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A Technique for Estimation of Box Dimension about Fractional Integral

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Abstract

This paper discusses further the roughness of Riemann-Liouville fractional integral on an arbitrary fractal continuous functions that follows Rfs. [1]. A novel method is used to reach a similar result for an arbitrary fractal function $f(x)$, $\overline{\dim}_B \mathcal{G}(D^{-\nu} f, J) \le \overline{\dim}_B \mathcal{G}(f, J)$, where $D^{-\nu} f(x)$ is the Riemann-Liouville fractional integral. Furthermore, a general result $\dim_B \mathcal{G}(D^{-\nu} f, J) = \dim_B \mathcal{G}(f, J) = 1$ is arrived at for 1-dimensional fractal functions such as with unbounded variation and(or) infinite lengths, which can infer all previous studies such as [2] [3]. This paper's estimation reveals that the fractional integral does not increase the fractal dimension of $f(x)$, *i.e.* fractional integration does not increase at least the fractal roughness. And the result has partly answered the fractal calculus conjecture and completely answered this conjecture for all 1-dimensional fractal function (Xiao has not answered). It is significant with a comparison to the past researches that the box dimension connection between a fractal function and its Riemann-Liouville integral has been carried out only for Weierstrass type and Besicovitch type functions, and at most Hlder continuous. Here the proof technique for Riemann-Liouville fractional integral is possibly of methodology to other fractional integrals.

Keywords

Upper Box Dimension, Riemann-Liouville Fractional Integral, Fractal Continuous Function, Box Dimension

1. Introduction

There are many formulas for fractional integrals and differentiations, since Riemann-Liouville integral of order ν (See [4] [5] [6] [7])

$$D^{-\nu}f(x)=\frac{1}{\Gamma(\nu)}\int_0^x(x-t)^{\nu-1}f(t)\mathrm{d}t,\ 0<\nu<1,\ x\in[0,1] \tag{1.1}$$

and Weyl-Marchaud derivative of order α (See [6] [7] [8])

$$D^{\alpha}f(x)=\frac{\alpha}{\Gamma(1-\alpha)}\int_0^{\infty}\frac{f(x)-f(x-t)}{t^{1+\alpha}}\mathrm{d}t,\ 0<\alpha<1(\text{left-sided}) \tag{1.2}$$

are widely applied in physics such as Rfs. [6] [7] [9] [10] [11] [12] and deeply studied in theory such as Rfs. [5] [6] [13]. Particularly, according to five criteria for definition of the fractional integration and differentiation provided by Ross [12], (1.1) and (1.2) may be one of the best definitions of fractional integrations and differentiations. Hence, the research is restricted to the Riemann-Liouville fractional integral to discuss its box dimension.

For classic calculus, it is well known that the smoothness of a function would increase after integration and decrease after differentiation. For fractal calculus, one believes that similar behaviour holds. So the following conjecture has been put forward, which may be mentioned firstly by Tatom [4] and Zähle and Ziezold [8].

Conjecture 1.1. [1] [4] [8] [14] [15] [16] *Let* $f(x)$ *be a fractal function on* $J=[0,1]$. *Its graph is indicated by* $\dim_B\mathcal{G}(f,J)$, *and the graph of its integral and derivative by* $\mathcal{G}(D^{-\nu}f,J)$ *and* $\mathcal{G}(D^{\alpha}f,J)$ *respectively.*

1) *If* $\dim_B\mathcal{G}(f,J)=1$, *then*

$$\dim_B\mathcal{G}(D^{-\nu}f,J)=\dim_B\mathcal{G}(f,J)=1.$$

2) *If the box dimension does not exist, then*

$$\overline{\dim}_B\mathcal{G}(D^{-\nu}f,J)\le\overline{\dim}_B\mathcal{G}(f,J),\ \overline{\dim}_B\mathcal{G}(D^{\alpha}f,J)\ge\overline{\dim}_B\mathcal{G}(f,J).$$

