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Some New Transformation Formulas for *q*-Series through the Bailey Transform

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Abstract

In the literature, the Bailey transform has many applications in basic hypergeometric series. In this paper, we derive many new transformation formulas for *q*-series by means of the Bailey transform. Meanwhile, We also obtain some new terminated identities. Furthermore, we establish a companion identity to the Rogers-Ramanujan identity labelled by number (23) on Slater's list.

Keywords

q-Series, Bailey Transform, Transformation Formulas, Rogers-Ramanujan Identities

1. Introduction

Throughout this paper, a, x and q are complex number with $|q|<1$. Here and in what follows, we adopt the standard q-series notation [1]. For any positive integer n,

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{k=0}^{n-1}\left(1-aq^k\right), \quad (a;q)_\infty := \prod_{k=0}^{\infty}\left(1-aq^k\right),$$

$$(a_1,a_2,a_3,\cdots,a_m;q)_n := (a_1;q)_n (a_2;q)_n (a_3;q)_n \cdots (a_m;q)_n,$$

$$(a_1,a_2,a_3,\cdots,a_m;q)_\infty := (a_1;q)_\infty (a_2;q)_\infty (a_3;q)_\infty \cdots (a_m;q)_\infty.$$

For convenience, we use $(a)_n$ to denote $(a;q)_n$. We will often use basic properties without reference, such as

$$(a;q)_{n+k} = (a;q)_n \left(aq^n;q\right)_k, \quad (a;q)_\infty = (a;q)_n \left(aq^n;q\right)_\infty.$$

The q-binomial coefficient is given for any nonnegative integers M and N by

$$\begin{bmatrix} N \\ M \end{bmatrix}_q := \begin{bmatrix} N \\ M \end{bmatrix} = \begin{cases} \dfrac{(q;q)_N}{(q;q)_M (q;q)_{N-M}}, & \text{if } N \geq M, \\ 0, & \text{otherwise.} \end{cases}$$

Following Gasper and Rahman [1], the bilateral basic hypergeometric series is defined by

$${}_r\psi_s\left[\begin{matrix} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{matrix}; q, z\right] = \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, \cdots, a_r; q)_n}{(b_1, b_2, \cdots, b_s; q)_n} \left\{(-1)^n q^{\binom{n}{2}}\right\}^{s-r} z^n,$$

and the unilateral basic hypergeometric series is defined by

$${}_r\phi_s\left[\begin{matrix} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{matrix}; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \cdots, a_r; q)_n}{(q, b_1, b_2, \cdots, b_s; q)_n} \left\{(-1)^n q^{\binom{n}{2}}\right\}^{1+s-r} z^n.$$

The Ramanujan's ${}_1\psi_1$ sum ([1], Appendix (II. 29)), and the sum of a ${}_1\phi_1$ series, ([1], Appendix (II.5)) are stated as follows.

$${}_1\psi_1\left[\begin{matrix} a \\ b \end{matrix}; q, z\right] = \frac{(q, b/a, az, q/az; q)_\infty}{(b, q/a, z, b/az; q)_\infty}, \tag{1.1}$$

$${}_1\phi_1\left[\begin{matrix} a \\ c \end{matrix}; q, \frac{c}{a}\right] = \frac{(c/a; q)_\infty}{(c; q)_\infty}. \tag{1.2}$$

Among the other formulas needed for this paper, we have separately documented the q-binomial formula and its consequence ([1] (1.3.2)) in the sequel,

$$\sum_{n\geq 0} \frac{(a;q)_n}{(q;q)_n} x^n = \frac{(ax;q)_\infty}{(x;q)_\infty}. \tag{1.3}$$

Setting $q \to q^2, a \to \infty, x \to xq/a$ in (1.3), we have

$$\sum_{n\geq 0} \frac{(-1)^n q^{n^2} x^n}{(q^2;q^2)_n} = (xq;q^2)_\infty. \tag{1.4}$$

