T_w, m)$, $T_t^m = cut(T_t, m)$, containing both trees without through paths and through trees (called the first class). By virtue of the results of [1], the first class of trees exists. Trees with double vertices T_{wt}^m are introduced, representing isomorphisms of the trees T_w^m, T_t^m (and for simplicity, identified with them). The tree T_t is fixed and various overlays (isomorphisms) of T_{wt} (and their cuts) are considered.

For every two trees T_w^m, T_t^m from the first class of the same height *m*, a tree of impositions (isomorphisms) $\boldsymbol{T}_{wt}^m$ is introduced, the vertices of which at level *k* are all possible isomorphisms T_{wt}^k. Thus, a second class of trees is introduced into consideration—the class of trees of isomorphisms of trees of the first class. This class also contains both trees without through paths and through trees. The first ones are obtained if we take two non-isomorphic trees of the first class of height $\propto = \omega_1$, the second ones—in all other cases. For $m < \propto$ the tree $\boldsymbol{T}_{wt}^m$ is through and homogeneous.

The T_t tree is fixed as the tree on which the T_w trees of the first class are superimposed (when this can be done). The numbering of automorphisms T_{tt}^k is fixed in a certain way for each level *k* in T_t and the notation P_{ti}^k is introduced for numbered automorphisms. The operation of multiplying the isomorphism T_{wt}^k by the automorphism P_{ti}^k is introduced, the result of which is a new isomorphism. Numbering of isomorphisms is introduced, obeying the rule: $T_{wt,i}^k = T_{wt,0}^k \times P_{ti}^k$. Each numbering is determined by the choice of $T_{wt,0}^k$. The introduction of numbering makes it possible to place isomorphism trees $\boldsymbol{T}_{wt}^m$ on the place plane in various ways. We assume that the isomorphism $T_{wt,i}^k$ is superimposed on place n_i^k.

The characteristic property of isomorphism trees is established:

$$cut\left(T_{wt,i}^l \times P_{tj}^l, k\right) = cut\left(T_{wt,i}^l, k\right) \times cut\left(P_{tj}^l, k\right), k \le l \le m.$$

A set of automorphisms for isomorphism trees is introduced and the properties of these automorphisms are established. The following main result holds. Let

$W_a^m = \left(T_{wt,i_k}^k, k \le m\right)$, $W_b^m = \left(T_{wt,j_k}^k, k \le m\right)$ be two through paths in the through tree $\boldsymbol{T}_{wt}^m$. The imposition of W_b^m on W_a^m uniquely determines the automorphism $\boldsymbol{T}_{wt}^m$.

Each imposition of $\boldsymbol{T}_{wt}^m$ on the place plane naturally generates a place tree T_n^m, in which the order relation between places n_i^k and n_j^l copies the order relation between the vertices $T_{wt,i}^k$ and $T_{wt,j}^l$ in tree $\boldsymbol{T}_{wt}^m$. The tree T_n^m is trivially isomorphic to $\boldsymbol{T}_{wt}^m$. We will call the path in the place tree, on which the path from the tree $\boldsymbol{T}_{wt}^m$ is superimposed, the prototype of this path. The properties of prototypes copy the properties of the paths themselves. The place tree T_n corresponding to the isomorphism tree $\boldsymbol{T}_{wt}$ has through paths if and only if the tree $\boldsymbol{T}_{wt}$ has them. And the tree $\boldsymbol{T}_{wt}$ has through paths if and only if the trees T_w, T_t are isomorphic. We have now introduced the class of place trees. In this class there are T_n trees that do not have through paths, and there are through trees.

Since for $m < \propto$ all trees $\boldsymbol{T}_{wt}^m$ are isomorphic to each other, any $\boldsymbol{T}_{wt}^m$ can be superimposed on any T_n^m, and all T_n^m are through trees (for $m < \propto$). The concept of the disposition of the tree $\boldsymbol{T}_{wt}^m$ on the plane of places is introduced (reflecting the intuitive meaning of this concept). By disposition of $\boldsymbol{T}_{wt}^m$ we mean the set of all possible impositions of $\boldsymbol{T}_{wt}^m$ on some tree T_n^m (thus, for every two impositions, the second is obtained from the first one using the automorphism operation, leaving in place the prototypes of through paths). If $\boldsymbol{T}_{wt}^m$ is a through tree, then its disposition on the place plane is a set of overlaps corresponding to the same set of prototypes of through paths. The disposition of $\boldsymbol{T}_{wt}^m$ continues the disposition of $\boldsymbol{T}_{wt}^l$ if the corresponding T_n^m continues the corresponding T_n^l. By disposition of $\boldsymbol{T}_{wt}^{\propto-0}$ on the plane of places we mean the set of dispositions of $\boldsymbol{T}_{wt}^m$ ($m < \propto$) corresponding to some T_n. This means that for all $m < \propto$ the disposition of $\boldsymbol{T}_{wt}^m$ corresponds to the place tree

$T_n^m = cut\left(T_n, m\right)$. If $\boldsymbol{T}_{wt}$ is given, then for all T_n there is a disposition $\boldsymbol{T}_{wt}^{\propto-0}$ corresponding to this T_n.

Let $\boldsymbol{T}_{wt}$ be a through tree. Note that further in Section 6, for greater clarity, instead of $\boldsymbol{T}_{wt}$ we take a splitting tree T_S isomorphic to $\boldsymbol{T}_{wt}$ with the isomorphism described in the proof of theorem 3 from [1]. If T_n is a through tree, then the disposition of $\boldsymbol{T}_{wt}^{\propto-0}$ on the place plane in accordance with T_n obviously implies the existence of a disposition $\boldsymbol{T}_{wt}^{\propto}$ that continues the disposition of $\boldsymbol{T}_{wt}^{\propto-0}$. But the entire difference between this case and any other comes down to a different disposition of the same mathematical object on the plane of places, and all dispositions of a mathematical object on the plane of places are isomorphic to each other due to the homogeneity of the place plane. Therefore, in the general case, the disposition of $\boldsymbol{T}_{wt}^{\propto-0}$ on the plane of places entails the existence of a disposition $\boldsymbol{T}_{wt}^{\propto}$, which continues the disposition of $\boldsymbol{T}_{wt}^{\propto-0}$. But then an arbitrary place tree T_n has through paths, and we get a contradiction in set theory.

3. Tree of Isomorphisms

We will further consider the class-set of almost through almost homogeneous trees of height $\propto = \omega_1$, almost isomorphic to each other, containing a strange tree T_{str} (a tree without through paths), see Section 3 in [1]. Note that the tree T_{str} is homogeneous. We will call this class of trees the *first class*. We will use the notations T_w, T_v, T_t, etc., for the trees of the first class (with or without additional indices), using notations like w_i^k, v_i^k, t_i^k, etc., for the vertices at level *k*. To denote an arbitrary representative of the first class, we will usually use the notation T_w. By virtue of theorem 1 from [1], we can assume that the first class of trees contains trees that have through paths. And by virtue of theorem 2 from [1], we can assume that it also contains through trees. Note that for T_{str} the 4° condition is satisfied (see Sections 2, 4 in [1]): final vertices cannot appear at tree levels with non-limit numbers. Therefore, this is executed for any tree T_w from the first class. Along with T_w, any tree $T_w^m = cut(T_w, m)$ can participate in the considerations.

Our ultimate goal is to show that such a situation leads to a contradiction.

For $m < \propto$ trees $T_w^m = cut(T_w, m)$ (from the first class) are homogeneous and through (even strongly homogeneous and strongly through). But the tree $T_w = T_w^{\propto}$ in the general case is only almost homogeneous and almost through and may not contain a through path, having an empty level $\propto$. For $m < \propto$ in the tree $T_w^m = cut(T_w, m)$ at level *m*, we will also divide (where necessary) the vertices into final and non-final, considering non-final those vertices that are non-final in the tree T_w.

Let us choose T_t as the tree onto which we will isomorphically superimpose the trees T_w in various ways. For $m < \propto$, the tree T_w^m can always be superimposed on the tree T_t^m while preserving the order relation between the vertices, but for $m = \propto$ the superposition is not always possible since T_w, T_t can be non-isomorphic (only almost isomorphic). We consider the result of the overlay as a tree T_{wt}^m or T_{wt} having *double vertices* (w_j^k, t_i^k) at level *k*, which represents some isomorphism of the trees T_w^m, T_t^m or T_w, T_t. Let us define w_j^k as the *first subvertices*, and t_i^k as the *second ones*. With this isomorphism, vertices w_j^k go to vertices t_i^k: $w_j^k \to t_i^k$. We will also say that w_j^k are superimposed on t_i^k, which makes the isomorphism clearer. The number of possible overlaps determines the number of isomorphisms. In the case when $T_w^m = T_t^m$ ($T_w = T_t$), we are dealing with automorphisms.

