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Uniqueness of Viscosity Solutions to the Dirichlet Problem Involving Infinity Laplacian

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Abstract

In this paper, we study the Dirichlet boundary value problem involving the highly degenerate and h-homogeneous quasilinear operator associated with the infinity Laplacian, where the right hand side term is $F(x,t,p) \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ and the boundary value is $\varphi \in C(\partial\Omega)$. First, we establish the comparison principle by the double variables method based on the viscosity solutions theory for the general equation $\Delta_\infty^h u = F(x,u,Du)$ in Ω. We propose two different conditions for the right hand side $F(x,u,Du)$ and get the comparison principle results under different conditions by making different perturbations. Then, we obtain the uniqueness of the viscosity solution to the Dirichlet boundary value problem by the comparison principle. Moreover, we establish the local Lipschitz continuity of the viscosity solution.

Keywords

Infinity Laplacian, Comparison Principle, Uniqueness, Local Lipschitz Continuity

1. Introduction

In this paper, we study the following Dirichlet boundary value problem

$$\begin{cases} \Delta_\infty^h u = F(x,u,Du), & \text{in } \Omega, \\ u = \varphi, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where the domain $\Omega \subseteq \mathbb{R}^n$, the function $\varphi \in C(\partial\Omega)$ and

$$\Delta_\infty^h u := |Du|^{h-3} \langle D^2 u Du, Du \rangle = |Du|^{h-3} \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j}, \; h > 1 \tag{2}$$

is the h-homogeneous quasilinear operator related to the infinity Laplacian.

The choice $h=1$ reduces the operator (2) to the normalized infinity Laplacian

$$\Delta_\infty^N u := |Du|^{-2}\left\langle D^2uDu, Du\right\rangle = |Du|^{-2}\sum_{i,j=1}^{n} u_{x_i}u_{x_j}u_{x_ix_j}. \tag{3}$$

The operator (3) has been investigated extensively, see for papers [1] [2] [3] [4] [5] and the references therein. Peres *et al.* [5] obtained the uniqueness of the viscosity solutions of the following Dirichlet problem corresponding to the normalized infinity Laplacian by a "tug-of-war" game theory

$$\begin{cases} \Delta_\infty^N u = f(x), & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where $f(|f|>0)$ and g are continuous functions. In [3], Lu and Wang established the existence and uniqueness results of the solution to the problem (4) based on the partial differential equation's methods. The normalized infinity Laplacian equations associated with some "tug-of-war" game have attracted much attention. One can see López-Soriano *et al.* [6] and Peres *et al.* [7].

Another operator is the infinity Laplacian

$$\Delta_\infty u := \left\langle D^2uDu, Du\right\rangle = \sum_{i,j=1}^{n} u_{x_i}u_{x_j}u_{x_ix_j},$$

which is the case of $h=3$. The operator Δ_∞ first appeared in Aronsson's studies of the absolutely minimizing Lipschitz extension (AMLE) [8] [9] [10] [11] in the 1960s. For a bounded domain Ω, a function $u \in C(\Omega)$ is said to be an AMLE function in Ω if for any $\Omega' \subset\subset \Omega$ and any $v \in C(\overline{\Omega'})$ with $u=v$ on $\partial\Omega'$, there holds

$$\|u\|_{\mathrm{Lip}(\Omega')} \le \|v\|_{\mathrm{Lip}(\Omega')}.$$

For more details on AMLE, one can refer to Aronsson *et al.* [12].