Furthermore, Euler's formulas (cf. [1], Corollary 2.2)

$$\sum_{n\geq 0} \frac{q^{\binom{n}{2}} x^n}{(q;q)_n} = (-x,q)_\infty \quad \text{and} \quad \sum_{n\geq 0} \frac{x^n}{(q;q)_n} = \frac{1}{(x,q)_\infty}. \tag{1.5}$$

Cauchy's identity (cf. [2], Theorem 3.3)

$$(x;q)_n = \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} x^k, \tag{1.6}$$

and the formula ([1], Exercise 1.16) which is a special case of the Bailey-Daum sum ([1], (1.8.1))

$$\sum_{n\geq 0} \frac{(x;q)_n q^{\binom{n+1}{2}}}{(q;q)_n} = (-q;q)_\infty (xq;q^2)_\infty. \tag{1.7}$$

Besides, if $a \neq 0, 1 - bq^n \neq 0$, the following identity was given by Ramanujan

[3],

$$\sum_{n\geq 0}\frac{(-b/a;q)_n a^n q^{n(n+1)/2}}{(q;q)_n (bq;q)_n}=\frac{(-aq;q)_\infty}{(bq;q)_\infty}. \quad (1.8)$$

A pair of sequences (α_n,β_n) is called a Bailey pair relative to *a* if

$$\beta_n=\sum_{r=0}^{n}\frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}}. \quad (1.9)$$

And a conjugate Bailey pair relative to a is a pair of sequences (δ_n,γ_n) satisfying

$$\gamma_n=\sum_{k=n}^{\infty}\frac{\delta_k}{(q;q)_{k-n}(aq;q)_{k+n}}. \quad (1.10)$$

In fact, the Bailey pair and the conjugate Bailey pair are the special cases of the following Bailey transform.

Lemma 1.1 ([4]). *Let n be a nonnegative integer and* $\{A_n\}_{n=0}^{\infty},\{B_n\}_{n=0}^{\infty},\{C_n\}_{n=0}^{\infty},\{D_n\}_{n=0}^{\infty}$ *be sequences of complex numbers. Assuming convergence of the series, if*

$$B_n=\sum_{j=0}^{n}A_jU_{n-j}V_{n+j} \quad \text{and} \quad C_n=\sum_{j=n}^{\infty}D_jU_{j-n}V_{j+n}, \quad (1.11)$$

then

$$\sum_{n=0}^{\infty}A_nC_n=\sum_{n=0}^{\infty}B_nD_n.$$

This terminology was first proposed by Slater [5]. Bailey transform is widely used in mathematics for a long time, especially, in the area of basic hypergeometric series. For example, Andrews [6], Kim and Lovejoy [7], and Lovejoy [8] established multiple sums Rogers-Ramanujan type identities and partial theta identities. Andrews and Warnaar [9] applied the Bailey transform to give another proof of false theta functions. Bailey [4] [10], Bressoud [2], and Slater [11] [12] used this transform to derive a number of identities of Rogers-Ramanujan type identities. Ji and Zhao [13] established the Hecke-Rogers identities for the universal mock theta functions by means of the Bailey transform.

Notice that if we take

$$U_n=\frac{1}{(q;q)_n},V_n=\frac{1}{(aq;q)_n}$$

in lemma 1.1, the pair of sequences (A_n,B_n) is a Bailey pair relative to a, and the pair of sequences (C_n,D_n) is a conjugate Bailey pair relative to a.

The main motivation for this work came from some Bailey's transform of M. E. Bachraoui which appear in [14], specifically the (V_n) is the constant sequences with value 1. Applying the ([14], Theorem 3, 4, 6) and choosing appropriate D_n, we establish some new transformation formulas for q-series.

