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Convergence of a Sinusoidal Series $\sum \frac{\sin n^{\alpha}}{n^{p}}$ with an Infinite Integral $\int_{1}^{+\infty} \frac{\sin x^{\alpha}}{x^{p}} \mathrm{d}x$

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Abstract

In this paper, we study the relationship between the convergence of the sinusoidal series $\sum \frac{\sin n^{\alpha}}{n^{p}}$ and the infinity integrals $\int_{1}^{+\infty} \frac{\sin x^{\alpha}}{x^{p}} \mathrm{d}x$ (any real number $\alpha \in [0,1]$, parameter $p > 0$). First of all, we study the convergence of the series $\sum \frac{\sin n^{\alpha}}{n^{p}}$ (any real number $\alpha \in [0,1]$, parameter $p > 0$), mainly using the estimation property of the order to obtain that the series diverges when $0 < p \le 1-\alpha$, the series converges conditionally when $1-\alpha < p \le 1$, and the series converges absolutely when $p > 1$. In the next part, we study the convergence state of the infinite integral $\int_{1}^{+\infty} \frac{\sin x^{\alpha}}{x^{p}} \mathrm{d}x$ (any real number $\alpha \in [0,1]$, parameter $p > 0$), and get that when $0 < p \le 1-\alpha$, the infinite integral diverges; when $1-\alpha < p \le 1$, the infinite integral conditionally converges; when $p > 1$, the infinite integral absolutely converges. Comparison of the conclusions of the above theorem, it is not difficult to derive the theorem: the level of $\sum \frac{\sin n^{\alpha}}{n^{p}}$ and the infinity integral $\int_{1}^{+\infty} \frac{\sin x^{\alpha}}{x^{p}} \mathrm{d}x$ with the convergence of the state (any real number $\alpha \in [0,1]$, the parameter $p > 0$), thus promoting the textbook of the two with the convergence of the state requires the function of the general term or the product of the function must be monotonically decreasing conditions.

Keywords

Sinusoidal Series, Estimation of Order, Convergent States, Infinite Integrals, Discriminant Method

1. Introduction

Usually there is no necessary connection between the convergence of series and infinite integrals [1], but when the product function has certain characteristics, the two are in the same convergence state, for example, the infinite integral of a monotonically decreasing function has the same convergence as its corresponding series [2]. In the fourth edition of the eight cups competition, held on 1 August 2022, the eighth question of the mathematical group B appeared the problem of determining the convergence of a sinusoidal series: given the parameter $p>0$, try to discuss the convergence of the series $\sum_{n=1}^{\infty}\frac{\sin\sqrt{n}}{n^p}$ with respect to the value of the parameter p (when converging, it should be determined whether it is absolutely convergent). We find that the level when $0<p\le\frac{1}{2}$, the level of dispersion; when $\frac{1}{2}<p\le 1$, the level of conditional convergence; when $p>1$, the level of absolute convergence, and the infinite integral $\int_1^{+\infty}\frac{\sin\sqrt{x}}{x^p}\mathrm{d}x$ has the same convergence. It can be seen that the class of non-monotonic functions has the same convergence of the series and the corresponding infinite integral under certain conditions. The above shows that the sinusoidal series $\sum\frac{\sin n^{\alpha}}{n^p}$ has the same convergence as the infinite integral $\int_1^{+\infty}\frac{\sin x^{\alpha}}{x^p}\mathrm{d}x$ when $\alpha=\frac{1}{2}$. In the following, we try to extend this conclusion to the general $\alpha\in[0,1]$ case, to begin with we study the convergence of the series $\sum\frac{\sin n^{\alpha}}{n^p}$, then we discuss the convergence of the infinite integrals $\int_1^{+\infty}\frac{\sin x^{\alpha}}{x^p}\mathrm{d}x$, and by comparing the two, we conclude that the series $\sum\frac{\sin n^{\alpha}}{n^p}$ and the infinite integrals $\int_1^{+\infty}\frac{\sin x^{\alpha}}{x^p}\mathrm{d}x$ have the same convergence in this paper.