Furthermore,

$$\overline{\dim}_B\mathcal{G}(D^{-\nu}f,J)\le\overline{\dim}_B\mathcal{G}(f,J)-\nu,\ \overline{\dim}_B\mathcal{G}(D^{\alpha}f,J)\le\overline{\dim}_B\mathcal{G}(f,J)+\alpha.$$

3) *If the box dimension exists, then*

$$\dim_B\mathcal{G}(D^{-\nu}f,J)\le\dim_B\mathcal{G}(f,J),\ \dim_B\mathcal{G}(D^{\alpha}f,J)\ge\dim_B\mathcal{G}(f,J).$$

Furthermore,

$$\dim_B\mathcal{G}(D^{-\nu}f,J)=\dim_B\mathcal{G}(f,J)-\nu,\ \dim_B\mathcal{G}(D^{\alpha}f,J)=\dim_B\mathcal{G}(f,J)+\alpha. \tag{1.3}$$

The best answer is Formula (1.3), which includes descriptions of smoothness about traditional calculus: the smoothness of a function would be decreased one after derivative, and increased one after integration, *i.e.* if $f(x)\in C^n$ (n-th differentiable functions), then $f'(x)\in C^{n-1}$ and $\int f(x)\mathrm{d}x\in C^{n+1}$. Here, take $\nu=\alpha=1$, understand $\dim_B\mathcal{G}(f,J)$ as the reasonable description about smooth of a function $f(x)$. In this view Conjecture 1.1 is believed true.

For this problem, the research works can be classified into three branches according to the box dimension of fractal integrand $\dim_B\mathcal{G}(f,J)=s(1\le s\le 2)$.

1) $s=\dim_B\mathcal{G}(f,J)=1$

The box dimension of Riemann-Liouville fractional integral $\dim_B\mathcal{G}(D^{-\nu}f,J)$

is 1, same as the integrand's box dimension, including the integrand with bounded variation [17] or at most finite unbounded variation points [18]. If 1-box dimension function is continuous with countable unbounded variation points, the box dimension of its Riemann-Liouville fractional integral is still 1 (See [2] [19]). Liang [3] and Liu [20] investigated the relationship of 1-dimensional continuous function $f(x)$ with its Riemann-Liouville integral, and proved that $\dim_B \mathcal{G}(D^{-v} f, J) = 1$ for some special constructed functions. All these discussions show that

$$\dim_B \mathcal{G}(D^{-v} f, J) = \dim_B \mathcal{G}(f, J) = 1,$$

which seems to indicate that Riemann-Liouville integral does not increase the box dimension.

2) $1 < s = \dim_B \mathcal{G}(f, J) < 2$

Since the integrand is of any box dimension $s(1 < s < 2)$, estimation of $\dim_B \mathcal{G}(D^{-v} f, J)$ is sophisticated. Usually Weierstrass type functions and Hlder functions are considered such as in references [21] [22] [23]. No other literature appears about these identities (1.3). For Von Koch curve, Liang [24] proved that

$$\dim_B \mathcal{G}(D^{-v} f, J) \le \dim_B \mathcal{G}(f, J). \tag{1.4}$$

And Liang [14] also proved (1.4) holds when the integrand $f(x)$ is α-Hlder continuous. But Wu [25] reached a better estimation for α-Hlder continuous function:

$$\overline{\dim}_B \mathcal{G}(D^{-v} f, J) \le \overline{\dim}_B \mathcal{G}(f, J) - v, \tag{1.5}$$

under conditions $2 - \alpha > 1$ and $\alpha + v < 1$. Rfs. [26] presented the same estimation of (1.5) for Weyl fractional integral. All these seem to show that Riemann-Liouville integral decreases box dimension linearly.