The double-vertex tree T_{wt}^m gives a visual representation of how the corresponding isomorphism works. We will call T_{wt}^m the superposition of T_w^m on T_t^m and for simplicity we will identify T_{wt}^m with the isomorphism itself.

Let us consider, for example, the trees T_w^2, T_t^2 of height 2, shown in **Figure 1**.

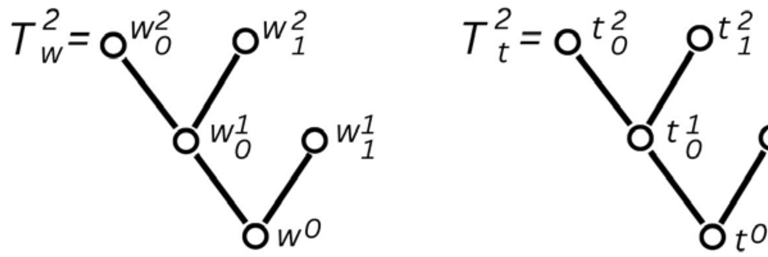

Figure 1. Example of trees of height 2.

In the tree T_w^2, $w^0 \le w_0^1$, $w^0 \le w_1^1$, $w_0^1 \le w_0^2$, $w_0^1 \le w_1^2$ and w_1^1 at level 1 is final. A similar thing occurs for T_t^2 and trees in **Figures 2-6**.

We will talk about *multiple impositions* of T_w^m on T_t^m, meaning by this the set of all possible impositions of T_w^m on T_t^m, and in accordance with this concept we will introduce a *tree of isomorphisms* $\boldsymbol{T}_{wt}^m$. When T_w^m and T_t^m are isomorphic (this is always the case if $m < \propto$), for each $k \le m$ in $\boldsymbol{T}_{wt}^m$ at level k the vertices are isomorphism trees T_{wt}^k (all such trees in a single design). And the order relation between vertices is defined as the relation of continuation of isomorphisms: $T_{wt,a}^k \le T_{wt,b}^l$ means that $T_{wt,a}^k = cut\left(T_{wt,b}^l, k\right)$ (see Section 2 in [1]). If T_w^m and T_t^m are not isomorphic (this can be the case if $m = \propto$), then in the definition of $\boldsymbol{T}_{wt}^m$ we replace $k \le m$ with $k < m$. The definition for T_{wt}^{m-0} is introduced accordingly. As one can easily see, the described structure is indeed tree.

Recall (see Section 2 in [1]) that we call the isomorphism of trees $T_w^m = cut\left(T_w, m\right)$, $T_t^m = cut\left(T_t, m\right)$ *strong* if it transforms non-final vertices of the tree T_w^m at the level *m* to non-final vertices of the tree T_t^m.

Lemma 20. A vertex-isomorphism T_{wt}^l in a tree $\boldsymbol{T}_{wt}^m$ for $l < m$ is non-final if and only if the isomorphism T_{wt}^l is strong. Thus, the set of non-final vertices at level $l < m$ in the tree $\boldsymbol{T}_{wt}^m$ coincides with the set of strong vertex-isomorphisms at this level.

The proof is straightforward.

In what follows, the concepts strong and non-final in relation to vertex-isomorphisms in a tree $\boldsymbol{T}_{wt}^m$ will be considered synonymous.

In the example under consideration, the isomorphism $T_{wt,0}^1$ is strong and the isomorphism $T_{wt,1}^1$ is not strong.

In our example, the isomorphism tree will be looked as shown in **Figure 4**.

Lemma 21. The imposition of T_w on T_t (*i.e.*, the isomorphism of trees T_w, T_t) exists if and only if the isomorphism tree $\boldsymbol{T}_{wt}$ has a through path. If we are talking about an automorphism ($T_w = T_t$), then the tree $\boldsymbol{T}_{tt}$ always has a through path.

Indeed, if T_w can be superimposed on T_t, then this superposition forms a through path in the tree $\boldsymbol{T}_{wt}$. On the other hand, each through path in $\boldsymbol{T}_{wt}$ is a continuing sequence of isomorphisms (T_{wt}^m, $m<\propto$), introducing the isomorphism of trees T_w, T_t (see lemma 10 in [1]). The second statement is obvious, since the tree T_t can always be superimposed on itself.

Lemma 22. Let trees T_w, T_t of height $\propto$ be almost isomorphic and almost homogeneous. Then the isomorphism tree $\boldsymbol{T}_{wt}$ is almost homogeneous and almost through, and the tree $\boldsymbol{T}_{wt}^m = cut(\boldsymbol{T}_{wt}, m)$ ($m<\propto$) is homogeneous and through. If there is no isomorphism of the trees T_w, T_t, the tree $\boldsymbol{T}_{wt}$ will be strange. If point 4° of the tree definition is satisfied for the trees T_w, T_t (see Section 2 in [1]), then it is also satisfied for the tree $\boldsymbol{T}_{wt}$. If T_w, T_t are isomorphic, then $\boldsymbol{T}_{wt}$ has a through path and vice versa. If at the same time T_t is homogeneous, then $\boldsymbol{T}_{wt}$ is a through tree.

The statements of the lemma are straightforward.

We will further assume that T_t is a homogeneous tree.

Corollary 1. The first class of almost homogeneous trees that are almost isomorphic to each other generates the second class of almost homogeneous almost isomorphic trees of isomorphisms. In both classes there are trees without through paths, there are through trees and the 4° condition is satisfied. In the future, all our interest will be focused on the study of trees from the second class (the set of trees $\boldsymbol{T}_{wt}$). If it is shown that in this class all trees have through paths, then the inconsistency of set theory will be shown.

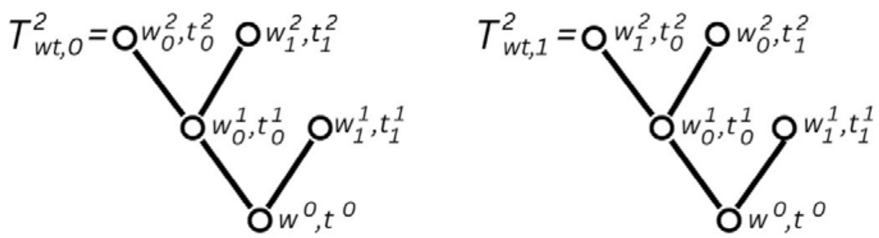

Figure 2. Trees representing isomorphisms (overlays) of the trees in **Figure 1**.

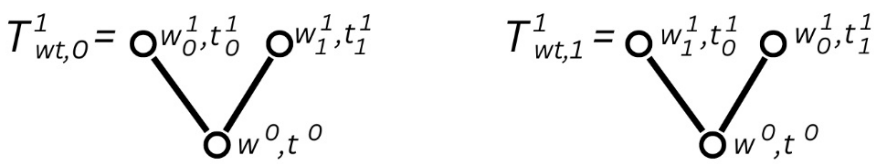

Figure 3. Trees representing isomorphisms of the trees in **Figure 1** for level 1.

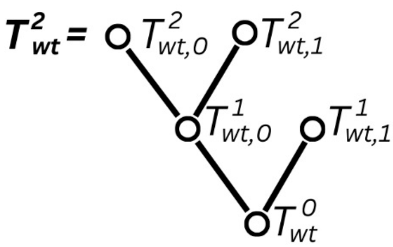

Figure 4. The tree of overlays (isomorphisms) $\boldsymbol{T}_{wt}^2$ for trees in **Figure 1**.

Let us introduce the numbering of vertex-isomorphisms at all levels of the tree $\boldsymbol{T}_{wt}^{m}$, satisfying certain rules. Looking ahead, we point out that we will need numbering in order to place $\boldsymbol{T}_{wt}^{m}$ trees on the plane of places. To do this, we first introduce the numbering of vertices in the automorphism tree $\boldsymbol{T}_{tt}$ for each level $m \le \propto$ using ordinals less than some cardinal, which we will call basic. Let the automorphisms ($T_{tt,i}^{m}$, $i < \beta_m$) of the tree $T_t^m = cut(T_t, m)$ be located at level *m*. Let us agree that for $i < \overline{\beta}_m \le \beta_m$ the automorphisms $T_{tt,i}^{m}$ are strong. Always $\overline{\beta}_m > 0$ since any $T_t^m = cut(T_t, m)$ has an identity automorphism, which is strong. We will assume that the automorphism $T_{tt,0}^{m}$ is identical. In the case of basic numbering, we will also use the notation P_{ti}^{m} instead of $T_{tt,i}^{m}$. Note that for non-limit m $\overline{\beta}_m = \beta_m$ (due to condition 4° from Section 2 in [1]).