2. Main Results

Theorem 2.1. *We have*

$$\left(aq^{\frac{1}{2}};q\right)_{\infty}\sum_{n\geq 0}\frac{\left(aq^{\frac{1}{2}}\right)^{n}\left(1-q^{n}\right)\left(x;q\right)_{n}}{\left(q;q\right)_{n}^{2}}=\sum_{n\geq 0}\frac{\left(-1\right)^{n}\left(ax\right)^{n}q^{\frac{n^{2}}{2}}\left(1-q^{n}\right)\left(x^{-1};q\right)_{n}}{\left(q;q\right)_{n}^{2}}. \quad (2.1)$$

Corollary 2.2. *There holds*

$$\frac{x^{n}\left(1-q^{n}\right)\left(x^{-1};q\right)_{n}}{\left(q;q\right)_{n}}=\sum_{k=0}^{n}\frac{\left(-1\right)^{k}q^{\binom{k+1}{2}-nk}\left(1-q^{k}\right)\left(x;q\right)_{k}}{\left(q;q\right)_{k}}\begin{bmatrix}n\\k\end{bmatrix}, \quad (2.2)$$

$$\left(q;q\right)_{\infty}\sum_{n\geq 1}\frac{q^{n}\left(-q;q\right)_{n-1}}{\left(q;q\right)_{n}\left(q;q\right)_{n-1}}=\sum_{n\geq 1}\frac{q^{\binom{n+1}{2}}\left(-q;q\right)_{n-1}}{\left(q;q\right)_{n}\left(q;q\right)_{n-1}}, \quad (2.3)$$

$$\left(-1;q\right)_{\infty}\sum_{n\geq 1}\frac{\left(-1\right)^{n}\left(-q;q\right)_{n-1}}{\left(q;q\right)_{n}\left(q;q\right)_{n-1}}=\sum_{n\geq 1}\frac{\left(-1\right)^{n}q^{\binom{n}{2}}\left(-q;q\right)_{n-1}}{\left(q;q\right)_{n}\left(q;q\right)_{n-1}}. \quad (2.4)$$

Theorem 2.3. *We have*

$$\frac{\left(c/a;q\right)_{\infty}}{\left(c;q\right)_{\infty}}\sum_{n\geq 0}\frac{\left(c/a\right)^{n}\left(1-q^{n}\right)\left(x,a;q\right)_{n}}{\left(q;q\right)_{n}^{2}}=\sum_{n\geq 0}\frac{\left(-1\right)^{n}\left(cx/a\right)^{n}q^{\binom{n}{2}}\left(1-q^{n}\right)\left(x^{-1},a;q\right)_{n}}{\left(q;q\right)_{n}^{2}\left(c;q\right)_{n}}. \quad (2.5)$$

Corollary 2.4. *There holds*

$$\frac{1}{\left(c;q\right)_{\infty}}\sum_{n\geq 0}\frac{\left(-1\right)^{n}c^{n}q^{\binom{n}{2}}\left(1-q^{n}\right)\left(x;q\right)_{n}}{\left(q;q\right)_{n}^{2}}=\sum_{n\geq 0}\frac{\left(cx\right)^{n}q^{n(n-1)}\left(1-q^{n}\right)\left(x^{-1};q\right)_{n}}{\left(q;q\right)_{n}^{2}\left(c;q\right)_{n}}, \quad (2.6)$$

$$\sum_{n\geq 0}\frac{q^{\binom{n}{2}}\left(1-q^{n}\right)}{\left(q;q\right)_{n}^{2}\left(-q^{n};q\right)_{\infty}}=\sum_{n\geq 0}\frac{q^{n(n-1)}\left(1-q^{n}\right)}{\left(q;q\right)_{n}^{2}}. \quad (2.7)$$

Theorem 2.5. *We have*

$$\left(aq^{\frac{1}{2}};q\right)_{\infty}\sum_{n\geq 0}\frac{\left(aq^{\frac{1}{2}}\right)^{n}\left(x;q\right)_{n}}{\left(q;q\right)_{n}}=\sum_{n\geq 0}\frac{\left(-1\right)^{n}\left(ax\right)^{n}q^{\frac{n^{2}}{2}}}{\left(q;q\right)_{n}}. \quad (2.8)$$