3) $\dim_B \mathcal{G}(f, J) = s = 2$

It seems that $\dim_B \mathcal{G}(f, J) = 2$ or $\overline{\dim}_B \mathcal{G}(f, J) = 2$ holds only for some special Weierstrass type functions with rapidly growing frequencies like in references [5] [21] [23] [27] [28] [29] (Some researchers called it Besicovitch function).

For Besicovitch function, Liang and Su [15] proved that (1.3) holds, and conjectured that (1.3) holds for any fractal function $f(x)$. All these seem to point out that Tatom's [4] assertion is true for 2-box dimension Weierstrass type functions.

In a word, the relation of fractal dimension between the fractal function and its integro-differentiation has been discussed in past years only for Weierstrass type functions, Besicovitch functions, Hlder continuous functions. However, it is generally believed that (1.3) holds although no theory proof appears. Xiao [1] proved that (1.4) is true for all fractal continuous functions, which is the first discussion the conjecture 1.1 for an arbitrary fractal functions. This paper tries to improve Rfs. [1] by taking a novel method that is completely different from Rfs. [1].

2. Preliminaries

For any set $\Omega \in R^n$, box dimensions are defined as follows.

Definition 2.1. [5] [13] *Let* $\Omega(\neq \varnothing)$ *be any bounded subset of* R^n *and* $N_\delta(\Omega)$ *the smallest number of sets coving* Ω *with diameters at most* δ*. Lower and upper box dimension of* Ω *are defined, respectively*

$$\underline{\dim}_B(\Omega) = \underline{\lim}_{\delta \to 0} \frac{\log N_\delta(\Omega)}{-\log \delta}$$

and

$$\overline{\dim}_B(\Omega) = \overline{\lim}_{\delta \to 0} \frac{\log N_\delta(\Omega)}{-\log \delta}.$$

Similarly, box dimension of Ω *is defined as*

$$\dim_B(\Omega) = \lim_{\delta \to 0} \frac{\log N_\delta(\Omega)}{-\log \delta}.$$

Let C_J be the set of all continuous functions on a closed interval $J = [0,1]$, $\mathcal{G}(f,J)$ the graph of $f(x)$ on *J*. Obviously $\mathcal{G}(f,J) \subset R^2$. The lower box dimension and upper box dimension and Box dimension of the graph of $f(x)$ on *J* denoted by

$$\underline{\dim}_B \mathcal{G}(f,J) \quad \text{and} \quad \overline{\dim}_B \mathcal{G}(f,J) \quad \text{and} \quad \dim_B \mathcal{G}(f,J),$$

respectively, which can be defined by Definition 2.1 with $\Omega = \mathcal{G}(f,J)$.

Write $R_{f,[a,b]}$ for the maximum range of $f(x)$ over $[a,b]$,

$$R_{f[a,b]} = R_f[a,b] = \sup_{a \le x < y \le b} |f(x) - f(y)|.$$

Assume that m is the least integer greater than or equal to δ^{-1}, $N_{f,\delta}$ represents the size of δ-mesh squares intersecting $\mathcal{G}(f,J)$. Note that the interval $J = [0,1]$ is divided into m subintervals with equal width δ. Write

$$\Delta_i = [i\delta, (i+1)\delta], i = 0,1,2,\cdots,m-1.$$

The following conclusion about $N_{f,\delta}$ is directly deduced from Definition 2.1 (or see books [5] [13]).

Lemma 2.2. [5] [13] *Assume* $f(x) \in C_J$, *then*

$$\sum_{i=0}^{m-1} \max\left\{\frac{R_f[i\delta,(i+1)\delta]}{\delta}, 1\right\} \le N_{f,\delta} \le \sum_{i=0}^{m-1}\left\{2 + \frac{R_f[i\delta,(i+1)\delta]}{\delta}\right\}.$$

For box dimension of fractal continuous function, some fundamental conclusions are listed in the following Lemma, interested readers can refer to [3] [5] [13] [16].