Lemma 23. Automorphisms (P_{ti}^{m}, $i < \beta_m$) for fixed *m* form a group. The set of strong (non-final) automorphisms (P_{ti}^{m}, $i < \overline{\beta}_m$) forms a subgroup of this group.

We will assume that $\left(P_{ti}^{m}, i < \beta_m, m \le \propto\right)$ is a homogeneous through tree. In particular, this will be the case if $T_t = T_{str}$.

Let us introduce the operation of multiplying the isomorphism T_{wv}^{m} by the isomorphism T_{vt}^{m}. The result of the multiplication will be the isomorphism T_{wt}^{m}, obtained as follows. "Glue" the second subvertices of T_{wv}^{m} at all levels with the corresponding first subvertices of T_{vt}^{m} to obtain a tree with triple vertices, and then remove the second subvertices from it. In a particular case, the operation of multiplying the isomorphism T_{wt}^{m} by the automorphism P_{ti}^{m} will take place. We will consider this operation as an isomorphism transformation operation: T_{wt}^{m} is transformed into $T_{wt}^{m} \times P_{ti}^{m}$. It is this subcase that will interest us in the future. If $T_{wt}^{m} = P_{tj}^{m}$, then we have the operation of multiplying automorphisms: $P_{tj}^{m} \times P_{ti}^{m}$.

The operation of multiplying T_{wt}^{m} by P_{ti}^{m} satisfies, as is easy to see, the law of associativity:

$$T_{wt}^{m} \times \left(P_{ti}^{m} \times P_{tj}^{m}\right) = T_{wt}^{m} \times P_{ti}^{m} \times P_{tj}^{m}. \tag{1}$$

Also

$$\text{if } T_{wt,b}^{m} = T_{wt,a}^{m} \times P_{t}^{m}, \text{ then } T_{wt,a}^{m} = T_{wt,b}^{m} \times \left(P_{t}^{m}\right)^{-1} \tag{2}$$

where the automorphism $\left(P_{t}^{m}\right)^{-1}$, as a tree with double vertices, is obtained from P_{t}^{m} when we swap the first and second subvertices.

Next, we will proceed as follows. For each $k \le m$, we choose some non-final isomorphism T_{wt}^{k} as the *main isomorphism* for a given *k*, and give number 0 to it. After this, we introduce the numbering of vertex-isomorphisms: $T_{wt,i}^{k} = T_{wt,0}^{k} \times P_{ti}^{k}$. In this case, the following will occur: $T_{wt,i}^{k} \times P_{tj}^{k} = T_{wt,0}^{k} \times \left(P_{ti}^{k} \times P_{tj}^{k}\right)$ and $T_{wt,0}^{k} = T_{wt,i}^{k} \times \left(P_{ti}^{k}\right)^{-1}$ (see (1) and (2)).

In the case of $\boldsymbol{T}_{tt}^{m}$, when choosing identical automorphisms as $T_{tt,0}^{k}$, we arrive at the tree $\left(P_{ti}^{k}, i<\beta_k, k\le m\right)$.

It is obvious that the numbering of vertices at the levels of the tree $\boldsymbol{T}_{wt}^{m}$ is completely determined by the choice of the main isomorphisms $T_{wt,0}^{k}$ for $k\le m$ and different numberings will take place for different choices.

To illustrate what has been said, let us turn again to the trees in **Figure 1**. In **Figure 5** and **Figure 6** the automorphisms P_{ti}^{2} and P_{ti}^{1} are shown.

For automorphisms the following equalities hold:

$$P_{t0}^{2}\times P_{t0}^{2}=P_{t0}^{2},\quad P_{t1}^{2}\times P_{t1}^{2}=P_{t0}^{2},\quad P_{t0}^{2}\times P_{t1}^{2}=P_{t1}^{2}\times P_{t0}^{2}=P_{t1}^{2};$$

$$P_{t0}^{1}\times P_{t0}^{1}=P_{t0}^{1},\quad P_{t1}^{1}\times P_{t1}^{1}=P_{t0}^{1},\quad P_{t0}^{1}\times P_{t1}^{1}=P_{t1}^{1}\times P_{t0}^{1}=P_{t1}^{1}.$$

Accordingly, multiplying isomorphisms by automorphisms leads to the equalities:

$$T_{wt,0}^{2}\times P_{t0}^{2}=T_{wt,0}^{2},\quad T_{wt,1}^{2}\times P_{t1}^{2}=T_{wt0}^{2},\quad T_{wt,0}^{2}\times P_{t1}^{2}=T_{wt,1}^{2}\times P_{t0}^{2}=T_{wt,1}^{2};$$

$$T_{wt,0}^{1}\times P_{t0}^{1}=T_{wt,0}^{1},\quad T_{wt,1}^{1}\times P_{t1}^{1}=T_{wt,0}^{1},\quad T_{wt,0}^{1}\times P_{t1}^{1}=T_{wt,1}^{1}\times P_{t0}^{1}=T_{wt,1}^{1}.$$

Lemma 24. Let for $k\le m$ $T_{wt,r}^{k}$ be an arbitrary non-final vertex-isomorphism: $r<\overline{\beta}_k$. Non-final vertices at level k are obtained by multiplication $T_{wt,r}^{k}$ by non-final automorphisms P_{tj}^{k} from the group of non-final automorphisms at level *k*. Thus, $T_{wt,r}^{k}\times\left(P_{tj}^{k}, j<\overline{\beta}_k\right)$ forms a subset of non-final vertex-isomorphisms. Knowledge of one non-final vertex at level *k* gives knowledge of all.

Corollary 2. If in the tree $\boldsymbol{T}_{wt}^{m}$ at level *k* the main isomorphism is non-final, then the isomorphisms $T_{wt,i}^{k}, i<\overline{\beta}_k$ will be non-final.

We will study trees $\boldsymbol{T}_{wt}^{m}$ when for all $k\le m$ the isomorphisms $T_{wt,i}^{k}, i<\overline{\beta}_k$ are non-final. Obviously, this limitation is justified.

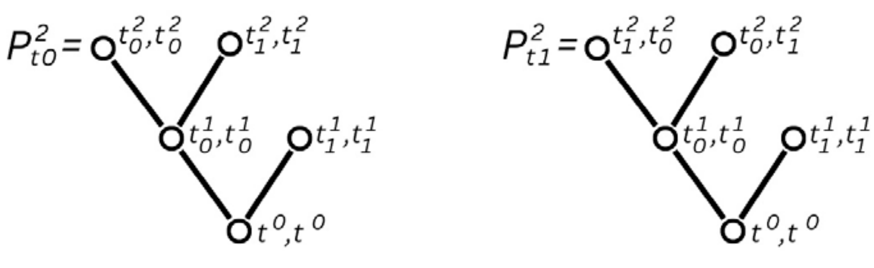

Figure 5. Trees representing automorphisms of the tree T_t^2.

Figure 6. Trees representing automorphisms of the tree T_t^1.

The fundamental property of an isomorphism tree is the property reflected in lemma 25.

Lemma 25. The *decomposition rule* is satisfied in the isomorphism tree $\boldsymbol{T}_{wt}^{m}$:

$$cut\left(T_{wt,i}^{l} \times P_{tj}^{l}, k\right) = cut\left(T_{wt,i}^{l}, k\right) \times cut\left(P_{tj}^{l}, k\right),\ k \le l \le m. \tag{3}$$

In particular, $cut\left(P_{ti}^{l} \times P_{tj}^{l}, k\right) = cut\left(P_{ti}^{l}, k\right) \times cut\left(P_{tj}^{l}, k\right)$.

The statement of the lemma follows from lemma 7 (see Section 2 in [1]) and the definition of multiplication of isomorphism by automorphism.

The essence of the lemma is that if it is known that $T_{wt,r}^{k} \le T_{wt,s}^{l}$, then by this relation all other relations between the vertices in rows k and l are determined purely algebraic.

Another formulation of the statement of lemma 25 looks like this: if $T_{wt,i_2}^{l} \ge T_{wt,i_1}^{k}$ and $P_{tj_2}^{l} \ge P_{tj_1}^{k}$, then $T_{wt,i_2}^{l} \times P_{tj_2}^{l} \ge T_{wt,i_1}^{k} \times P_{tj_1}^{k}$.

Corollary 3. If $cut\left(T_{wt,i}^{l} \times P_{tj_2}^{l}, k\right) = cut\left(T_{wt,i}^{l}, k\right) \times P_{tj_1}^{k}$, then $P_{tj_1}^{k} = cut\left(P_{tj_2}^{l}, k\right)$.

Indeed, we have

$cut\left(T_{wt,i}^{l} \times P_{tj_2}^{l}, k\right) = cut\left(T_{wt,i}^{l}, k\right) \times cut\left(P_{tj_2}^{l}, k\right) = cut\left(T_{wt,i}^{l}, k\right) \times P_{tj_1}^{k}$ and, therefore $P_{tj_1}^{k} = cut\left(P_{tj_2}^{l}, k\right)$.