The infinity Laplacian is quasilinear and highly degenerate, and we usually consider the viscosity solutions of the infinity Laplacian equation which defined by Crandall and Lions [13]. The viscosity solutions to the homogeneous infinity Laplacian equation $\Delta_\infty u = 0$ is said to be the infinity harmonic functions. Jensen [14] proved that the AMLE functions are equivalent to the infinity harmonic functions and proved the existence and uniqueness of AMLE. Crandall *et al.* [15] showed that the infinity harmonic functions, the AMLE functions and the property comparison with cones are equivalent. The property comparison with cones from above (below) is as follows: For any $\Omega' \subset \Omega$, $x_0 \in \Omega$ and any $a, b \in \mathbb{R}$, if

$$u(x) \le (\ge) a + b|x - x_0|, \quad \text{for } x \in \partial(\Omega' \setminus \{x_0\}),$$

then

$$u(x) \le (\ge) a + b|x - x_0|, \quad \text{for } x \in \Omega'.$$

A function u enjoys comparison with cones in Ω' if u enjoys comparison with cones both from above and below. For more results on the infinity Laplacian, one can see [15]-[22] etc.

We also mention that the Dirichlet boundary value problems involving the infinity Laplacian have been studied extensively and the comparison principles have proved to be useful tools in the investigation of existence and uniqueness of solutions to the Dirichlet boundary value problems.

In [23], Lu and Wang proved the comparison principle of the equation

$$\Delta_\infty u = f(x) \tag{5}$$

if the continuous function $f(x)$ has one sign. They also showed the existence and uniqueness of viscosity solutions for (5) under the Dirichlet boundary condition. Bhattacharya and Mohammed [24] proved the comparison principle of the equation

$$\Delta_\infty u = f(x,u)$$

when the continuous function $f(x,t)$ has one sign and is non-decreasing in *t*. They also established the local Lipschitz continuity, existence and nonexistence of viscosity solutions to the corresponding Dirichlet boundary value problem. For the local Lipschitz continuity results, one can also see [25]. Liu and Yang [26] gave the comparison principle of the equation

$$\Delta_\infty^h u = f(x), \quad 1 \le h \le 3 \tag{6}$$

and established the existence and uniqueness results of viscosity solutions of (6) under the Dirichlet boundary condition $u = \varphi$ on $\partial\Omega$, where $\varphi \in C(\bar{\Omega})$. In [27], Li and Liu established the comparison principle of the equation

$$\Delta_\infty^h u = f(x,u), \quad h > 1$$

when the right hand side $f(x,t)$ is non-decreasing in t and has one sign. In addition, it is also necessary to prove the comparison principle during the studies of the Dirichlet eigenvalue problem related to the infinity Laplacian, see for example [28] [29] [30].

In this paper, we study the Dirichlet boundary value problem (1) involving the strongly degenerate operator Δ_∞^h.

Now we state the comparison principle for the equation

$$\Delta_\infty^h u = F(x,u,Du), \quad \text{in } \Omega, \tag{7}$$

where $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is continuous. We propose some basic hypothetical conditions for the right hand side $F(x,t,p)$.

(F-1): $F(x,t,p)$ is positive and the map $\tau \mapsto F(x,t,\tau p)$ is non-increasing in $[1,\rho)$ for each $(x,t,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, where $\rho > 1$.

(F-2): $F(x,t,p)$ is negative and the map $\tau \mapsto F(x,t,\tau p)$ is non-decreasing in $(\rho,1]$ for each $(x,t,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, where $0 < \rho < 1$.

Theorem 1. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. Suppose that the function $F(x,t,p) \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ is non-decreasing in t and satisfies the condition **(F-1)** or **(F-2)**. Assume that $u \in C(\bar{\Omega})$ and $v \in C(\bar{\Omega})$ satisfy

$$\Delta_\infty^h u \ge F(x,u,Du), \quad x \in \Omega$$

and

$$\Delta_\infty^h v \le F(x, v, Dv), \quad x \in \Omega$$

in the viscosity sense, respectively. If $u \le v$ on $\partial\Omega$, then $u \le v$ in Ω.

We prove the comparison principle Theorem 1 based on the double variables method in the viscosity solution theory. Clearly, the result reduces to Li and Liu [27] if the nonhomogeneous term $F(x, u, Du)$ is independent of the gradient *Du*. It is worth pointing out that, unlike the case $h = 1$, the operator Δ_∞^h is quasilinear even in 1-dimension. Thus, we must make more subtle analysis. Due to the strong degeneracy of the operator Δ_∞^h and the dependence of the nonlinear term $F(x, t, p)$ on *p*, we have to perturb twice to make the Jensen's method useful [14] and consider the monotonicity of *F* with respect to the variable *p*.