Lemma 2.3. *Let* $f(x), g(x) \in C_J$ *be a continuous function with box dimension* $s(1 < s < 2)$, *the following statements obviously hold.*

1) *If* $\dim_B \mathcal{G}(f,J) > \dim_B \mathcal{G}(g,J)$, *then* $\dim_B \mathcal{G}(f+g,J) = \dim_B \mathcal{G}(f,J)$.

2) *For a constant function* $f(x) = c$, $\dim_B \mathcal{G}(c,J) = 1$.

3) $\dim_B \mathcal{G}(f+c,J) = \dim_B \mathcal{G}(f,J)$, *where* $f(x) = c$ *is a constant function.*

4) $1 \le \dim_B \mathcal{G}(f,J) \le 2$, *or* $1 \le \underline{\dim}_B \mathcal{G}(f,J) \le \overline{\dim}_B \mathcal{G}(f,J) \le 2$.

Since $\dim_B \mathcal{G}(f+c,J) = \dim_B \mathcal{G}(f,J)$ by Lemma 2.3, it is reasonable to assume $f(0)=0$ without losing generality. Note that C represents an absolute constant and are possibly different values even at the same line in this paper.

Lemma 2.4. *Assume that* $x, y \in \Delta_i = [i\delta, (i+1)\delta]$ $(i \ge 1)$, *then for any* $1 \le k \le i-1$,

$$\sup_{x,y\in\Delta_i} \int_{\Delta_k} (y-t)^{\nu-1} \left| f(t) - f(x-y+t) \right| \mathrm{d}t \le C\delta^\nu (i-k)^{\nu-1} \left(R_{f\Delta_{k-1}} + R_{f\Delta_k} \right).$$

Proof: Since $k\delta \le t \le (k+1)\delta$ and $i\delta \le x \le y \le (i+1)\delta$, so $(k-1)\delta \le x-y+t \le (k+1)\delta$. Denote $\Lambda = \{(x,y) \mid x, y \in \Delta_i\} \subset R^2$. Now Λ and Δ_k are split into two parts as

$$\Lambda_1 = \left\{(x,y) \mid (k-1)\delta \le x-y+t \le k\delta, \text{ for some } t \in \Delta_k^1 \subset \Delta_k, x, y \in \Delta_i \right\},$$

$$\Lambda_2 = \left\{(x,y) \mid k\delta \le x-y+t \le (k+1)\delta, \text{ for some } t \in \Delta_k^2 \subset \Delta_k, x, y \in \Delta_i \right\},$$

Apparently,

$$\Lambda_1 \cup \Lambda_2 \subseteq \Lambda, \text{ and } \Delta_k^1 \cup \Delta_k^2 \subseteq \Delta_k.$$

Hence, we arrive at

$$\begin{aligned}
&\sup_{x,y\in\Delta_i} \int_{\Delta_k} (y-t)^{\nu-1} \left| f(t) - f(x-y+t) \right| \mathrm{d}t \\
&\le \sup_{x,y\in\Lambda_1} \int_{\Delta_k^1} (y-t)^{\nu-1} \left| f(t) - f(x-y+t) \right| \mathrm{d}t \\
&\quad + \sup_{x,y\in\Lambda_2} \int_{\Delta_k^2} (y-t)^{\nu-1} \left| f(t) - f(x-y+t) \right| \mathrm{d}t \\
&\le \sup_{x,y\in\Delta_1} \int_{\Delta_k^1} (y-t)^{\nu-1} \left| f(t) - f(k\delta) + f(k\delta) - f(x-y+t) \right| \mathrm{d}t \\
&\quad + \sup_{x,y\in\Lambda_2} \int_{\Delta_k^2} (y-t)^{\nu-1} \left| f(t) - f(x-y+t) \right| \mathrm{d}t \\
&\le \left(R_{f,\Delta_k} + R_{f,\Delta_{k-1}} + R_{f,\Delta_k} \right) \int_{\Delta_k} (y-t)^{\nu-1} \mathrm{d}t \\
&\le \frac{2}{\nu} \left(R_{f,\Delta_{k-1}} + R_{f,\Delta_k} \right) \delta^\nu \left[(y-k\delta)^\nu - (y-(k+1)\delta)^\nu \right] \\
&\le \frac{2}{\nu} \left(R_{f,\Delta_{k-1}} + R_{f,\Delta_k} \right) \delta^\nu \left[(i-k+1)^\nu - (i-k-1)^\nu \right] \\
&\le 2\left(R_{f,\Delta_{k-1}} + R_{f,\Delta_k} \right) \delta^\nu \xi^{\nu-1} (\text{where } i-k-1 \le \xi \le i-k+1 \text{ by mean value theorem}) \\
&\le C\left(R_{f,\Delta_{k-1}} + R_{f,\Delta_k} \right) \delta^\nu (i-k)^{\nu-1}.
\end{aligned}$$