Lemma 26. Let T_{wt,i_k}^{k} be given for all $k \le m$. Then $\boldsymbol{T}_{wt}^{m}$ is uniquely defined by the conditions: $T_{wt,i}^{k} = T_{wt,i_k}^{k} \times \left(P_{ti_k}^{k}\right)^{-1} \times P_{ti}^{k}$, $i < \beta_k$, $k \le m$.

The next lemma follows from the above statements.

Lemma 27. Let $I^{m} = \left(i_k, k \le m\right)$ be the sequence of numbers for which $i_k < \overline{\beta}_k$. For isomorphic T_t^m, T_w^m there is a through tree $\boldsymbol{T}_{wt}^{m}$, for which $\left(T_{wt,i_k}^{k}, k \le m\right)$ is a through path. By specifying the pair $\left(T_{wt}^{m}, I^{m}\right)$, where $T_{wt}^{m} = T_{wt,i_m}^{m}$ is a non-final isomorphism, the tree $\boldsymbol{T}_{wt}^{m}$ is uniquely determined.

Lemma 27 is essential for us. If it is shown that $\boldsymbol{T}_{wt}$ always has a through path, then from this the inconsistency of set theory will follow.

4. Isomorphism of Trees of Isomorphisms (and Automorphisms)

The introduced numbering of vertex-isomorphisms makes it possible to add to the trees under consideration such a characteristic as their superposition on the plane of places. Let us introduce the *plane of places* and place trees of isomorphisms (and automorphisms) in its part bounded by places $\left(\left(n_i^k, 0 \le i < \beta_k\right), 0 \le k \le \propto\right)$. Places $\left(n_i^k, 0 \le i < \overline{\beta}_k\right)$ are reserved for non-final isomorphisms (and automorphisms). For clarity, we will assume that $T_{wt,i}^{k}$ is superimposed on n_i^k, and say that the numbering of vertex-isomorphisms in the tree $\boldsymbol{T}_{wt}^{m}$ determines the placement of the tree on the place plane. With a different numbering, we will have a different placement of $\boldsymbol{T}_{wt}^{m}$. In what follows, when speaking about the placement (or disposition, see Section 6) of a tree on the place plane, we will mean the placement on the part of the place plane limited by places $\left(\left(n_i^k, 0 \le i < \beta_k\right), 0 \le k \le \propto\right)$.

Let us make some additions and modifications to the terminology used. This will make it possible to present previously obtained results in a more complete and visual way and obtain a number of new ones.

Recall that P_{ti}^k are automorphisms of the tree T_t^k, onto which the trees T_w^k are superimposed in different ways, P_{t0}^k is the identical automorphism and P_{ti}^k are non-final automorphisms when $i < \overline{\beta}_k$. We consider the automorphism tree $\left(P_{ti}^k, i < \beta_k, k \le \propto\right)$ to be through and homogeneous (see the remark after lemma 23). For visibility, we assume that P_{ti}^k are superimposed on n_i^k.

Let a sequence of place numbers be given $I^m = \left(n_{i_k}^k, k \le m\right)$. If in the isomorphism tree $\boldsymbol{T}_{wt}^m$ there is a through path $W^m = \left(T_{wt,i_k}^k = cut\left(T_{wt,i_m}^m, k\right), k \le m\right)$, then we will say that I^m *defines* this path. For $k < m$, $i_k < \overline{\beta}_k$, and $i_m < \overline{\beta}_m$ holds for non-final I^m, defining non-final paths $\left(T_{wt,i_k}^k, k \le m\right)$. Let us introduce the operation of multiplying places n_i^k by automorphisms P_{tj}^k: $n_0^k \times P_{ti}^k = n_i^k$, $n_i^k \times P_{tj}^k = n_0^k \times \left(P_{ti}^k \times P_{tj}^k\right)$. The sets $\left(n_i^k, i < \beta_k\right)$, $\left(P_{ti}^k, i < \beta_k\right)$ and $\left(T_{wt,i}^k, i < \beta_k\right)$ are *trivially isomorphic* with respect to the operation of multiplying their elements by P_{tj}^k.

Lemma 28. For a given *k*, $n_i^k \times P_{tj}^k = n_l^k$ if and only if $P_{ti}^k \times P_{tj}^k = P_{tl}^k$ and $T_{wt,i}^k \times P_{tj}^k = T_{wt,l}^k$.

Let us denote by PI^m the continuing sequence of automorphisms $\left(P_{tj_k}^k, k \le m\right)$ superimposed on the plane of places. By definition, $I^m \times PI^m = \left(n_{i_k}^k \times P_{tj_k}^k, k \le m\right)$, while $W^m \times PI^m = \left(T_{wt,i_k}^k \times P_{tj_k}^k, k \le m\right)$. $PI^l = \left(P_{tj_k}^k, k \le l\right) = cut\left(PI^m, l\right)$ for $l \le m$. In the continuing sequence of automorphisms $\left(P_{tj_k}^k, k \le m\right)$ all $P_{tj_k}^k$ for $k < m$ are non-final. If also $P_{tj_m}^m$ is non-final, then PI^m will be called non-final. Otherwise, we will talk about the final PI^m. Also by definition, $PI_a^m \times PI_b^m = \left(P_{ti_k}^k \times P_{tj_k}^k, k \le m\right)$ where $PI_a^m = \left(P_{ti_k}^k, k \le m\right)$, $PI_b^m = \left(P_{tj_k}^k, k \le m\right)$. The multiplication operation "×" turns the set $\left(PI_i^m, i < \beta_m\right)$ into a group, and the set $\left(PI_i^m, i < \overline{\beta}_m\right)$ into a subgroup of this group.

The sequence $PI_0^m = \left(P_{t0}^k, k \le m\right)$ is a continuing sequence of identical automorphisms.

Lemma 29. For any non-final sequence $I^m = \left(n_{i_k}^k, k \le m\right)$ there is a through tree $\boldsymbol{T}_{wt}^m$ for which I^m defines a strongly through path $W^m = \left(T_{wt,i_k}^k, k \le m\right)$. If I^m defines the path W^m in the tree $\boldsymbol{T}_{wt}^m$, then $I^m \times PI^m$ defines the path $W^m \times PI^m$.

The validity of the first statement follows from lemma 27, and the second—

from the trivial isomorphisms introduced above.

Remark 1. Each through tree $\boldsymbol{T}_{wt}^{m}$ is uniquely determined by the set of its through paths $\left(W_i^m, i = 0,1,\cdots\right)$ where each through path is a continuing sequence of tree vertices superimposed on the plane of places. Therefore, it is convenient and visual to characterize order relations in a tree with the help of through paths, using continuing sequences of automorphisms and keeping in mind that each sequence $PI^m = \left(P_{ti_k}^k, k \le m\right)$ is completely specified by its upper term $P_{ti_m}^m$: $P_{ti_k}^k = cut\left(P_{ti_m}^m, k\right)$.

Note that the second statement of lemma 29 is a "vector" paraphrase of the decomposition rule (3). This becomes quite obvious if the multiplications $I^m \times PI^m$ and $W^m \times PI^m$ are written as $I^m \times \left(cut\left(P_{ti_m}^m, k\right), k \le m\right)$ and $W^m \times \left(cut\left(P_{ti_m}^m, k\right), k \le m\right)$.

We formulate the decomposition rule and its corollary in vector notation (see lemma 25 and corollary 3) in the form of lemma 30.

Lemma 30. Let W^m be a path in $\boldsymbol{T}_{wt}^{m}$.Then $W^m \times PJ^m$ is a path in $\boldsymbol{T}_{wt}^{m}$ if and only if $PJ^m = \left(P_{ti_k}^k, k \le m\right)$ is a continuing sequence of automorphisms ($P_{ti_k}^k = cut\left(P_{ti_m}^m, k\right)$) for all $k \le m$.

We will call sequences I^m that define paths in the tree $\boldsymbol{T}_{wt}^{m}$ the *prototypes* of these paths. Due to the trivial isomorphisms introduced above (see lemma 28), this term is justified, and we can treat the prototypes of paths in the same way as the paths themselves. In particular, we have: if I^m is the prototype of a path, then $I^m \times PI^m$ is also the prototype of a path.

Following the order introduced for automorphisms in the rows of each level, we introduce the order for continuing sequences of automorphisms: $PI_i^m = \left(P_{tj_k}^k \le m\right)$, where $j_m = i$. For non-final sequences of automorphisms we have: $i < \overline{\beta}_m$.