Our work is divided as follows: In Section 2, we recall the definition of the viscosity solutions. In Section 3, we establish the local Lipschitz continuity of the viscosity solution. Then, we present a proof of the comparison principle for the Equation (7) by the double variables method based on the viscosity solutions theory. Based on the comparison principle, we give the uniqueness theorem of the corresponding Dirichlet problem.

2. Definition of Viscosity Solutions

In this section, we first list some notations that appear in the paper.

$B_r(x)$: the ball of radius r centered at the point x.

$|x|$: the Euclidean norm of x.

$\text{diam}(\Omega)$: the diameter of the domain Ω, that is, the maximum of the distance between all two points in Ω.

$d(x)$: the distance from the point $x \in \Omega$ to the boundary $\partial\Omega$, that is, the minimum of the distance between x and the all points on $\partial\Omega$.

$\text{USC}(\Omega)$ and $\text{LSC}(\Omega)$: for any $\Omega \subset \mathbb{R}^n$,

$$\text{USC}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ is upper semi-continuous}\},$$
$$\text{LSC}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ is lower semi-continuous}\}.$$

I: the $n \times n$ identity matrix.

Now we introduce the definition of viscosity solutions to the Equation (7).

It is worth noting that the operator Δ_∞^h is highly degenerate and singular at the points where the gradient vanishes, one should give a reasonable explanation at these points. Here we adopt the definition of viscosity solutions based on the semi-continuous extension [13] [29] [31]. Hence, one can rewrite the Equation (7) as

$$G_h(D^2u, Du) = F(x, u, Du), \quad x \in \Omega,$$

where $G_h : \mathbb{S} \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}$, $G_h(X, p) := |p|^{h-3}(Xp) \cdot p$ and $\mathbb{S}$ is the set of all $n \times n$ real symmetric matrices. When $h > 1$, we have $\lim_{p \to 0} G_h(X, p) = 0$ for any $X \in \mathbb{S}$. Thus, we can define the following continuous extension of G_h:

$$\bar{G}_h(X, p) = \begin{cases} G_h(X, p), & \text{if } p \ne 0, \\ 0, & \text{if } p = 0. \end{cases}$$

Now we give the definition of viscosity solutions to the Equation (7).

Definition 1. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. We say that $u \in \mathrm{USC}(\Omega)$ is a viscosity subsolution of (7) if and only if for any $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ such that $u(x_0) = \varphi(x_0)$ and $u(x) \le \varphi(x)$ for all $x \in \Omega$ near x_0, there holds

$$\bar{G}_h\left(D^2\varphi(x_0), D\varphi(x_0)\right) \ge F\left(x_0, \varphi(x_0), D\varphi(x_0)\right).$$

Similarly, we say that $u \in \mathrm{LSC}(\Omega)$ is a viscosity supersolution of (7) if and only if for any $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ such that $u(x_0) = \varphi(x_0)$ and $u(x) \ge \varphi(x)$ for all $x \in \Omega$ near x_0, there holds

$$\bar{G}_h\left(D^2\varphi(x_0), D\varphi(x_0)\right) \le F\left(x_0, \varphi(x_0), D\varphi(x_0)\right).$$

If a continuous function u is both a viscosity supersolution and viscosity subsolution of (7), then we say that u is a viscosity solution of (7).

We can define the viscosity subsolutions and viscosity supersolutions equivalently by super-jets and sub-jets [13].