Lemma 2.4 is done.

Remarks to Lemma 2.4: This lemma plays key role for the later proof of Theorem 3.1. The technique for treating this kind of integration is possibly of general methodology, which is why we claim that our paper is a new general method to deal with other similar fractional integrals.

3. On All Fractal Continuous Functions

It has been studied in the past days that Box counting dimension of fractional integral on only particular fractal functions such as Weierstrass type, Besicovitch type, and Hlder continuous functions (see references and there in, or our introduction). This paper takes an attempt to estimate the fractal dimension of fractional integral on all fractal continuous functions and arrives at a weaker answer for conjecture 1.1.

Theorem 3.1. *Let* $f(x)\in C_J$ *with* $f(0)=0$, *then*

$$\overline{\dim}_B \mathcal{G}\left(D^{-\nu} f,J\right)\le \overline{\dim}_B \mathcal{G}(f,J).$$

Proof: If we try to estimate $N_{D^{-\nu}f,\delta}$ of Riemann-Liouville fractional integral (1.1), by Lemma 2.2, we need calculate the oscillation of $D^{-\nu}f(x)$ on all subinterval $\Delta_i=\left[i\delta,(i+1)\delta\right](i=0,1,2,\cdots,m-1)$. For any $x,y\in\Delta_i$, suppose that $x\le y$ without loss generality. For convenience to deal with the difference

$$D^{-\nu}f(y)-D^{-\nu}f(x)=\frac{1}{\Gamma(\nu)}\int_0^y (y-\tau)^{\nu-1} f(\tau)\mathrm{d}\tau-\frac{1}{\Gamma(\nu)}\int_0^x (x-\tau)^{\nu-1} f(\tau)\mathrm{d}\tau,$$

Let $x-\tau=y-t$ for the second term above to obtain that

$$\frac{1}{\Gamma(\nu)}\int_0^x (x-\tau)^{\nu-1} f(\tau)\mathrm{d}\tau=\frac{1}{\Gamma(\nu)}\int_{y-x}^y (y-t)^{\nu-1} f(x-y+t)\mathrm{d}t.$$