Lemma 31. Let $I^m = \left(n_{i_k}^k, k \le m\right)$ be the prototype of a non-final path in the through tree $\boldsymbol{T}_{wt}^{m}$. Then $I^m \times PI_i^m$, when PI_i^m runs over all sequences (non-final sequences) of automorphisms, forms the set of all prototypes of paths (non-final paths) in $\boldsymbol{T}_{wt}^{m}$. If $I^m = \left(n_{i_k}^k, i_k < \overline{\beta}_k, k \le m\right)$ and a non-final isomorphism T_{wt}^m are given, then their combination uniquely determines the tree $\boldsymbol{T}_{wt}^{m}$, for which I^m is the prototype of the through non-final path, and the isomorphism T_{wt}^m is placed on $n_{i_m}^m$.

See lemma 27.

Lemma 32. Let I_a^m, I_b^m be the prototypes of through non-final paths in a through tree $\boldsymbol{T}_{wt}^{m}$.The one-to-one correspondence $I_a^m \times PI_i^m \to I_b^m \times PI_i^m$, when PI_i^m runs over all continuing sequences of automorphisms, defines a strong automorphism of the tree $\boldsymbol{T}_{wt}^{m}$ such that I_a^m goes to I_b^m. Conversely, each strong automorphism of the through tree $\boldsymbol{T}_{wt}^{m}$ is determined by a pair of prototypes of non-final paths I_a^m, I_b^m in accordance with the formula $I_a^m \times PI_i^m \to I_b^m \times PI_i^m$, when PI_i^m runs over all continuing sequences of automorphisms. If we fix I_a^m and vary I_b^m (or vice versa), we obtain all existing strong automorphisms of $\boldsymbol{T}_{wt}^m$.

Let $I_b^m = I_a^m \times PI_r^m$, $r < \overline{\beta}_m$. Therefore $I_b^m \times PI_i^m = I_a^m \times PI_r^m \times PI_i^m = I_a^m \times PI_j^m$, where $PI_j^m = PI_r^m \times PI_i^m$. Thus, we have a one-to-one correspondence $I_{ai}^m = I_a^m \times PI_i^m \to I_{aj}^m = I_a^m \times PI_j^m$, where PI_j^m (following PI_i^m) runs over all continuing sequences of automorphisms: $j = j(i)$. Let $cut(PI_i^m, l) = cut(PI_j^m, l)$. Then $cut(PI_i^m, k) = cut(PI_j^m, k)$ for all $k \le l$ and so $cut(I_{ai}^m, k) = cut(I_{aj}^m, k)$ for all $k \le l$ (see lemma 30). Therefore $\left(I_{ai}^m \to I_{aj(i)}^m, i < \beta_m\right)$ defines automorphism of $\boldsymbol{T}_{wt}^m$.

To prove the second part of the lemma, it is enough to restrict ourselves to the case when *m* is a limit ordinal, assuming that the converse statement holds for all $\boldsymbol{T}_{wt}^k = cut(\boldsymbol{T}_{wt}^m, k)$ for $k < m$. Let Aut^m be the automorphism of $\boldsymbol{T}_{wt}^m$ described in the condition of the lemma, under which the prototype of the through path I_a^m goes to the prototype of the through path I_b^m, and Aut_1^m is another automorphism of $\boldsymbol{T}_{wt}^m$, also moving I_a^m into I_b^m. $Aut^k = cut(Aut^m, k)$ converts $I_a^k = cut(I_a^m, k)$ to $I_b^k = cut(I_b^m, k)$. And the same is true for Aut_1^k. For $k < m$ $Aut^k = Aut_1^k$ to the assumption made. But then (due to the uniqueness of the limit) $Aut^m = Aut_1^m$.

The third statement of the lemma is obvious.

Remark 2. Careful analysis shows that the statements of lemma 32 are the logical consequence of the fact that specifying one prototype of a through non-final path in the tree $\boldsymbol{T}_{wt}^m$ uniquely determines all through paths (see lemma 27).

Lemma 32 can easily be carried over to a more general case.

Lemma 33. Let I_a^m, I_b^m be the prototypes of through non-final paths in the isomorphic through trees $\boldsymbol{T}_{wt}^m$ and $\boldsymbol{T}_{vt}^m$.The one-to-one correspondence $I_a^m \times PI_i^m \to I_b^m \times PI_i^m$, when PI_i^m runs over all continuing sequences of automorphisms, defines an isomorphism $\boldsymbol{T}_{wt}^m$ on $\boldsymbol{T}_{vt}^m$, for which I_a^m goes into I_b^m. The converse statement is also true.

5. Related Sequences of Places and the Tree of Places

Let us continue our research.

We will consider sequences of places $I^m = \left(n_{i_k}^k, k \le m\right)$, for which $i_k < \overline{\beta}_k \le \beta_k$ for $k < m$, $i_m < \beta_m$. For non-limit *k*, $\overline{\beta}_k = \beta_k$ (by virtue of the condition 4° in [1]). We will call the sequence I^m, *m* is a limit ordinal, non-final if also $i_m < \overline{\beta}_m \le \beta_m$. For each *k*, the multiplication operation $n_i^k \times P_{tj}^k$ is introduced: $n_0^k \times P_{ti}^k = n_i^k$, $n_i^k \times P_{tj}^k = n_0^k \times \left(P_{ti}^k \times P_{tj}^k\right)$. As a consequence of this introduction, the multiplication operation $I^m \times PI^m = \left(n_{i_k}^k \times P_{tj_k}^k, k \le m\right)$ turns out to be introduced (where $PI^m = \left(P_{tj_k}^k, k \le m\right)$), in which $I_0^m \times PI_i^m = I_i^m$, $I_i^m \times PI_j^m = I_0^m \times \left(PI_i^m \times PI_j^m\right)$, algebraically similar to the operation of multiplication when both components are continuing sequences of automorphisms. Note that for PI^m, the multiplication operation "×" turns the set of all $\left(PI_i^m, i < \beta_m\right)$ into a group, and the set of non-final $\left(PI_i^m, i < \overline{\beta}_m\right)$—into a subgroup of this group (see Section 3).

We will call sequences I_a^m, I_b^m *related* if $I_b^m = I_a^m \times PI^m$ with some PI^m (in this case also $I_a^m = I_b^m \times \left(PI^m\right)^{-1}$). We will call them *strongly related* if $I_b^m = I_a^m \times PI^m$ with some non-final PI^m. We will talk about *classes of related* (*strongly related*) I^m, meaning by them non-expandable sets of sequences of places I^m related (strongly related) to each other.

In a similar way, we can talk about classes of related and strongly related sequences of objects of arbitrary type (for which the operation of multiplication by PI^m is introduced).

We will further denote by $\boldsymbol{I}^m$ и $\overline{\boldsymbol{I}}^m$ the classes of related and strongly related sequences of places. By virtue of the choice of the basic numbering of automorphisms, the set of non-final place sequences is one of the classes of strongly related sequences. Let us introduce a special notation for this class- $\boldsymbol{J}^m$. For non-limit *m* we will assume that $\boldsymbol{I}^m = \boldsymbol{J}^m$. We will also assume that $\overline{\beta}_\infty = \beta_\infty$.

The set $\boldsymbol{I}^m$ defines the following order relation between places: $n_i^k \le n_j^l$ ($k \le l$) if and only if there exists $I^m \in \boldsymbol{I}^m$ such that n_i^k and n_j^l belong to I^m.

Lemma 34. The order relation between places in $\boldsymbol{I}^m$ induces a through tree (we will call it a *place tree*) T_n^m, for which $\boldsymbol{I}^m$ is simultaneously the set of through paths and the set of prototypes of through paths. In this tree lemma 30 holds (with W^m replaced by I^m). As a set of prototypes of through paths, $\boldsymbol{I}^m$ determines the placement of T_n^m on the place plane. We have $cut\left(\boldsymbol{I}^m, l\right) = \boldsymbol{J}^l$ when $l < m$.

In the tree of places T_n^m we consider as non-final those vertices n_i^m for which $i < \overline{\beta}_m$.

We thus have introduced the class of place trees.

Each $\boldsymbol{I}^m$ uniquely represents some T_n^m. In the following, if we talk about the set $\boldsymbol{I}^m$ as a tree, then we mean the tree T_n^m, which $\boldsymbol{I}^m$ represents. And when we talk about the isomorphism of $\boldsymbol{I}_a^m$ and $\boldsymbol{I}_b^m$, we mean, of course, isomorphism with respect to the order relation between places on the plane of places, *i.e.*, in fact we are talking about the isomorphism of T_{na}^m and T_{nb}^m. The isomorphism of $\boldsymbol{I}_a^m, \boldsymbol{I}_b^m$ means the possibility of superimposing $\boldsymbol{I}_a^m$ on $\boldsymbol{I}_b^m$ while preserving the order relation between places. We will talk about strong isomorphism if non-final sequences overlap non-final ones.