Theorem 2.6. *There holds*

$$\frac{\left(c/a;q\right)_{\infty}}{\left(c;q\right)_{\infty}}\sum_{n\geq 0}\frac{\left(c/a\right)^{n}\left(x,a;q\right)_{n}}{\left(q;q\right)_{n}}=\sum_{n\geq 0}\frac{\left(-1\right)^{n}\left(cx/a\right)^{n}q^{\binom{n}{2}}\left(a;q\right)_{n}}{\left(q,c;q\right)_{n}}. \quad (2.9)$$

Corollary 2.7. *We have*

$$\sum_{n\geq 0}\frac{\left(c/a\right)^{n}\left(aq^{k};q\right)_{n}}{\left(q;q\right)_{n}}=\frac{\left(cq^{k};q\right)_{\infty}}{\left(c/a;q\right)_{\infty}}, \quad (2.10)$$

$$\frac{1}{\left(c;q\right)_{\infty}}\sum_{n\geq 0}\frac{\left(-1\right)^{n}c^{n}q^{\binom{n}{2}}\left(x;q\right)_{n}}{\left(q;q\right)_{n}}=\sum_{n\geq 0}\frac{\left(cx\right)^{n}q^{n(n-1)}}{\left(c,q;q\right)_{n}}. \quad (2.11)$$

Remark. Setting $c=-q, x=q$ in (2.11), we derive

$$\frac{1}{(-q;q)_\infty}\sum_{n=0}^{\infty} q^{\binom{n+1}{2}} = \sum_{n\ge 0}\frac{(-1)^n q^{n^2+n}}{(q^2;q^2)_n}.$$

Furthermore, we have

$$\sum_{n=0}^{\infty} q^{\binom{n+1}{2}} = (q^2;q^2)_\infty (-q;q)_\infty .$$

Combining the above two identities, we arrive at

$$\sum_{n\ge 0}\frac{(-1)^n q^{n^2+n}}{(q^2;q^2)_n} = (q^2;q^2)_\infty .$$

The above identity is a companion to number (23) on Slater's list [12] as follows.

$$\sum_{n\ge 0}\frac{(-1)^n q^{n^2}}{(q^2;q^2)_n} = (q;q^2)_\infty .$$

Theorem 2.8. *We have*

$$(-q^2;q^2)_\infty \sum_{n\ge 0}\frac{q^{n^2}(x;q^2)_n (xq^{2n+2};q^4)_\infty}{(q^2;q^2)_n} = \sum_{n\ge 0}\frac{q^{\binom{n+1}{2}}(x;q^2)_n}{(q;q)_n}. \tag{2.12}$$

Corollary 2.9. *There holds*

$$(-q^2;q^2)_\infty \sum_{n\ge 0} q^{n^2}(q^{2n+4};q^4)_\infty = \sum_{n\ge 0} q^{\binom{n+1}{2}}(-q;q)_n , \tag{2.13}$$

$$(-q^2;q^2)_\infty \sum_{n\ge 0}\frac{q^{n^2}(q;q^2)_n (q^{2n+3};q^4)_\infty}{(q^2;q^2)_n} = \sum_{n\ge 0}\frac{q^{\binom{n+1}{2}}(q;q^2)_n}{(q;q)_n}. \tag{2.14}$$

Theorem 2.10. *We have*

$$\frac{(-aq^3;q^2)_\infty}{(bq^2;q^2)_\infty}\sum_{n\ge 0}\frac{a^n q^{n^2+n}\left(-\frac{b}{aq};q^2\right)_n}{(q^2;q^2)_n} = \sum_{n\ge 0}\frac{a^n q^{\binom{n+2}{2}-1}\left(-\frac{b}{aq};q^2\right)_n}{(q;q)_n (bq^2;q^2)_n}. \tag{2.15}$$