Definition 2. The second-order super-jet of $u \in \mathrm{USC}(\Omega)$ at $x_0 \in \Omega$ is the set

$$\mathcal{J}^{2,+}u(x_0) = \Big\{\left(D\varphi(x_0), D^2\varphi(x_0)\right) : \varphi \in C^2(\Omega) \text{ and } u - \varphi \text{ has a local maximum at } x_0\Big\},$$

and the closure of $\mathcal{J}^{2,+}u(x_0)$ is

$$\bar{\mathcal{J}}^{2,+}u(x_0) := \Big\{(p, X) \in \mathbb{R}^n \times \mathbb{S} : \exists (x_n, p_n, X_n) \in \Omega \times \mathbb{R}^n \times \mathbb{S} \text{ such that } (p_n, X_n) \in \mathcal{J}^{2,+}u(x_0) \text{ and } (x_n, u(x_n), p_n, X_n) \to (x_0, u(x_0), p, X)\Big\}.$$

Similarly, the second-order sub-jet of $u \in \mathrm{LSC}(\Omega)$ at $x_0 \in \Omega$ is the set

$$\mathcal{J}^{2,-}u(x_0) = \Big\{\left(D\varphi(x_0), D^2\varphi(x_0)\right) : \varphi \in C^2(\Omega) \text{ and } u - \varphi \text{ has a local minimum at } x_0\Big\},$$

and the closure of $\mathcal{J}^{2,-}u(x_0)$ is

$$\bar{\mathcal{J}}^{2,-}u(x_0) := \Big\{(p, X) \in \mathbb{R}^n \times \mathbb{S} : \exists (x_n, p_n, X_n) \in \Omega \times \mathbb{R}^n \times \mathbb{S} \text{ such that } (p_n, X_n) \in \mathcal{J}^{2,-}u(x_0) \text{ and } (x_n, u(x_n), p_n, X_n) \to (x_0, u(x_0), p, X)\Big\}.$$

Definition 3. We say that $u \in \mathrm{USC}(\Omega)$ is a viscosity subsolution of (7) if

$$\bar{G}_h(X, p) \ge F\left(x_0, u(x_0), p\right), \quad \forall x_0 \in \Omega, \forall (p, X) \in \bar{\mathcal{J}}^{2,+}u(x_0).$$

Similarly, we say that $u \in \mathrm{LSC}(\Omega)$ is a viscosity supersolution of (7) if

$$\bar{G}_h(X, p) \le F\left(x_0, u(x_0), p\right), \quad \forall x_0 \in \Omega, \forall (p, X) \in \bar{\mathcal{J}}^{2,-}u(x_0).$$

A function $u \in C(\Omega)$ is a viscosity solution of the Equation (7) if u is both a viscosity supersolution and viscosity subsolution of (7).

3. Comparison Principle

In this section, we mainly prove the comparison principle of the Equation (7), which immediately implies the uniqueness theorem.

First, we establish the local Lipschitz continuity of a viscosity solution to $\Delta_\infty^h u = C$, where C is a constant. One can refer to [23] [25] etc. for more regularity results of the infinity Laplacian.

Lemma 2. Let C be a constant. If $u \in C(\Omega) \cap L^\infty(\Omega)$ satisfies $\Delta_\infty^h u \ge C$ in the viscosity sense, then u is locally Lipschitz continuous in Ω. Moreover, for any given $x_0 \in \Omega$, there exists a constant L such that

$$|u(x) - u(y)| \le L|x - y|, \quad \forall x, y \in B_{\frac{d(x_0)}{4}}(x_0),$$

where L depends on $x_0, \operatorname{diam}(\Omega), |C|$ and $\|u\|_{L^\infty(\Omega)}$.

Proof. Set

$$k(x_0) := \frac{4(M - m)}{3d(x_0)} + |C|\operatorname{diam}(\Omega) + 1, \tag{8}$$

where $M := \max_{\Omega} u$ and $m := \min_{\Omega} u$. For any $y \in B_{\frac{d(x_0)}{4}}(x_0)$, we consider the function

$$\psi(w) := u(y) + k|w - y| - \frac{|C|}{2}|w - y|^2, \quad \forall w \in \Omega,$$

where $k := k(x_0)$ is defined in (8). It is clear that $\psi \in C^\infty(\mathbb{R}^n - \{y\})$. For $w \ne y$, it is easy to check that

$$\Delta_\infty^h \psi(w) = -|C|\left(k - |C||w - y|\right)^{h-1}.$$

Since $k \ge 1 + |C|\operatorname{diam}(\Omega)$, we have $\Delta_\infty^h \psi \le C$ in $\Omega \setminus \{y\}$.