2. Convergence States of Sinusoidal Term Levels

Proposition: For the sinusoidal series $\sum\frac{\sin\sqrt{n}}{n^p}$ (parameter $p>0$), the series diverges when $0<p\le\frac{1}{2}$, converges conditionally when $\frac{1}{2}<p\le 1$, and converges absolutely when $p>1$.

Corollary: Let any real number $\alpha \in [0,1]$ and parameter $p > 0$ be the convergence state of the sinusoidal series $\sum \frac{\sin n^\alpha}{n^p}$: when $0 < p \le 1-\alpha$, the series diverges; when $1-\alpha < p \le 1$, the series converges conditionally; when $p > 1$, the series converges absolutely (provided that $\{p \mid m < p \le m, m \in R\} = \varnothing$).

Proof:

Using the sum and difference product formula, express $\sin n^\alpha$ as

$$\sin n^\alpha = \frac{\cos\frac{(n+1)^\alpha - n^\alpha}{2}}{\sin\frac{(n+1)^\alpha - n^\alpha}{2}} \frac{\cos n^\alpha - \cos(n+1)^\alpha}{2} + \frac{\sin n^\alpha - \sin(n+1)^\alpha}{2} \tag{1}$$

According to Taylor's formula, it is possible to obtain

$$\frac{\cos x}{\sin x} = \frac{1}{x} + o(x), x \to 0 \tag{2}$$

$$\frac{(n+1)^\alpha - n^\alpha}{2} = \frac{n^\alpha}{2}\left(\left(1+\frac{1}{n}\right)^\alpha - 1\right) = \frac{n^\alpha}{2}\left(1+\frac{\alpha}{n} + o\left(\frac{1}{n^2}\right) - 1\right) = \frac{\alpha}{2n^{1-\alpha}}\left(1 + o\left(\frac{1}{n}\right)\right) \tag{3}$$

The association of (1), (2) and (3) yields

$$\frac{\sin n^\alpha}{n^p} = \frac{\cos n^\alpha - \cos(n+1)^\alpha}{\alpha n^{p+\alpha-1}} + \frac{\sin n^\alpha - \sin(n+1)^\alpha}{2n^p} + o\left(\frac{1}{n^{p+1}}\right) \tag{4}$$

Therefore, when $p = 1-\alpha$ is found

$$\frac{\sin n^\alpha}{n^{1-\alpha}} = \frac{\cos n^\alpha - \cos(n+1)^\alpha}{\alpha} + \frac{\sin n^\alpha - \sin(n+1)^\alpha}{2n^{1-\alpha}} + o\left(\frac{1}{n^{2-\alpha}}\right) \tag{5}$$

According to by the Cauchy criterion $\lim\limits_{n\to\infty} \cos n^\alpha$ does not exist, so the level $\sum \frac{\cos n^\alpha - \cos(n+1)^\alpha}{\alpha}$ diverges, and by the A-D discriminant, the levels $\sum \frac{\sin n^\alpha - \sin(n+1)^\alpha}{2n^{1-\alpha}}$ all converge, and the p-levels also converge, so the level $\sum \frac{\sin n^\alpha}{n^p}$ diverges.

When $p > 1$, the level $\sum \left|\frac{\sin n^\alpha}{n^p}\right| \le \sum \left|\frac{1}{n^p}\right|$, it is easy to know that the p level converges, there is a comparative discriminant method to get the level $\sum \frac{\sin n^\alpha}{n^p}$ absolute convergence.

When $1-\alpha < p \le 1$, by the A-D discriminant, the level $\sum \frac{\sin n^\alpha}{n^p}$ converges, the following consider the convergence of the level $\sum \left|\frac{\sin n^\alpha}{n^p}\right|$, for $\forall n \in N_+$, there is $\left|\frac{\sin n^\alpha}{n^p}\right| \ge \frac{\sin^2 n^\alpha}{n^p} = \frac{1-\cos 2n^\alpha}{2n^p}$, it is not difficult to see that the p level

$\sum \frac{1}{n^p}$ diverges, and $\sum \frac{\cos 2n^\alpha}{2n^p}$ converges, so $\sum \left| \frac{\sin n^\alpha}{n^p} \right|$ diverges, so $\sum \frac{\sin n^\alpha}{n^p}$ conditional convergence.