3. Proofs of Theorem 2.1 and Corollary 2.2

Proof of Theorem 2.1. Setting

$$A_n = \frac{(-1)^n q^{\frac{n}{2}}(1-q^n)(x;q)_n}{(q;q)_n^2}, \quad U_n = \frac{q^{\frac{n^2}{2}}}{(q;q)_n},$$

$$B_n = \frac{x^n q^{\frac{n^2}{2}}(1-q^n)(x^{-1};q)_n}{(q;q)_n^2}, \quad D_n = (-1)^n a^n$$

and $V_n = 1$ in Lemma 1.1, we obtain

$$C_n=\sum_{k=n}^{\infty}D_kU_{k-n}V_{k+n}=\sum_{k\ge0}D_{n+k}U_k=\sum_{k\ge0}(-a)^{n+k}\frac{q^{\frac{k^2}{2}}}{(q;q)_k}$$
$$=(-a)^n\sum_{k\ge0}\frac{(-1)^k a^k q^{\frac{k^2}{2}}}{(q;q)_k}=(-1)^n a^n\left(aq^{\frac{1}{2}};q\right)_\infty, \tag{3.1}$$

where the last step follows by (1.4).

Thus,

$$\sum_{n\ge0}A_nC_n=\left(aq^{\frac{1}{2}};q\right)_\infty\sum_{n\ge0}\frac{\left(aq^{\frac{1}{2}}\right)^n(1-q^n)(x;q)_n}{(q;q)_n^2}$$
$$=\sum_{n\ge0}B_nD_n=\sum_{n\ge0}\frac{(-1)^n(ax)^n q^{\frac{n^2}{2}}(1-q^n)(x^{-1};q)_n}{(q;q)_n^2}.$$

This completes the proof. □

Proof of Corollary 2.2. Based on (1.4), we can rewrite (2.1) as follows

$$\sum_{m\ge0}\frac{(-1)^m q^{\frac{m^2}{2}}a^m}{(q;q)_m}\sum_{k\ge0}\frac{q^{\frac{k}{2}}(1-q^k)(x;q)_k a^k}{(q;q)_k^2}=\sum_{n\ge0}\frac{(-1)^n x^n q^{\frac{n^2}{2}}(1-q^n)(x^{-1};q)_n a^n}{(q;q)_n^2}.$$

Equating terms of the corresponding powers of a^n, we achieve

$$\sum_{k=0}^{n}\frac{(-1)^{n-k}q^{\frac{(n-k)^2}{2}}}{(q;q)_{n-k}}\cdot\frac{q^{\frac{k}{2}}(1-q^k)(x;q)_k}{(q;q)_k^2}=\frac{(-1)^n x^n q^{\frac{n^2}{2}}(1-q^n)(x^{-1};q)_n}{(q;q)_n^2},$$

which is (2.2) by some basic simplifications. (2.3) (2.4) follow from (2.1) upon letting $a=q^{\frac{1}{2}},x=-1$ ($a=-q^{-\frac{1}{2}},x=-1$), respectively. □

4. Proofs of Theorem 2.3 and Corollary 2.4

Proof of Theorem 2.3. We now apply Lemma 1.1 with

$$A_n=\frac{(-1)^n q^{\frac{n}{2}}(1-q^n)(x;q)_n}{(q;q)_n^2},$$
$$U_n=\frac{q^{\frac{n^2}{2}}}{(q;q)_n},$$
$$B_n=\frac{x^n q^{\frac{n^2}{2}}(1-q^n)(x^{-1};q)_n}{(q;q)_n^2},$$
$$D_n=(-1)^n(c/a)^n q^{-\frac{n}{2}}\frac{(a;q)_n}{(c;q)_n},$$