Obviously, we have $d(y) \ge \frac{3d(x_0)}{4}$ for any $y \in B_{\frac{d(x_0)}{4}}(x_0)$. For any $w \in \partial B_{d(y)}(y)$, one can verify that

$$\begin{aligned}
\psi(w) &= u(y) + kd(y) - \frac{|C|}{2}d^2(y) \\
&\ge m + \frac{3d(x_0)}{4}\left(k - \frac{|C|}{2}d(y)\right) \\
&\ge m + \frac{3d(x_0)}{4}\left(k - \frac{|C|}{2}\operatorname{diam}(\Omega)\right) \ge M \ge u(w),
\end{aligned}$$

where we have used (8). Thus, $u \le \psi$ on $\partial\left(B_{d(y)}(y) \setminus \{y\}\right)$. Since $\Delta_\infty^h \psi \le C$ and $\Delta_\infty^h u \ge C$ in $B_{d(y)}(y) \setminus \{y\}$, we have $u \le \psi$ in $B_{d(y)}(y)$ by the comparison principle in [27]. Therefore, for any $y \in B_{\frac{d(x_0)}{4}}(x_0)$ and any $z \in B_{d(y)}(y)$, we get

$$u(z) \le u(y) + k|z - y| - \frac{|C|}{2}|z - y|^2. \tag{9}$$

Note that $B_{\frac{d(x_0)}{4}}(x_0) \subseteq B_{d(p)}(p)$ for any $p \in B_{\frac{d(x_0)}{4}}(x_0)$. According to (9), for any $x, y \in B_{\frac{d(x_0)}{4}}(x_0)$, we have

$$u(y) \le u(x) + k|x-y| - \frac{|C|}{2}|x-y|^2$$

and

$$u(x) \le u(y) + k|x-y| - \frac{|C|}{2}|x-y|^2.$$

That is,

$$|u(x)-u(y)| \le \left(k - \frac{|C|}{2}|x-y|\right)|x-y| \le k|x-y|, \quad \forall x, y \in B_{\frac{d(x_0)}{4}}(x_0).$$

Therefore, for any $x_0 \in \Omega$, we have

$$|u(x)-u(y)| \le L|x-y|, \forall x, y \in B_{\frac{d(x_0)}{4}}(x_0),$$

where L depends on $x_0, \operatorname{diam}(\Omega), |C|$ and $\|u\|_{L^\infty(\Omega)}$. □

Remark. Let C be a constant. If $u \in C(\Omega) \cap L^\infty(\Omega)$ satisfies $\Delta_\infty^h u \le C$ in the viscosity sense, then the similar result is also valid.

Next we give the proof of the comparison principle by the double variables method based on the viscosity solutions theory.

Proof of Theorem 1. Suppose that $F(x,t,p)$ satisfies the condition **(F-1)**. Define

$$u_\varepsilon := u(1+\varepsilon) - \varepsilon \sup_{\bar{\Omega}} u, \quad 0 < \varepsilon < \rho - 1.$$

Since $F(x,t,p)$ is non-decreasing in t and satisfies the condition **(F-1)**, we have

$$\begin{aligned} \Delta_\infty^h u_\varepsilon &= (1+\varepsilon)^h \Delta_\infty^h u \\ &\ge (1+\varepsilon)^h F(x,u,Du) \\ &\ge (1+\varepsilon)^h F(x,u_\varepsilon,Du_\varepsilon) \\ &> F(x,u_\varepsilon,Du_\varepsilon) \end{aligned}$$

in the viscosity sense. That is, u_ε is a viscosity subsolution of the Equation (7).