When $0 < p < 1-\alpha$, the level $\sum \frac{\sin n^\alpha}{n^p}$ is divergent, also use the inverse method to prove that the level is divergent. First of all, assume that there is a point $p_0 \in \left(0, \frac{1}{2}\right)$, so that the level of $\sum \frac{\sin n^\alpha}{n^p}$ convergence, to find out the level of *n* before the part of the term and $S_n = \sum_{k=1}^{n} \frac{\sin k^\alpha}{k^{p_0}}$ is bounded on *n*, according to the A-D method of discrimination can be $\sum \frac{\sin n^\alpha}{n^{1-\alpha}} = \sum \frac{\sin n^\alpha}{n^{p_0}} \cdot \frac{1}{n^{1-\alpha-p_0}}$ convergence, which is contradictory to the $\sum \frac{\sin n^\alpha}{n^{1-\alpha}}$ divergence, so when $0 < p < \frac{1}{2}$, the level of $\sum \frac{\sin n^\alpha}{n^p}$ divergence.

In summary, the sinusoidal series $\sum \frac{\sin n^\alpha}{n^p}$ ($\alpha \in (0,1)$, $p > 0$), when $0 < p \le 1-\alpha$, the series diverges; when $1-\alpha < p \le 1$, the series converges conditionally; when $p > 1$, the series converges absolutely. (*a*)

Consider below the case where α takes values at the endpoints $\alpha = 0$ and $\alpha = 1$:

When $\alpha = 0$ is the level $\sum \frac{\sin 1}{n^p}$, $\sin 1$ is a constant, so it is in the same convergence state as the level *p*. That is, when is the level converges and is absolutely convergent; when is the level diverges. That is, when $p > 1$, the level $\sum \frac{\sin 1}{n^p}$ converges and absolutely converges; when $p \le 1$, the level $\sum \frac{\sin 1}{n^p}$ diverges. (*b*)

When $\alpha = 1$, the convergence state of the level $\sum \frac{\sin n}{n^p}$ is discussed below:

When $p > 1$, the *p*-series converges due to $\left| \frac{\sin n}{n^p} \right| \le \frac{1}{n^p}$, and the series $\sum \frac{\sin n}{n^p}$ converges absolutely by the comparative discriminant;

When $p = 1$ is used, by the product to sum formula, we know that

$$\sum_{k=1}^{n} \sin k = \frac{1}{2\sin\frac{1}{2}} \sum_{k=1}^{n} 2\sin\frac{1}{2}\sin k = -\frac{\sum_{k=1}^{n}\left(\cos\left(k+\frac{1}{2}\right) - \cos\left(k-\frac{1}{2}\right)\right)}{2\sin\frac{1}{2}} = \frac{\cos\frac{1}{2} - \cos\frac{2n+1}{2}}{2\sin\frac{1}{2}} \tag{6}$$

Using the A-D discriminant method, $\sum \sin n$ part of the sum series is bounded, and $\left\{\frac{1}{n}\right\}$ monotonically decreasing and tends to 0, so $\sum \frac{\sin n}{n}$

convergence, the following proof of $\sum\frac{\sin n}{n}$ conditional convergence, because $\left|\frac{\sin n}{n}\right|\geq\frac{\sin^2 n}{n}=\frac{1-\cos 2n}{2n}=\frac{1}{2n}-\frac{\cos 2n}{2n}$, and by the A-D discriminant method of the series $\sum\frac{\cos 2n}{2n}$ convergence, but the sum series $\sum\frac{1}{n}$ divergence, so $\sum\frac{\sin n}{n}$ conditional convergence.

When $0<p<1$, by the A-D discriminant method, the part of the series $\sum\sin n$ and the series is bounded, and $\left\{\frac{1}{n^p}\right\}$ monotonically decreasing and tends to 0, so $\sum\frac{\sin n}{n}$ convergence. And $\left|\frac{\sin n}{n^p}\right|\geq\frac{\sin^2 n}{n^p}=\frac{1-\cos 2n}{2n^p}=\frac{1}{2n^p}-\frac{\cos 2n}{2n^p}$, by the product and difference formula to get

$$\sum_{k=1}^{n}\cos 2k=\frac{\sum_{k=1}^{n}2\sin 1\cos 2k}{2\sin 1}=\frac{\sum_{k=1}^{n}\sin(2k+1)-\sin(2k-1)}{2\sin 1}=\frac{\sin(2n+1)-\sin 1}{2\sin 1} \quad (7)$$

Therefore, the part of $\sum\cos 2n$ and the series are bounded, and $\frac{1}{n^p}$ is monotonically decreasing and tends to 0, so the series $\sum\frac{\cos 2n}{2n^p}$ converges, but the series $\sum\frac{1}{n^p}$ diverges, so the series $\sum\frac{\sin n}{n^p}$ converges conditionally.