and $V_n=1$. We compute

$$\begin{aligned}C_n &= \sum_{k\ge 0}\frac{(a;q)_{n+k}}{(c;q)_{n+k}}(-1)^{n+k}(c/a)^{n+k}q^{-\frac{n+k}{2}}\cdot\frac{q^{\frac{k^2}{2}}}{(q;q)_k}\\ &=(-1)^n(c/a)^n q^{-\frac{n}{2}}\frac{(a;q)_n}{(c;q)_n}\sum_{k\ge 0}\frac{(aq^n;q)_k}{(cq^n,q;q)_k}(-1)^k q^{\binom{k}{2}}(c/a)^k \\ &=(-1)^n(c/a)^n q^{-\frac{n}{2}}(a;q)_n\cdot\frac{(c/a;q)_\infty}{(c;q)_\infty},\end{aligned} \tag{4.1}$$

where the last step follows by (1.2). Thus, we arrive at

$$\begin{aligned}\sum_{n\ge 0}A_nC_n &= \frac{(c/a;q)_\infty}{(c;q)_\infty}\sum_{n\ge 0}\frac{(c/a)^n(1-q^n)(x,a;q)_n}{(q;q)_n^2}\\ &=\sum_{n\ge 0}B_nD_n=\sum_{n\ge 0}\frac{(-1)^n(cx/a)^n q^{\binom{n}{2}}(1-q^n)(x^{-1},a;q)_n}{(q;q)_n^2(c;q)_n},\end{aligned}$$

which completes the proof.

Proof of Corollary 2.4. (2.6) follows from (2.5) upon letting $a\to\infty$ and (2.7) follows from (2.6) by letting $x=c=-1$, which proves the desired formula.

5. Proof of Theorem 2.5

Proof of Theorem 2.5. Setting

$$A_n=\frac{(-1)^n q^{\frac{n}{2}}(x;q)_n}{(q;q)_n},\quad U_n=\frac{q^{\frac{n^2}{2}}}{(q;q)_n},\quad B_n=\frac{x^n q^{\frac{n^2}{2}}}{(q;q)_n},\quad D_n=(-1)^n a^n,$$

and V_n is the constant sequences with value 1 in Lemma 1.1. Due to (3.1), we have

$$C_n=(-1)^n a^n\left(aq^{\frac{1}{2}};q\right)_\infty.$$

Thus,

$$\begin{aligned}\sum_{n\ge 0}A_nC_n &= \left(aq^{\frac{1}{2}};q\right)_\infty\sum_{n\ge 0}\frac{\left(aq^{\frac{1}{2}}\right)^n(x;q)_n}{(q;q)_n}\\ &=\sum_{n\ge 0}B_nD_n=\sum_{n\ge 0}\frac{(-1)^n(ax)^n q^{\frac{n^2}{2}}}{(q;q)_n},\end{aligned}$$

which is (2.8). □

Notice that we use (1.4) to express the left-hand side of (2.8)

$$\sum_{k\ge 0}\frac{(-1)^k q^{\frac{k^2}{2}}a^k}{(q;q)_k}\sum_{m\ge 0}\frac{q^{\frac{m}{2}}(x;q)_m a^m}{(q;q)_m}=\sum_{n\ge 0}\frac{(-1)^n x^n q^{\frac{n^2}{2}}a^n}{(q;q)_n}. \tag{5.1}$$

Now we equate the terms corresponding to a^n in (5.1) to obtain

$$\sum_{k=0}^{n}\frac{(-1)^{n-k}q^{\frac{(n-k)^2}{2}}}{(q;q)_{n-k}}\cdot\frac{q^{\frac{k}{2}}(x;q)_k}{(q;q)_k}=\frac{(-1)^n x^n q^{\frac{n^2}{2}}}{(q;q)_n},$$

which gives the following identity after straightforward simplifications.

$$x^n=\sum_{k=0}^{n}(-1)^k q^{\binom{k+1}{2}-nk}(x;q)_k\begin{bmatrix}n\\k\end{bmatrix}.$$

The above identity appears in ([4], p. 157).