Lemma 35. Each $\boldsymbol{I}^m$ is uniquely determined by specifying any sequence $I^m \in \boldsymbol{I}^m$: $\boldsymbol{I}^m = \boldsymbol{I}^m\left(I^m\right)$. And each I^m uniquely determines the class $\boldsymbol{I}^m$ for which $I^m \in \boldsymbol{I}^m$. Therefore, each T_n^m is uniquely determined by specifying the sequence $I^m \in \boldsymbol{I}^m$.

Indeed, if $I^m \in \boldsymbol{I}^m$ is given, then the set $I^m \times PI^m$, when PI^m runs over all continuing sequences of automorphisms, contains all different sequences of places related to I^m (in one version and only such sequences). Therefore, it coincides with $\boldsymbol{I}^m$ and is uniquely determined.

Obviously, for each $\boldsymbol{J}^m$ there is a unique $\boldsymbol{I}^m$ that extends it.

We will call $\boldsymbol{I}^m$, obtained by expanding $\boldsymbol{J}^m$, *accompanying*. The correspondence between $\boldsymbol{J}^m$ and accompanying $\boldsymbol{I}^m$ is one-to-one. Due to this circumstance, statements for $\boldsymbol{I}^m$ are easily transferred to $\boldsymbol{J}^m$ and vice versa, and there is no need for modified duplication of statements.

Lemma 36. Given m any class of related sequences of places is the union of disjoint classes of strongly related sequences: $\boldsymbol{I}^m = \bigcup_i \overline{\boldsymbol{I}}_i^m$, $\overline{\boldsymbol{I}}_i^m \cap \overline{\boldsymbol{I}}_j^m = \varnothing$, if $i \neq j$.

Let us save the definition of related and strongly related sequences given above for the case when $m-0$ is taken instead of m (m is the limit ordinal). To do this, let us introduce the necessary clarification of what we mean by a non-final sequence of automorphisms PI^{m-0}. Let $PI^{m-0} = \left(P_{ti_k}^k, k < m\right)$. There is a unique $P_{ti_m}^m$, which is the limit of $\left(P_{ti_k}^k, k < m\right)$. The sequence PI^{m-0} will be called non-final if $P_{ti_m}^m$ is a non-final automorphism. Lemma 36 remains valid if m is replaced by $m-0$.

Let us move on.

Lemma 37. For any through tree $\boldsymbol{T}_{wt}^m$, the set of prototypes of through paths is the class of related sequences of places.

In fact, both sets are determined from one of their representatives I^m by a formula: $I^m \times PI^m$, when PI^m runs over all continuing sequences of automorphisms. See lemmas 31, 35.

The same is true for through non-final paths.

The following statement is also true.

Lemma 38. For any $\boldsymbol{I}^m$ and isomorphic T_w^m, T_t^m, there is a through tree $\boldsymbol{T}_{wt}^m$ for which $\boldsymbol{I}^m$ is the set of the prototypes of through paths.

In fact, let $I^m = \left(n_{i_k}^k, k \leq m\right) \in \boldsymbol{I}^m$, $n_{i_m}^m = n_0^m$. Let us place the non-final isomorphism T_{wt}^m on $n_{i_m}^m$. By this, we define $\boldsymbol{T}_{wt}^m$, in which $W^m = \left(T_{i_k}^k = cut\left(T_{wt}^m, k\right), k \leq m\right)$ is a through path, for which I^m is the prototype, see lemma 27. Sequences related to $I^m = \left(n_{i_k}^k, k \leq m\right)$ are obtained from I^m by multiplying I^m by different continuing sequences PI^m. They will be the prototypes of through paths of $\boldsymbol{T}_{wt}^m$, which are obtained from W^m by multiplying W^m by PI^m. See lemma 37.

Lemma 39. Let I_a^m, I_b^m be non-final sequences, $I_a^m \in \boldsymbol{I}_a^m$, $I_b^m \in \boldsymbol{I}_b^m$, $m < \propto$. The classes $\boldsymbol{I}_a^m, \boldsymbol{I}_b^m$ are isomorphic, and the one-to one correspondence $I_a^m \times PI_i^m \to I_b^m \times PI_i^m$, when PI_i^m runs over all continuing sequences of automorphisms, determines uniquely a strong isomorphism of $\boldsymbol{I}_a^m, \boldsymbol{I}_b^m$, in which I_a^m goes into I_b^m. The converse is also true: every strong isomorphism is uniquely determined by the choice of non-final I_a^m, I_b^m (and of course, in the general case there can be many such choices).

See lemmas 32, 33 and 38.

Let us introduce in a natural way the order relation for T_n^l, T_n^m: $T_n^l \le T_n^m$ means that $T_n^l = cut\left(T_n^m, l\right)$. The order relation for T_n^l, T_n^m induces an order relation for the corresponding $\boldsymbol{J}^l, \boldsymbol{J}^m$, where $\boldsymbol{J}^l \le \boldsymbol{J}^m$ (for $l \le m$) means that all sequences of places included in the set $\boldsymbol{J}^l$, are obtained from sequences included in $\boldsymbol{J}^m$ using the cutting operation at level *l*: $\boldsymbol{J}^l = cut\left(\boldsymbol{J}^m, l\right)$. Of course, also we have $\boldsymbol{J}^l = cut\left(\boldsymbol{I}^m, l\right)$. In this case lemma 40 is carried out.

Lemma 40. $\boldsymbol{J}^l \le \boldsymbol{J}^m$ if and only if there is a sequence $I^m \in \boldsymbol{J}^m$ for which $cut\left(I^m, l\right) \in \boldsymbol{J}^l$. If $l < m$ and $I^m \in \boldsymbol{I}^m$, then also $cut\left(I^m, l\right) \in \boldsymbol{J}^l$.

The introduced order relations allow us to talk about continuing sequences of sets of non-final sequences of places: $\left(\boldsymbol{J}^k, k < m\right)$ (as well as about continuing sequences $\left(\boldsymbol{I}^k, k < m\right)$). Let us make an addition for the uncovered case, when for the limit m $\boldsymbol{I}^m$ does not exist (in this case $\boldsymbol{I}^{m-0}$ also does not exist), but there is a continuing sequence $\left(\boldsymbol{I}^k, k < m\right)$ that does not have final limits at level *m* (this can happen for $m = \propto$). The sequence $\left(\boldsymbol{I}^k, k < m\right)$ introduces in an obvious way the order relations between places n_i^k and n_j^l and thereby introduces T_n^{m-0}. If $\boldsymbol{I}^m$ does not exist, then $\left(\boldsymbol{I}^k, k < m\right)$ and the corresponding T_n^{m-0} do not have through paths.

Lemma 41. Let $\left(\boldsymbol{I}^k, k < m\right)$ be a continuing sequence, $m < \propto$ a limit ordinal, and $\left(T_n^k, k < m\right)$ the corresponding continuing sequence of place trees. The limit tree T_n^{m-0} has the set $\boldsymbol{I}^{m-0}$ as the set of through paths, which is the limit of the sequence $\left(\boldsymbol{I}^k, k < m\right)$.

Lemma 42. Let there be $\boldsymbol{I}^{m-0}$ and $I^{m-0} = \left(n_{i_k}^k, k < m\right) \in \overline{\boldsymbol{I}}_i^{m-0}$, where $\overline{\boldsymbol{I}}_i^{m-0} \subseteq \boldsymbol{I}^{m-0}$. Let us expand I^{m-0} to $I^m = \left(n_{i_k}^k, k \le m\right)$ with $i_m < \overline{\beta}_m$. I^m defines $\boldsymbol{I}^m$ in which non-final paths are continuations of paths from $\overline{\boldsymbol{I}}_i^{m-0}$.

Lemma 43. Let $\boldsymbol{T}_{wt}^m$ be an isomorphism tree placed on the place plane. The tree $\boldsymbol{T}_{wt}^m$ introduces a continuing sequence $\left(\boldsymbol{I}^k, k < m\right)$, in which each $\boldsymbol{I}^k$ is the set of prototypes of the through paths of the tree $\boldsymbol{T}_{wt}^k = cut\left(\boldsymbol{T}_{wt}^m, k\right)$.

Theorem 4. Among the continuing sequences $\left(\boldsymbol{I}^m, m < \propto\right)$ there are both sequences with through paths and sequences without through paths.

In fact, the former are obtained if we use lemma 43 with the tree $\boldsymbol{T}_{wt}^{m} = \boldsymbol{T}_{wt}^{\propto}$, when the trees T_w and T_t are isomorphic, and the latter—when they are not isomorphic (only almost isomorphic).

Theorem 4 is the final result of this section, reflecting the fact that in the first class of almost through almost homogeneous trees almost isomorphic to each other (with height $\propto = \omega_1$) there are both trees without through paths and trees with through paths.