Next we want to show $u_\varepsilon \le v$ in Ω when $F > 0$. Instead, suppose that $u_\varepsilon > v$ at some point $x_0 \in \Omega$ and

$$M = \sup_{\Omega}(u_\varepsilon - v) = u_\varepsilon(x_0) - v(x_0) > 0.$$

According to [13], we double the variables

$$z_j(x,y) := u_\varepsilon(x) - v(y) - \frac{j}{4}|x-y|^4, \quad (x,y) \in \Omega \times \Omega, \ j = 1, 2, \cdots.$$

Let z_j attain its maximum at $(x_j, y_j) \in \bar{\Omega} \times \bar{\Omega}$. According to ([13], Proposition 3.7), we obtain

$$\lim_{j\to\infty} M_j = \lim_{j\to\infty}\left(u_\varepsilon(x_j) - v(y_j) - \frac{j|x_j - y_j|^4}{4}\right) = M$$

and

$$\lim_{j\to\infty}\frac{j|x_j - y_j|^4}{4} = 0.$$

Clearly, we have $x_j \to x_0$, $y_j \to x_0$ as $j\to\infty$. Due to $M > 0 \geq \sup_{\partial\Omega}(u_\varepsilon - v)$, there exists an open set Ω_0 such that x_0, x_j and $y_j \in \Omega_0 \subseteq \Omega$ as $j \to \infty$.

Set

$$\varphi(x) = \frac{j|x - y_j|^4}{4} \quad \text{and} \quad \psi(y) = -\frac{j|x_j - y|^4}{4}.$$

Note that the function $u_\varepsilon - \varphi$ has a local maximum at x_j and $v - \psi$ has a local minimum at y_j.

We discuss the following two cases: either $x_j = y_j$ or $x_j \neq y_j$ for j large enough.

Case 1: When $x_j = y_j$, we have $D\varphi(x_j) = 0$ and $D^2\varphi(x_j) = 0$. Since u_ε is a viscosity subsolution, we get

$$F(x_j, \varphi(x_j), D\varphi(x_j)) = F(x_j, u_\varepsilon(x_j), D\varphi(x_j)) \leq 0.$$

It is contrary to $F > 0$.

Case 2: When $x_j \neq y_j$, we apply the jets and maximum principle for semi-continuous functions ([13], Theorem 3.2). There exist $n\times n$ symmetric matrices X_j and Y_j such that

$$(p_j, X_j) \in \bar{\mathcal{J}}^{2,+}u_\varepsilon(x_j), \quad (p_j, Y_j) \in \bar{\mathcal{J}}^{2,-}v(y_j)$$

and

$$-\frac{3}{\varepsilon}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X_j & 0 \\ 0 & -Y_j \end{pmatrix} \leq \frac{3}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \tag{10}$$

where $p_j = j|x_j - y_j|^2(x_j - y_j)$. Following from the inequality (10), we have $X_j \leq Y_j$. Since $\Delta_\infty^h u_\varepsilon \geq (1+\varepsilon)^h F(x, u_\varepsilon, Du_\varepsilon)$ and $\Delta_\infty^h v \leq F(x, v, Dv)$ in the viscosity sense, by the definition of the viscosity subsolution and supersolution, we obtain

$$\begin{aligned} 0 &\leq |p_j|^{h-3}\langle X_j p_j, p_j\rangle - (1+\varepsilon)^h F(x_j, u_\varepsilon(x_j), p_j) \\ &\leq |p_j|^{h-3}\langle Y_j p_j, p_j\rangle - (1+\varepsilon)^h F(x_j, u_\varepsilon(x_j), p_j) \\ &\leq F(y_j, v(y_j), p_j) - (1+\varepsilon)^h F(x_j, u_\varepsilon(x_j), p_j), \end{aligned} \tag{11}$$

where we have used $X_j \leq Y_j$. Due to z_j attains its maximum at $(x_j, y_j) \in \bar{\Omega}\times\bar{\Omega}$, we get

$$u_\varepsilon(x) - v(y) - \frac{j}{4}|x - y|^4 \leq u_\varepsilon(x_j) - v(y_j) - \frac{j|x_j - y_j|^4}{4}, \quad \forall x, y \in \bar{\Omega}. \tag{12}$$