In summary, when $\alpha=1$, the level $\sum\frac{\sin n^\alpha}{n^p}$ When $p>1$, the level converges absolutely; when $0<p\leq 1$, the level converges conditionally. (*c*)

Summing up at (*a*), (*b*) and (*c*), we have that the series diverges when $0<p\leq 1-\alpha$; the series converges conditionally when $1-\alpha<p\leq 1$; and the series converges absolutely when $p>1$. The corollary is proved. ***QED***

3. Convergence States of Infinite Integrals

Lemma: Infinite integrals $\int_1^{+\infty}\frac{\sin x^\alpha}{x^p}\mathrm{d}x$, for any real number $\alpha\in[0,1]$, parameter $p>0$, at that time $0<p\leq 1-\alpha$, the infinite integrals diverge; at that time $1-\alpha<p\leq 1$, the infinite integrals converge conditionally; at that time $p>1$, the infinite integrals converge absolutely.

Proof.

Consider first the case of $\alpha=0$. The infinite integral is transformed into $\int_1^{+\infty}\frac{\sin 1}{x^p}\mathrm{d}x$, and $\sin 1$ is a positive constant, so the level is in the same convergence state as $\int_1^{+\infty}\frac{1}{x^p}\mathrm{d}x$. So when $p>1$, the series converges and converges absolutely; when $p\leq 1$, the series diverges.

Next consider $\alpha=1$. The infinite integral is transformed to $\int_1^{+\infty}\frac{\sin x}{x^p}\mathrm{d}x$.

When $p>1$ is $\left|\frac{\sin x}{x^p}\right| \le \frac{1}{x^p}$, while $\int_1^{+\infty}\frac{1}{x^p}\mathrm{d}x$ converges, by the comparative discriminant, we know that $\int_1^{+\infty}\frac{\sin x}{x^p}\mathrm{d}x$ converges and is absolutely convergent.

Consider $p=1$ when $F(A)=\int_1^A \sin x\mathrm{d}x=-(\cos A-\cos 1)$ is bounded on $[1,+\infty)$ and $g(x)=\frac{1}{x}$ is monotone on $[1,+\infty)$ and $\lim_{x\to+\infty}\frac{1}{x}=0$, so $\int_1^{+\infty}\frac{\sin x}{x}\mathrm{d}x$ converges. Since $\left|\frac{\sin x}{x}\right| \ge \frac{\sin^2 x}{x}=\frac{1}{2x}-\frac{\cos 2x}{2x}$, combined with the A-D discriminant, we know that $\int_1^{+\infty}\frac{\cos 2x}{2x}\mathrm{d}x$ converges and $\int_1^{+\infty}\frac{1}{2x}\mathrm{d}x$ diverges. So $\int_1^{+\infty}\frac{\sin x}{x}\mathrm{d}x$ converges conditionally.

When $0<p<1$, the same process as above, consider that $F(A)=\int_1^A \sin x\mathrm{d}x$ is bounded on $[1,+\infty)$ and $g(x)=\frac{1}{x^p}$ is monotone on $[1,+\infty)$ and $\lim_{x\to+\infty} g(x)=0$, so $\int_1^{+\infty}\frac{\sin x}{x^p}\mathrm{d}x$ converges. But $\left|\frac{\sin x}{x^p}\right| \ge \frac{\sin^2 x}{x^p}=\frac{1}{2x^p}-\frac{\cos 2x}{2x^p}$, combined with the A-D discriminant, we know that $\int_1^{+\infty}\frac{\cos 2x}{2x^p}\mathrm{d}x$ converges while $\int_1^{+\infty}\frac{1}{2x^p}\mathrm{d}x$ diverges. So $\int_1^{+\infty}\frac{\sin x}{x^p}\mathrm{d}x$ converges conditionally.