6. Proofs of Theorem 2.6 and Corollary 2.7

Proof of Theorem 2.6. We apply Lemma 1.1 with

$$A_n=\frac{(-1)^n q^{\frac{n}{2}}(x;q)_n}{(q;q)_n},\quad U_n=\frac{q^{\frac{n^2}{2}}}{(q;q)_n},\quad B_n=\frac{x^n q^{\frac{n^2}{2}}}{(q;q)_n},$$

$$D_n=(-1)^n(c/a)^n q^{-\frac{n}{2}}\frac{(a;q)_n}{(c;q)_n},$$

and V_n is the constant sequences with value 1. Then due to (4.1), we get

$$C_n=(-1)^n(c/a)^n q^{-\frac{n}{2}}(a;q)_n\cdot\frac{(c/a;q)_\infty}{(c;q)_\infty}.$$

Thus,

$$\sum_{n\ge0}A_nC_n=\frac{(c/a;q)_\infty}{(c;q)_\infty}\sum_{n\ge0}\frac{(c/a)^n(a,x;q)_n}{(q;q)_n}=\sum_{n\ge0}B_nD_n=\frac{(-1)^n(cx/a)^n q^{\binom{n}{2}}(a;q)_n}{(q,c;q)_n},$$

which yields the desired formula. □

Proof of Corollary 2.7. To prove (2.10), we first use (1.6) to express the left-hand side of (2.9) as powers series in *x*. Then

$$\text{L. H. S. of (2.9)}=\frac{(c/a;q)_\infty}{(c;q)_\infty}\sum_{n\ge0}\frac{(c/a)^n(a;q)_n}{(q;q)_n}\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}}\begin{bmatrix}n\\k\end{bmatrix}x^k$$

$$=\frac{(c/a;q)_\infty}{(c;q)_\infty}\sum_{k\ge0}\frac{(-1)^k q^{\binom{k}{2}}x^k}{(q;q)_k}\sum_{n=k}^{\infty}\frac{(c/a)^n(a;q)_n}{(q;q)_{n-k}}$$

$$=\frac{(c/a;q)_\infty}{(c;q)_\infty}\sum_{k\ge0}\frac{(-1)^k q^{\binom{k}{2}}x^k}{(q;q)_k}\sum_{n=0}^{\infty}\frac{(c/a)^{n+k}(a;q)_{n+k}}{(q;q)_n}$$

$$=\frac{(c/a;q)_\infty}{(c;q)_\infty}\sum_{k\ge0}\left(\sum_{n\ge0}\frac{(-1)^k q^{\binom{k}{2}}(c/a)^{n+k}(a;q)_{n+k}}{(q;q)_k(q;q)_n}\right)x^k.$$

Now equate the terms corresponding to x^k in (2.9) to obtain

$$\sum_{n\ge0}\frac{(-1)^k q^{\binom{k}{2}}(c/a)^{n+k}(a;q)_{n+k}}{(q;q)_k(q;q)_n}=\frac{(c;q)_\infty}{(c/a;q)_\infty}\cdot\frac{(-1)^k(c/a)^k q^{\binom{k}{2}}(a;q)_k}{(c,q;q)_k},$$

which by some basic calculations down to (2.10). (2.11) follows easily from (2.9) upon letting $a \to \infty$.

7. Proofs of Theorem 2.8 and Corollary 2.9

Proof of Theorem 2.8. Let us use Lemma 1.1 with a Bailey transform as follows:

$$A_n = \frac{q^{n^2+n}}{\left(q^2;q^2\right)_n},\ U_n = \frac{q^{n^2+2n}}{\left(q^2;q^2\right)_n},\ B_n = \frac{q^{\binom{n+2}{2}-1}}{(q;q)_n},\ D_n = q^{-n}\left(x;q^2\right)_n,\ V_n = 1.$$