If it is shown that all continuing sequences $\left(\boldsymbol{I}^m, m < \propto\right)$ have through paths, then the inconsistency of set theory will follow.

6. Disposition of Isomorphism Trees on the Place Plane

It is convenient for us to assume that the mathematical objects under study are placed on the part of a homogeneous place plane (see the beginning of Section 4). In this case, we will distinguish between the concept of "imposition (overlay, placing) an object on the plane of places" and that of "disposition of an object on the plane of places". This will now be the essential point.

Let there be a continuing sequence $\left(\boldsymbol{I}_a^m, m \le \propto\right)$ $\left(\boldsymbol{I}_a^m = \left(I_{ai}^m = I_{a0}^m \times PI_i^m, i < \beta_m\right)\right)$, generated by a through tree $\boldsymbol{T}_{wt,a}$, and the place tree T_{na} corresponds to it. T_{na} is a through tree isomorphic to $\boldsymbol{T}_{wt,a}$. The set $\boldsymbol{I}_a^m$ ($m \le \propto$) is the set of prototypes of through paths in the tree $\boldsymbol{T}_{wt,a}^m = cut\left(\boldsymbol{T}_{wt,a}, m\right)$ and $T_{na}^m = cut\left(T_{na}, m\right)$ (in the latter case $\boldsymbol{I}_a^m$ is also the set of through paths of the tree). It gives a full description of T_{na}^m and can be identified with it. For greater clarity, using the results of Section 8 in [1] (see theorem 3), we introduce a through splitting tree T_S with a root set S (realizing successive partitions of S into disjoint subsets), isomorphic to the tree $\boldsymbol{T}_{wt,a}$, with the isomorphism described in the proof of theorem 3 in [1]. Next, we will consider various impositions of T_S on the plane of places.

The tree $T_S^m = cut\left(T_S, m\right)$ has the sets S_i^k, $i < \beta_k$, as vertices at level k, and for $i < \overline{\beta}_k$ we have non-final vertices when $S_i^k \ne \varnothing$. S_i^k is the final vertex when $S_i^k = \varnothing$. As its through paths, the tree T_S^m contains non-increasing sequences of sets $W^m = \left(S_{i_k}^k, k \le m\right)$, where $S_{i_0}^0 = S$, $S_{i_{k+1}}^{k+1} \subseteq S_{i_k}^k$. At the limit m, if $S_{i_m}^m = \bigcap_{k \le m} S_{i_k}^k = \varnothing$, then the path W^m is final and terminates. The tree T_S^m ($m \le \propto$) under the isomorphism of theorem 3 turns out to be superimposed on T_{na}^m. The set of sequences $\boldsymbol{I}_a^m = \left(I_{ai}^m, i < \beta_m\right)$ is the set of prototypes of the set of through paths $\boldsymbol{W}^m = \left(W_i^m, i < \beta_m\right)$ of the tree T_S^m (I_{ai}^m is the prototype of W_i^m). The following multiplication operation is introduced: $n_0^k \times P_{ti}^k = n_i^k$, $n_i^k \times P_{tj}^k = n_0^k \times \left(P_{ti}^k \times P_{tj}^k\right)$. Also, we have: $S_0^k \times P_{ti}^k = S_i^k$, $S_i^k \times P_{tj}^k = S_0^k \times \left(P_{ti}^k \times P_{tj}^k\right)$. Accordingly, for all k we have the operation of multiplying W_i^k on PI_j^k: $W_0^k \times PI_i^k = W_i^k$, $W_i^k \times PI_j^k = W_0^k \times \left(PI_i^k \times PI_j^k\right)$. So, $W_i^k \times PI_j^k = W_l^k$ if and only if $I_{ai}^k \times PI_j^k = I_{al}^k$.

The set $\boldsymbol{W}^m$ (the set of through paths of T_S^m) adequately represents the tree T_S^m and is identified with it. And the set $\boldsymbol{I}_a^m$ (the set of through paths of T_{na}^m) adequately represents the tree T_{na}^m and is identified with T_{na}^m.

The sequence $\left(\boldsymbol{I}_a^m, m \le \infty\right)$ is a special case of the sequence of sets of related sequences $\left(\boldsymbol{I}^m, m \le \infty\right)$. In general case, the sequence $\left(\boldsymbol{I}^m, m \le \infty\right)$ (and the corresponding place tree T_n^∞) a priory may have no through paths. The set $\boldsymbol{I}^m = \left(I_i^m = I_0^m \times PI_i^m, i < \beta_m\right)$ (when it exists) is the set of through paths (and at the same time the set of prototypes of through paths) for the tree T_n^m.

Under the $\boldsymbol{WI}^m$-overlay of T_S^m ($m \le \infty$) on the plane of places, we mean an isomorphic (taking into account, also, the distinguish between non-final and final sequences) superposition of $\boldsymbol{W}^m = \left(W_i^m, i < \beta_m\right)$ with $\boldsymbol{I}^m = \left(I_j^m, j < \beta_m\right)$. We will say that T_S^m is superimposed on the plane of places according to $\boldsymbol{I}^m$ or that there is an $\boldsymbol{I}^m$-overlay of T_S^m on the plane of places. This overlay is the union of individual overlays $W_{i(j)}^m$ on I_j^m into one set, where the one-to-one function $i(j)$ must ensure the isomorphism of the overlay: $\boldsymbol{WI}^m = \left(\left(W_{i(j)}^m, I_j^m\right), j < \beta_m\right)$. For $i(j) = j$ we have the *basic overlay*. The superposition $\boldsymbol{WI}^m$ represents some isomorphism of T_S^m and T_n^m and can be identified with it. The sequences $\left(W_{i(j)}^m, I_j^m\right)$ will be called *overlay paths*. Overlay paths are *double paths*. Each double path consists of two *subpaths*, of which $W_{i(j)}^m$ is the *first subpath* and I_j^m is *the second* one. We follow here the formalism introduced in Sections 3, 4.

Note that the sets $\left(W_i^m, i < \overline{\beta}_m\right)$ and $\left(I_j^m, j < \overline{\beta}_m\right)$ are classes of strongly related sequences.

By overlay of T_S^m on the plane of places, we mean any one-to-one correspondence between S_i^k and n_j^k, in which the paths of T_S^m turn into paths of T_n^m.

Lemma 44. Any overlay is an $\boldsymbol{WI}^m$-overlay where $\boldsymbol{I}^m$ is some set of prototypes of through paths of T_S^m.

Lemma 45. Let under the $\boldsymbol{WI}^m$-overlay W_i^m be superimposed on I_j^m, where $i, j < \overline{\beta}_m$. Overlays of $W_i^m \times PI_r^m$ on $I_j^m \times PI_r^m$, when PI_r^m runs over all continuing sequences of automorphisms, define all individual path overlays that form the given $\boldsymbol{WI}^m$-overlay. Varying *i* and *j*, we will get all possible $\boldsymbol{WI}^m$-overlays (isomorphisms of $\boldsymbol{W}^m$ and $\boldsymbol{I}^m$).

See lemmas 32, 33 and 37.

The pair $\left(W_i^m, I_j^m\right)$ will be called the *leading pair*. The $\boldsymbol{WI}^m$-overlay is uniquely determined by the choice of the leading pair.

Lemma 46. Let $\left(W_i^m, I_j^m\right)$ be the leading pair of the $\boldsymbol{WI}^m$-overlay. Then every pair $\left(W_i^m \times PI_r^m, I_j^m \times PI_r^m\right)$, where $r < \overline{\beta}_m$, is a leading pair, and every leading pair has such a representation.

If it is necessary to indicate that in the overlay under consideration W_i^m is superimposed on I_j^m, we will use the notation $\boldsymbol{WI}_{ij}^m$.

$\boldsymbol{WI}_{ij}^m = \left(\left(W_i^m \times PI_r^m, I_j^m \times PI_r^m\right), r < \beta_m\right)$. In the overlay $\boldsymbol{WI}_{ij}^m$, $\left(W_i^m, I_j^m\right)$ is the leading pair.

The following lemma is true.

Lemma 47. Let I_j^m be fixed and W_i^m vary (or vice versa). Each $\boldsymbol{WI}^m$-overlay coincides with one of the overlays obtained in this way.

The assertion of the lemma follows from the fact that each $\boldsymbol{WI}^m$-overlay is determined by the choice of a leading pair.

The set of all possible overlays for a given *m* can be described, for example, by the following formula (which represents the set of $\boldsymbol{WI}_{i0}^m$-overlays):

$$\left(W_i^m \times PI_r^m, I_r^m = I_0^m \times PI_r^m\right), r < \beta_m, i < \overline{\beta}_m.$$

In the overlay $\left(W_i^m \times PI_r^m, I_r^m\right), r < \beta_m$, pair $\left(W_i^m, I_0^m\right)$ is the leading pair.