Finally, consider the case of $\alpha\in(0,1)$. Let $t=x^\alpha$, then $x=t^{\frac{1}{\alpha}}$, $\mathrm{d}x=\mathrm{d}t^{\frac{1}{\alpha}}=\frac{1}{\alpha}t^{\frac{1}{\alpha}-1}\mathrm{d}t$, so that

$$\int_1^{+\infty}\frac{\sin x^\alpha}{x^p}\mathrm{d}x=\frac{1}{\alpha}\int_1^{+\infty}\frac{\sin t}{t^{\frac{p}{\alpha}}}t^{\frac{1}{\alpha}-1}\mathrm{d}t=\frac{1}{\alpha}\int_1^{+\infty}\frac{\sin t}{t^{\frac{p-1}{\alpha}+1}}\mathrm{d}t \tag{8}$$

When $1<\frac{p-1}{\alpha}+1$ is $p>1$, $\left|\frac{\sin t}{t^{\frac{p-1}{\alpha}+1}}\right| \le \frac{1}{t^{\frac{p-1}{\alpha}+1}}$, $t\in[1,+\infty)$, we know that $\int_1^{+\infty}\left|\frac{\sin t}{t^{\frac{p-1}{\alpha}+1}}\right|\mathrm{d}t$ converges by the comparative discriminant, and thus $\int_1^{+\infty}\frac{\sin x^\alpha}{x^p}\mathrm{d}x$ converges absolutely.

When $0<\frac{p-1}{\alpha}+1\le 1$ is $1-\alpha<p\le 1$. On the one hand, $\forall u\ge 1$, has $\int_1^u \sin x\mathrm{d}x=|\cos 1-\cos u|\le 2$, and $\frac{1}{x^{\frac{p-1}{\alpha}+1}}$ is monotonic and tends to 0 when $\frac{p-1}{\alpha}+1>0$ ($x\to+\infty$). It can be deduced from the fact that $\int_1^{+\infty}\frac{\sin x}{x^{\frac{p-1}{\alpha}+1}}\mathrm{d}x$ converges according to the A-D discriminant. On the other hand, since

$$\left|\frac{\sin x}{x^{\frac{p-1}{\alpha}+1}}\right| \ge \frac{\sin^2 x}{x^{\frac{p-1}{\alpha}+1}}=\frac{1}{2x^{\frac{p-1}{\alpha}+1}}-\frac{\cos 2x}{2x^{\frac{p-1}{\alpha}+1}},\quad x\in[1,+\infty) \tag{9}$$

where $\int_1^{+\infty}\frac{\cos 2x}{2x^{\frac{p-1}{\alpha}+1}}\mathrm{d}x=\frac{1}{2}\int_2^{+\infty}\frac{\cos t}{2^{-\frac{p-1}{\alpha}}t^{\frac{p-1}{\alpha}+1}}\mathrm{d}t$. According to the A-D discriminant condition, it is known that $\int_1^{+\infty}\frac{\cos 2x}{2x^{\frac{p-1}{\alpha}+1}}\mathrm{d}x$ is convergent and $\int_1^{+\infty}\frac{\mathrm{d}x}{2x^{\frac{p-1}{\alpha}+1}}$ is divergent ($1-\alpha<p\le 1$), so the infinite integral $\int_1^{+\infty}\left|\frac{\sin x}{x^{\frac{p-1}{\alpha}+1}}\right|\mathrm{d}x$ is divergent when $0<\frac{p-1}{\alpha}+1\le 1$ and thus $\int_1^{+\infty}\frac{\sin x^{\alpha}}{x^p}\mathrm{d}x$ is conditionally convergent.