Note that $x,y\in\Delta_i$ and $x\le y$, so $0\le y-x\le\delta$. We have

$$\begin{aligned}
R_{D^{-\nu}f,\Delta_i} &= \sup_{x,y\in\Delta_i}\left|D^{-\nu}f(y)-D^{-\nu}f(x)\right| \\
&=\frac{1}{\Gamma(\nu)}\sup_{x,y\in\Delta_i}\left|\int_0^y (y-\tau)^{\nu-1} f(\tau)\mathrm{d}\tau-\int_0^x (x-\tau)^{\nu-1} f(\tau)\mathrm{d}\tau\right| \\
&=\frac{1}{\Gamma(\nu)}\sup_{x,y\in\Delta_i}\left|\int_0^y (y-t)^{\nu-1} f(t)\mathrm{d}t-\int_{y-x}^y (y-t)^{\nu-1} f(x-y+t)\mathrm{d}t\right| \\
&\le\frac{1}{\Gamma(\nu)}\sup_{x,y\in\Delta_i}\left(\int_{y-x}^y (y-t)^{\nu-1}\left|f(t)-f(x-y+t)\right|\mathrm{d}t+\int_0^{y-x}(y-t)^{\nu-1}\left|f(t)\right|\mathrm{d}t\right) \\
&\le\frac{1}{\Gamma(\nu)}\sup_{x,y\in\Delta_i}\left(\int_\delta^{i\delta}(y-t)^{\nu-1}\left|f(t)-f(x-y+t)\right|\mathrm{d}t\right. \\
&\quad\left.+\int_{i\delta}^y (y-t)^{\nu-1}\left|f(t)-f(x-y+t)\right|\mathrm{d}t+\int_0^{y-x}(y-t)^{\nu-1}\left|f(t)\right|\mathrm{d}t\right) \\
&\le\frac{1}{\Gamma(\nu)}\left(\sup_{x,y\in\Delta_i}\int_\delta^{i\delta}(y-t)^{\nu-1}\left|f(t)-f(x-y+t)\right|\mathrm{d}t\right. \\
&\quad\left.+\sup_{x,y\in\Delta_i}\int_{i\delta}^y (y-t)^{\nu-1}\left|f(t)-f(x-y+t)\right|\mathrm{d}t+\sup_{y\in\Delta_i}\int_0^\delta (y-t)^{\nu-1}\left|f(t)\right|\mathrm{d}t\right) \\
&=:\frac{1}{\Gamma(\nu)}\left(I_1+I_2+I_3\right).
\end{aligned}$$

From Lemma 2.4,

$$\begin{aligned}
I_1 &= \sup_{x,y\in\Delta_i}\int_\delta^{i\delta}(y-t)^{\nu-1}\left|f(t)-f(x-y+t)\right|\mathrm{d}t \\
&=\sum_{k=0}^{i-1}\sup_{x,y\in\Delta_i}\int_{\Delta_k}(y-t)^{\nu-1}\left(f(t)-f(x-y+t)\right)\mathrm{d}t \\
&\le C\sum_{k=0}^{i-1}\delta^\nu (i-k)^{\nu-1}\left(R_{f\Delta_{k-1}}+R_{f\Delta_k}\right).
\end{aligned}$$

For the calculation I_2, take a substitution $t-i\delta=u$, and note that

$\delta \le x \le y \le (i+1)\delta$, then

$$\begin{aligned} I_2 &= \sup_{x,y\in\Delta_i} \int_{i\delta}^{y} (y-t)^{\nu-1} \left| f(t) - f(x-y+t) \right| \mathrm{d}t \\ &= \sup_{x,y\in\Delta_i} \int_{0}^{y-i\delta} (y-i\delta-u)^{\nu-1} \left| f(i\delta+u) - f(x-y+i\delta+u) \right| \mathrm{d}u \\ &\le R_{f,\Delta_i} \int_0^{y-i\delta} (y-i\delta-u)^{\nu-1} \mathrm{d}u \\ &= \frac{1}{\nu} R_{f,\Delta_i} (y-i\delta)^{\nu} \\ &\le \frac{1}{\nu} \delta^{\nu} R_{f,\Delta_i}. \end{aligned}$$

Take direct calculations for I_3,

$$\begin{aligned} I_3 &= \sup_{y\in\Delta_i} \int_0^{\delta} (y-t)^{\nu-1} \left| f(t) \right| \mathrm{d}t \\ &= \sup_{y\in\Delta_i} \int_0^{\delta} (y-t)^{\nu-1} \left| f(t) - f(0) \right| \mathrm{d}t \\ &\le \frac{1}{\nu} R_{f,\Delta_0} \left(y^{\nu} - (y-\delta)^{\nu} \right) \\ &\le C\delta^{\nu} i^{\nu-1} R_{f,\Delta_0}. \end{aligned}$$