The overlay $\boldsymbol{WI}^m$ continues the overlay $\boldsymbol{WI}^l$ if the double paths of $\boldsymbol{WI}^m$ continue the double paths of $\boldsymbol{WI}^l$.

The notion of $\boldsymbol{WW}^m$-overlay (when the set $\boldsymbol{W}^m$ is automorphically superimposed on itself) is introduced in a similar way and has similar properties. The notation $\boldsymbol{WW}_{ij}^m$ means that this overlay is determined by the overlay of W_i^m on $W_j^m : \left(W_i^m, W_j^m\right)$ is the leading pair. We single out the formula $\left(\left(W_i^m \times PI_r^m, W_r^m\right), r < \beta_m, i < \overline{\beta}_m\right)$. This formula corresponds to the case when $\left(W_i^m, W_0^m\right)$ is the leading pair and covers all possible $\boldsymbol{WW}^m$-overlays. For fixed *i*, this formula describes some particular imposition of $\boldsymbol{W}^m$ on itself, $\left(\left(W_i^m \times PI_r^m, W_r^m\right), r < \beta_m\right)$, and accordingly describes some automorphism of the tree T_S^m.

Lemma 48. Let $\left(W_i^m, W_j^m\right)$ be the leading pair of the $\boldsymbol{WW}^m$-overlay. Then every pair $\left(W_i^m \times PI_r^m, W_j^m \times PI_r^m\right)$, where $r < \overline{\beta}_m$, is a leading pair, and every leading pair has such a representation.

Let us introduce the product of impositions. $\boldsymbol{WW}_a^m \times \boldsymbol{WW}_b^m$ means the following overlay $\boldsymbol{WW}_c^m$. Let $\boldsymbol{WW}_a^m = \boldsymbol{WW}_{ij}^m$ and $\boldsymbol{WW}_b^m = \boldsymbol{WW}_{jl}^m$. Then $\boldsymbol{WW}_c^m = \boldsymbol{WW}_{il}^m$.

Lemma 49. The definition is correct since the choice of any other consistent leading pairs leads to the same result (by virtue of lemma 48).

The introduction of the product operation turns the set $\boldsymbol{WW}^m$ into the group of automorphisms of $\boldsymbol{W}^m$.

The product $\boldsymbol{WW}^m \times \boldsymbol{WI}^m$ is defined in a similar way. This product is a new

$\boldsymbol{WI}^m$-overlay (a new isomorphism of $\boldsymbol{W}^m$ and $\boldsymbol{I}^m$). If in $\boldsymbol{WW}^m$ the pair $\left(W_i^m, W_j^m\right)$ is the leading one and in $\boldsymbol{WI}^m\left(W_j^m, I_l^m\right)$ is the leading pair, then $\left(W_i^m, I_l^m\right)$ will be the leading pair of the new overlay $\boldsymbol{WI}^m$.

The set of all possible $\boldsymbol{WI}^m$-overlays of the tree T_S^m on the place plane can be described by the following formula:
$\left(\boldsymbol{WI}_i^m, i<\bar{\beta}_m\right)=\left(\left(W_i^m \times PI_r^m, I_r^m\right), r<\beta_m, i<\bar{\beta}_m\right)$. The overlay $\boldsymbol{WI}_i^m$ has $\left(W_i^m, I_0^m\right)$ as a leading pair. Note that the above-mentioned set of all possible $\boldsymbol{WI}^m$-overlays can be obtained as follows. Let us take one overlay. Let it be, for example, the basic overlay $\boldsymbol{WI}_{00}^m$ (with the leading pair $\left(W_0^m, I_0^m\right)$). We have $\boldsymbol{WI}_i^m=\boldsymbol{WW}_{i0}^m \times \boldsymbol{WI}_{00}^m (i<\bar{\beta}_m)$. In essence, this means that we make transformations $\boldsymbol{W}^m$ inside the overlay $\boldsymbol{WI}_{00}^m$ using strong automorphisms with the leading pair $\left(W_i^m, W_0^m\right)$, and as a result the leading pair $\left(W_0^m, I_0^m\right)$ is replaced by the pair $\left(W_i^m, I_0^m\right)$ ($i<\bar{\beta}_m$).With different $i<\bar{\beta}_m$, we get all overlays $\boldsymbol{WI}_i^m$.

Let us define the *disposition* of T_S^m on the place plane (the designation is DT_S^m) as the maximum set of overlays when, for every two overlays, the second overlay is obtained from the first one using the automorphism transformation of T_S^m inside the first overlay. It is clear that DT_S^m exists for every $m \leq \propto$.

Lemma 50. Every disposition of T_S^m is a $\boldsymbol{WI}^m$-disposition: the set of all possible overlays when $\boldsymbol{I}^m$ is the set of prototypes of through paths in T_S^m. $\boldsymbol{WI}^m$-disposition exists if $\boldsymbol{I}^m$ exists. So, $\boldsymbol{WI}^m$-disposition exists for every $m<\propto$.

The disposition of T_S^m continues the disposition of T_S^l if the set of prototypes of through paths T_S^m continues the set of prototypes of through paths T_S^l: $\boldsymbol{I}^l=cut\left(\boldsymbol{I}^m, l\right)$. We will say in this case: the continuing sequence $\left(\boldsymbol{I}^m, m<\propto\right)$ determines the continuing sequence of $\boldsymbol{WI}^m$-dispositions of T_S^m on the place plane.

Thus, the disposition of T_S^m is defined as a collection of all possible overlays of T_S^m on the plane of places with a single set of prototypes of through paths. In other words, the disposition of T_S^m is an overlay up to an automorphic transformation that does not change the set of the prototypes of through paths of T_S^m. The disposition of T_S^m is determined uniquely by the set of the prototypes of through paths $\boldsymbol{I}^m$, and any set of related sequences $\boldsymbol{I}^m$ uniquely determines the disposition of T_S^m on the plane of places.

Let us introduce the concept of the disposition of $T_S^{\propto-0}$ on the plane of places. By the disposition of $T_S^{\propto-0}$ on the plane of places we mean the sequence of dispositions $\left(DT_S^m, m<\propto\right)$, when for all $k<l<\propto$ the disposition of T_S^l continues the disposition of T_S^k. In this case, the set of the prototypes of through paths of T_S^k is obtained using the *k*-cutting operation for the prototypes of through paths of T_S^l.

Lemma 51. Every continuing sequence $\left(\boldsymbol{I}^m, m < \propto\right)$ determines the disposition of $T_S^{\propto - 0}$ on the place plane and vice versa. And we have the continuing sequence of dispositions $\left(DT_S^m, m < \propto\right)$ such that each disposition DT_S^m has $\boldsymbol{I}^m$ as the set of prototypes of through paths of T_S^m.

The proof of the inconsistency of set theory will be completed if the following fundamental point is justified.

Fundamental point. The disposition of $T_S^{\propto - 0}$ on the plane of places entails the existence of the disposition of $T_S^{\propto}$ that continues the disposition of $T_S^{\propto - 0}$.

The fundamental point obviously holds in the case of the disposition $T_S^{\propto - 0}$ induced by the sequence $\left(\boldsymbol{I}_a^m, m < \propto\right)$, which has through paths due to our choice. But the entire difference between this case and any other comes down to a different disposition of $T_S^{\propto - 0}$ on the same homogeneous plane of places. Therefore, the fundamental point must be satisfied in the general case.

Let $\left(\boldsymbol{I}^m, m < \propto\right)$ be an arbitrary continuing sequence to which the place tree $T_n^{\propto - 0}$ corresponds. It determines for each $m < \propto$ the disposition of T_S^m on a homogeneous plane of places (in which $\boldsymbol{I}^m$ is the set of prototypes of through paths) and accordingly defines the disposition of $T_S^{\propto - 0}$ on the plane of places. Therefore, the disposition of $T_S^{\propto}$ on the plane of places, that continues $T_S^{\propto - 0}$, is determined. The latter means that an arbitrary continuing sequence $\left(\boldsymbol{I}^m, m < \propto\right)$ has through paths, and we arrive at a contradiction in set theory (see theorem 4 in the end of Section 5).

7. Conclusion

The article presents the chain of statements leading to the conclusion that set theory is inconsistent. The critical point in this chain is the last step, which affirms the possibility of transition from the disposition of $T_S^{\propto - 0}$ on a homogeneous plane of places to the disposition of $T_S^{\propto}$, which continues the disposition of $T_S^{\propto - 0}$. The feasibility of this step can hardly raise doubts (as we are talking about different dispositions of the same mathematical object $T_S^{\propto - 0}$ on a homogeneous plane of places and dispositions at which transition is possible demonstrably exist). But, of course, more subtle arguments are necessary here.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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