When $\frac{p-1}{\alpha}+1=0$ is $p=1-\alpha$, substitution yields

$$\int_1^{+\infty}\frac{\sin x}{x^{\frac{p-1}{\alpha}+1}}\mathrm{d}x=\int_1^{+\infty}\sin x\mathrm{d}x=\lim_{M\to+\infty}\left(1-\cos M\right) \tag{10}$$

It follows from the Cauchy convergence criterion that $\exists\varepsilon_0=\frac{1}{2}$, $\exists M'=2k\pi$ and $M''=2k\pi+\frac{\pi}{2}$, such that

$$\left|\cos M'-\cos M''\right|=\left|\cos\left(2k\pi\right)-\cos\left(2k\pi+\frac{\pi}{2}\right)\right|=1>\varepsilon_0 \tag{11}$$

Therefore $\lim\limits_{M\to+\infty}\cos M$ does not exist, so $\int_1^{+\infty}\sin x\mathrm{d}x$ diverges.

When $\frac{p-1}{\alpha}+1<0$ is $p<1-\alpha$, the infinite integral $\int_1^{+\infty}\frac{\sin x}{x^{\frac{p-1}{\alpha}+1}}\mathrm{d}x$ is divergent. Using the converse method, suppose the infinite integral $\int_1^{+\infty}\frac{\sin x}{x^{\frac{p-1}{\alpha}+1}}\mathrm{d}x$ converges, then since $x^{\frac{p-1}{\alpha}+1}$ is bounded on $x\in[1,+\infty)$, according to the A-D discriminant, there should be $\int_1^{+\infty}\sin x\mathrm{d}x$ convergence, a contradiction. Therefore, when $\frac{p-1}{\alpha}+1<0$, $\int_1^{+\infty}\frac{\sin x}{x^{\frac{p-1}{\alpha}+1}}\mathrm{d}x$ is divergent.

To sum up: the infinite integral $\int_1^{+\infty}\frac{\sin x^{\alpha}}{x^p}\mathrm{d}x$, for any real number $\alpha\in[0,1]$, parameter $p>0$, when $0<p\le 1-\alpha$, the infinite integral diverges; when $1-\alpha<p\le 1$, the infinite integral converges conditionally; when $p>1$, the infinite integral converges absolutely. *QED*

4. Theorem on the Convergence State of the Sine Term Hierarchy with the Infinite Integral Homology

Theorem: Arbitrarily $\alpha\in[0,1]$, with parameter $p>0$, the level $\sum\frac{\sin n^{\alpha}}{n^p}$ is convergent to the same state as the infinite integral $\int_1^{+\infty}\frac{\sin x^{\alpha}}{x^p}\mathrm{d}x$.

Proof:

According to the above corollary, for a sinusoidal series $\sum\frac{\sin n^{\alpha}}{n^{p}}$, any real number $\alpha\in[0,1]$, and the parameter $p>0$, the series diverges when $0<p\le 1-\alpha$, the series converges conditionally when $1-\alpha<p\le 1$, and the series converges absolutely when $p>1$.

According to the above lemma, for the infinite integral $\int_{1}^{+\infty}\frac{\sin x^{\alpha}}{x^{p}}\mathrm{d}x$, any real number $\alpha\in[0,1]$, parameter $p>0$, the infinite integral diverges when $0<p\le 1-\alpha$, the infinite integral converges conditionally when $1-\alpha<p\le 1$, and the infinite integral converges absolutely when $p>1$.

Accordingly, we obtain that the level $\sum\frac{\sin n^{\alpha}}{n^{p}}$ is homoconvergent with the infinite integral $\int_{1}^{+\infty}\frac{\sin x^{\alpha}}{x^{p}}\mathrm{d}x$ (any $\alpha\in[0,1]$, parameter $p>0$). The proof of the theorem is thus complete. ***QED***

5. Conclusion

Inspired by $\alpha=\frac{1}{2}$ when the sine series $\sum\frac{\sin n^{\alpha}}{n^{p}}$ and the infinite integral $\int_{1}^{+\infty}\frac{\sin x^{\alpha}}{x^{p}}\mathrm{d}x$ (parameter $p>0$) are in the same convergent state, we explore the convergence of the series $\sum\frac{\sin n^{\alpha}}{n^{p}}$ and the infinite integral $\int_{1}^{+\infty}\frac{\sin x^{\alpha}}{x^{p}}\mathrm{d}x$ when $\alpha\in[0,1]$ is in the same convergent state, and we extend the conditions of the function class of the two in the same convergent state, expanding from monotonically decreasing functions to the class of non-monotonous functions, and we will continue to explore the other classes of the two in the same convergent state in the future.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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