Combination of I_1, I_2, and I_3 leads to

$$R_{D^{-\nu}f,\Delta_i} \le C\delta^{\nu} \sum_{k=0}^{i} (k+1)^{\nu-1} R_{f,\Delta_k}.$$

Finally, we focus on the estimation of $N_{D^{-\nu}f,\delta}$, the size of δ-mesh squares intersecting $\mathcal{G}(D^{-\nu}f,J)$. Note $R_{D^{-\nu}f,\Delta_0}$ can be estimated like I_3. At last, from Lemma 2.2, we obtain that

$$\begin{aligned} N_{D^{-\nu}f,\delta} &\le \sum_{i=0}^{m-1} \left(2 + \delta^{-1} R_{D^{-\nu}f,\Delta_i} \right) \\ &\le C\delta^{\nu-1} \sum_{i=0}^{m-1} \sum_{k=0}^{i} (k+1)^{\nu-1} R_{f,\Delta_k} \\ &\le C\delta^{\nu-1} m^{\nu} \sum_{i=0}^{m-1} R_{f,\Delta_i} \sum_{k=1}^{m} \left(\frac{k}{m} \right)^{\nu-1} \frac{1}{m} \\ &\le C\delta^{\nu-1} \delta^{-\nu} \sum_{i=0}^{m-1} R_{f,\Delta_i} \int_0^1 x^{\nu-1} \mathrm{d}x \\ &\le C\delta^{-1} \sum_{i=0}^{m-1} R_{f,\Delta_i} \\ &\le CN_{f,\delta}. \end{aligned}$$

By Definition 2.1, we reach that

$$\begin{aligned} \overline{\dim}_B \mathcal{G}(D^{-\nu}f,J) &= \overline{\lim_{\delta\to 0}} \frac{\log N_{D^{-\nu}f,\delta}}{-\log\delta} \le \overline{\lim_{\delta\to 0}} \frac{\log (CN_{f,\delta})}{-\log\delta} \\ &= \overline{\lim_{\delta\to 0}} \frac{\log N_{f,\delta}}{-\log\delta} = \overline{\dim}_B \mathcal{G}(f,J), \end{aligned}$$

which completes the proof of Theorem 3.1.

Remarks: Some more statements are listed as follows.

1) Theorem 3.1 makes a positive answer to Conjecture 2.1 in [16] for Riemann-Liouville fractional integral and partly answers to the smoothness of fractional calculus like [4] [7].

2) After our discussion on smoothness of fractional calculus, we insist on the strong relationship of box dimension of fractal continuous function with its Riemann-Liouville fractional integral (this is part of Conjecture 1.1):

$$\overline{\dim}_B \mathcal{G}\left(D^{-\nu} f,[0,1]\right) \leq \overline{\dim}_B \mathcal{G}\left(f,[0,1]\right)-\nu.$$

Furthermore,

$$\dim_B \mathcal{G}\left(D^{-\nu} f,[0,1]\right) = \dim_B \mathcal{G}\left(f,[0,1]\right)-\nu.$$

However, it is beyond our abilities to solve these problems.

For an arbitrary fractal continuous function, Theorem 3.1 shows that a weaker estimation (2) in Conjecture 1.1 reaches. Can the exact quality estimation (3) in Conjecture 1.1 arrive at? We discuss this problem through 1-dimensional fractal continuous function in the sequent section.

4. On All 1-Dimensional Fractal Continuous Functions

There are many works to construct a curve of 1-dimensional fractal continuous function, then to calculate its fractional integral dimension [2] [14] [16] [17] [24] [30] [31]. These works show that a plane curve with unbounded variations and (or) infinite lengths is a really fractals with one fractal dimension. However, Devil's staircase [32] (see **Figure 1**) is regarded as 1-dimensional fracture though it is of bounded variation and finite length, and then is actually an "ordinary curve". Here, we discuss "a real fractal continuous function" with unbounded variation and infinite length.