Since u_ε is a viscosity subsolution, we see that u_ε is locally Lipschitz continuous according to Lemma 2. We take $x = y = y_j$ in (12) and obtain

$$\frac{j\left|x_j - y_j\right|^4}{4} \le u_\varepsilon\left(x_j\right) - u_\varepsilon\left(y_j\right) \le L\left|x_j - y_j\right|,$$

where L is the Lipschitz constant of u_ε. Then we have

$$\frac{j\left|x_j - y_j\right|^3}{4} \le L.$$

Therefore, upon taking a subsequence if necessary, we can assume $p_j \to p$. Taking the limit in (11), we get

$$F\left(x_0, v\left(x_0\right), p\right) - \left(1+\varepsilon\right)^h F\left(x_0, u_\varepsilon\left(x_0\right), p\right) \ge 0.$$

Thus,

$$F\left(x_0, v\left(x_0\right), p\right) \ge \left(1+\varepsilon\right)^h F\left(x_0, u_\varepsilon\left(x_0\right), p\right) > F\left(x_0, u_\varepsilon\left(x_0\right), p\right). \tag{13}$$

Since $F\left(x, t, p\right)$ is non-decreasing in t and $u_\varepsilon\left(x_0\right) > v\left(x_0\right)$, we obtain

$$F\left(x_0, u_\varepsilon\left(x_0\right), p\right) \ge F\left(x_0, v\left(x_0\right), p\right).$$

It is a contradiction to (13).

Thus, we have $u_\varepsilon \le v$ in Ω when $F > 0$. Letting $\varepsilon \to 0$, we have $u \le v$ in Ω.

Now suppose that $F\left(x, t, p\right)$ satisfies the condition **(F-2)**. Define

$$u_\varepsilon := u\left(1-\varepsilon\right) + \varepsilon \inf_{\Omega} u, \quad 0 < \varepsilon < 1-\rho.$$

Since $F\left(x, t, p\right)$ is non-decreasing in t and satisfies the condition **(F-2)**, one has

$$\begin{aligned} \Delta_\infty^h u_\varepsilon &= \left(1-\varepsilon\right)^h \Delta_\infty^h u \\ &\ge \left(1-\varepsilon\right)^h F\left(x, u, Du\right) \\ &\ge \left(1-\varepsilon\right)^h F\left(x, u_\varepsilon, Du_\varepsilon\right) \\ &\ge F\left(x, u_\varepsilon, Du_\varepsilon\right) \end{aligned}$$

in the viscosity sense. Thus, u_ε is a viscosity subsolution of the Equation (7). Then one can prove that $u_\varepsilon \le v$ in Ω by the similar procedure. We leave it to the reader. □

With the comparison principle in hand, the uniqueness theorem of the corresponding Dirichlet problem follows immediately.

Theorem 3. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. If the function $F\left(x, t, p\right) \in C\left(\Omega \times \mathbb{R} \times \mathbb{R}^n\right)$ is non-decreasing in t and satisfies the condition **(F-1)** or **(F-2)**, then there exists at most one viscosity solution to the Dirichlet problem (1).

When the right side hand $F\left(x, t, p\right)$ is independent of the variables t and p, Lu and Wang [23] constructed a counterexample to show that the uniqueness is invalid if F changes its sign. And the case $F = 0$ is covered by Jensen's theorem [14]. But for the case $F \ge (\le) 0$, the uniqueness is open.

Remark. If $\varphi = 0$ and $F < 0$ in the problem (1), then the viscosity solution to the problem (1) is positive. Similarly, if $\varphi = 0$ and $F > 0$ in the problem (1), then the viscosity solution to the problem (1) is negative.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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