We compute

$$\begin{aligned} C_n &= \sum_{k\ge0} q^{-n-k}\left(x;q^2\right)_{n+k} \cdot \frac{q^{k^2+2k}}{\left(q^2;q^2\right)_k} \\ &= q^{-n}\left(x;q^2\right)_n \sum_{k\ge0} \frac{q^{k(k+1)}}{\left(xq^{2n};q^2\right)_k}\left(q^2;q^2\right)_k \\ &= q^{-n}\left(x;q^2\right)_n \left(-q^2;q^2\right)_\infty \left(xq^{2n+2};q^4\right)_\infty, \end{aligned}$$

where in the last step, we used the (1.7) with $q \to q^2, x \to xq^{2n}$. Then by virtue of Lemma 1.1, we get

$$\begin{aligned} \sum_{n\ge0} A_n C_n &= \left(-q^2;q^2\right)_\infty \sum_{n\ge0} \frac{q^{n^2}\left(x;q^2\right)_n \left(xq^{2n+2};q^4\right)_\infty}{\left(q^2;q^2\right)_n} \\ &= \sum_{n\ge0} B_n D_n = \sum_{n\ge0} \frac{q^{\binom{n+1}{2}}\left(x;q^2\right)_n}{(q;q)_n}, \end{aligned}$$

which completes the proof.

Proof of Corollary 2.9. (2.13) and (2.14) follow from (2.12) by letting, respectively, $x = q^2$ and $x = q$.

8. Proof of Theorem 2.10

Proof of Theorem 2.10. Let us use Lemma 1.1 with a Bailey transform as follows:

$$A_n = \frac{q^{n^2+n}}{\left(q^2;q^2\right)_n},\ U_n = \frac{q^{n^2+2n}}{\left(q^2;q^2\right)_n},\ B_n = \frac{q^{\binom{n+2}{2}-1}}{(q;q)_n},\ D_n = \frac{a^n\left(-\frac{b}{aq};q^2\right)_n}{\left(bq^2;q^2\right)_n},\ V_n = 1.$$

We compute

$$\begin{aligned} C_n &= \sum_{k\ge0} \frac{a^{n+k}\left(-\frac{b}{aq};q^2\right)_{n+k}}{\left(bq^2;q^2\right)_{n+k}} \cdot \frac{q^{k^2+2k}}{\left(q^2;q^2\right)_k} \\ &= \frac{a^n\left(-\frac{b}{aq};q^2\right)_n}{\left(bq^2;q^2\right)_n} \sum_{k\ge0} \frac{a^k q^{k^2+2k}\left(-bq^{2n+1}/a;q^2\right)_k}{\left(q^2;q^2\right)_k \left(bq^{2n+2};q^2\right)_k} \\ &= a^n\left(-\frac{b}{aq};q^2\right)_n \frac{\left(-aq^3;q^2\right)_\infty}{\left(bq^2;q^2\right)_\infty}, \end{aligned}$$

where in the last step, we used (1.8) by letting $a = aq, b = bq^{2n}$. Then by virtue of Lemma 1.1, we get

$$\sum_{n\geq 0} A_n C_n = \frac{\left(-aq^3;q^2\right)_\infty}{\left(bq^2;q^2\right)_\infty}\sum_{n\geq 0}\frac{a^n q^{n^2+n}\left(-\frac{b}{aq};q^2\right)_n}{\left(q^2;q^2\right)_n}$$

$$= \sum_{n\geq 0} B_n D_n = \sum_{n\geq 0}\frac{a^n q^{\binom{n+2}{2}-1}\left(-\frac{b}{aq};q^2\right)_n}{(q;q)_n\left(bq^2;q^2\right)_n},$$

which completes the proof.

9. Conclusion

By choosing some sequences, we can derive many identities from the Bailey transform. Furthermore, we should study the generalized Bailey transform [14] deeply to establish the multiple parameterized identities. On the other hand, we can also study the mock theta functions or the Rogers-Ramanujan identities through the Bailey transform.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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