For a 1-dimensional fractal function, what is its box dimension of the graph of fractional integral? Appeared papers discussed some examples such as Koch curve and constructed curves. Here study Riemann-Liouville fractional integral of all fractal functions with one box dimension.

Theorem 4.2. *Assume* $f(x)\in C_{[0,1]}$ *and* $\dim_B \mathcal{G}\left(f,[0,1]\right)=1$ *such as fractal functions with unbounded variation and/or infinite lengths. Then we derive that*

$$\dim_B \mathcal{G}\left(D^{-\nu} f,[0,1]\right) = \dim_B \mathcal{G}\left(f,[0,1]\right)=1.$$

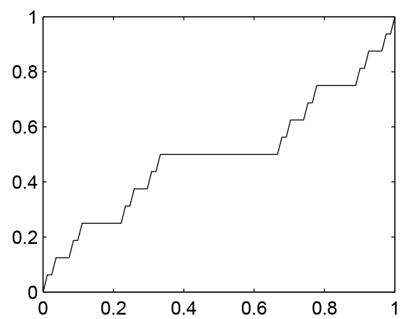

Figure 1. Devil's staircase.

Proof: Combining with Theorem 3.1 and Lemma 2.3 leads to

$$1 \le \underline{\dim}_B \mathcal{G}\left(D^{-\nu} f, [0,1]\right) \le \overline{\dim}_B \mathcal{G}\left(D^{-\nu} f, [0,1]\right)$$
$$\le \overline{\dim}_B \mathcal{G}\left(f, [0,1]\right) = \dim_B \mathcal{G}\left(f, [0,1]\right) = 1,$$

which means $\dim_B \mathcal{G}\left(D^{-\nu} f, [0,1]\right) = 1$. The proof is completed.

Remarks: Theorem 4.2 includes Theorem 3.1 in Rfs. [3], Theorem 1.2 and Theorem 2.2 in Rfs. [14]. Actually, these references discussed only some examples of 1-dimensional fractal functions such as Koch curve and constructing some particular unbounded functions. Theorem 4.2 answers (1) in Conjecture 1.1 completely. The published papers tested only examples of 1-dimensional functions.

5. Conclusions

For a long time, the smoothness of integro-differentiation has been focused. The first idea checking the fractal dimension is to execute numerical simulation other than rigid proof by mathematical theory. Zähle [8] designed a method to calculate the fractional derivative and believed (1.3) is true. And Tatom [4] made a systematic numerical calculation of fractal integration for many fractal Brownian motions and deterministic fractal functions like the Koch curve, and concluded that (1.3) is true. Mathematical proofs of (1.3) or weaker results (1.4) and (1.5) are believed sophisticatedly. Hence, this problem is considered in most cases by special functions like Weierstrass type functions or Besicovitch functions, and recently by Hlder continuous functions. We intend to claim that Conjecture 1.1 holds. This paper considers arbitrary fractal functions to reach a weaker result in comparison to the conjectured results in Rfs [4] [16], which is

$$\overline{\dim}_B \mathcal{G}\left(D^{-\nu} f, J\right) \le \overline{\dim}_B \mathcal{G}\left(f, J\right).$$

However, this estimation indicates at least that the fractional integration does not decrease the smoothness or not increase the fractal dimension for all continuous fractal functions. It is a very interesting conclusion that, for 1-dimensional fractal functions, we proved that $\dim_B \mathcal{G}\left(D^{-\nu} f, [0,1]\right) = \dim_B \mathcal{G}\left(f, [0,1]\right) = 1$, which answers completely the fractal calculus conjecture for one box dimensional fractals. We believe strongly that at least $\overline{\dim}_B \mathcal{G}\left(D^{-\nu} f, J\right) \le \overline{\dim}_B \mathcal{G}\left(f, J\right) - \nu$.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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