Advances in Pure Mathematics, 2023, 13, 635-641
https://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

Products of Odd Numbers or Prime Number Can Generate the Three Members' Families of Fermat Last Theorem and the Theorem Is Valid for Summation of Squares of More Than Two Natural Numbers

Susmita Pramanik[1], Deepak Kumar Das[1], Panchanan Pramanik[2*]

[1]Department of Chemistry and Nanoscience, GLA University, Mathura, India
[2]Agriculture and Ecology Research Unit, Indian Statistical Institute, Kolkata, India
Email: *pramanik1946@gmail.com

How to cite this paper: Pramanik, S., Das, D.K. and Pramanik, P. (2023) Products of Odd Numbers or Prime Number Can Generate the Three Members' Families of Fermat Last Theorem and the Theorem Is Valid for Summation of Squares of More Than Two Natural Numbers. *Advances in Pure Mathematics*, **13**, 635-641.
https://doi.org/10.4236/apm.2023.1310043

Received: July 26, 2023
Accepted: October 7, 2023
Published: October 10, 2023

Abstract

Fermat's last theorem, had the statement that there are no natural numbers A, B, and C such that $A^n + B^n = C^n$, in which n is a natural number greater than 2. We have shown that any product of two odd numbers can generate Fermat or Pythagoras triple (A, B, C) following $n = 2$ and also it is applicable $A^2 + B^2 + C^2 + D^2 + \text{so on} = A_n^2$ where all are natural numbers.

Keywords

Fermat Last Theorem, Generation of Fermat's Numbers, Extension of Fermat's Expression, Fermat's Expression from Products of Odd Numbers

1. Introduction

The Pythagorean equation, $x^2 + y^2 = z^2$, has an infinite number of positive integer solutions for x, y, and z; these solutions are known as Pythagorean triples (with the simplest example $3^2 + 4^2 = 5^2$). Around 1637, Fermat wrote in the margin of a book that the more general equation $x^n + y^n = z^n$, had no solutions in positive integers if n is an integer greater than 2. In theory, this statement is known as Fermat's Last Theorem (it is also called as Fermat's conjecture before 1995). The cases $n = 1$ and $n = 2$ have been known from Pythagoras time having infinite solutions [1] [2] [3].

The proposition was first stated as a theorem by Pierre de Fermat around

1637. It was written in the margin of a copy of *Arithmetica*. Fermat claimed that he had a proof and due length of the calculation, he was unable to fit in the margin of the copy. However, after his death no document was found to substantiate his claim. Consequently, the proposition became as a conjecture rather than a theorem. After 358 years of effort by mathematicians, the first successful proof was completed in 1994 by Andrew Wiles and formally published in 1995. It was described as a "stunning advance in mathematics" in the citation for Wiles's Abel Prize award in 2016 [2]. It also proved many parts of the Taniyama-Shimura conjecture. Afterward, it was defined as the modularity theorem, It opened up new approaches to numerous other problems and developed powerful technique known as modularity lifting in mathematics. It is among the most remarkable theorems in the history of mathematics Fermat himself proved the special case $n = 4$. Hilbert D (1897) elaborated some of the studies [3]. Alternative proofs of the case $n = 4$ were developed later by Frénicle de Bessy (1676) [4], Leonhard Euler (1738) [5], Kausler (1802) [6], Peter Barlow (1811) [7], Adrien-Marie Legendre (1830) [8], Joseph Bertrand (1851) [9], Victor Lebesgue (1853, 1859, 1862) [10], Tafelmacher (1893) [11], Gambioli (1901) [12], Bang (1905) [13], Sommer (1907) [14], Nutzhorn (1912) [15], Robert Carmichael (1913) [16], Hancock (1931) [17], Grant and Perella (1999) [18], and Barbara (2007) [19]. The conjecture was proved for only the primes 3, 5, and 7 during 1637 to 1839. In addition, some innovative proof was provided by Sophie Germain and that was very relevant to entire class of primes [20]. In mid-19th century Ernst Kummer extended the analysis. He proved the theorem for all regular primes, leaving irregular primes which were analyzed separately [21] [22]. Based on Kummer's work and using advanced computer, many mathematicians were able to extend the proof covering prime exponents up to four million [23]. It was understood that a proof for all exponents was unsolvable.

British mathematician Andrew Wiles showed that it is a special case of the modularity theorem for elliptic curves expressed in 170 pages in the year 1995.

For $x = 2$, it is preferred to define A, B, C as a member of Fermat triplet (FT). Here are some simple calculations to find A, B, C from product of any two odd numbers.

As theorem

$$A^2 + B^2 = C^2 \tag{1}$$

$$\rightarrow C^2 - B^2 = A^2 \tag{2}$$

$$\rightarrow (C - B) \times (C + B) = A^2 \tag{3}$$

Let it is assumed that A is product of two odd numbers X and Y. then $A^2 = X^2 \cdot Y^2$ as

$$A = X \times Y \tag{4}$$

where $X > Y$ (Y may be 1 whether A is prime or not).

This implies

$$C + B = X^2 \quad \text{and} \quad C - B = Y^2$$

Thus

$$C = \left(X^2 + Y^2\right)/2 \quad \text{and} \quad B = \left(X^2 - Y^2\right)/2 \tag{5}$$

To get B and C as integers X and Y should be odd numbers as ($X^2 + Y^2$) and ($X^2 - Y^2$) are divided by 2 to get B and C as integers.

On substitution of values of C and B in Equation (3)

We get,

$$\left[\left(X^2 + Y^2\right)/2\right]^2 - \left[\left(X^2 - Y^2\right)/2\right]^2$$

$$1/4\left[\left(X^4 + Y^4 + 2X^2Y^2\right) - \left(X^4 + Y^4 - 2X^2Y^2\right)\right]$$

$X^2Y^2 = A^2$ as per Equation (4).

Thus, any product two natural odd numbers can generate Fermat triplet (FT). When A is prime number then A can be presented as $1 \times A$.

So, all-natural number (NN) those may be prime or compound odd members can generate Fermat triplet (FT).

For simplest example:

If NN = 3 then NN = 3 × 1.

So, members are $\left(3^2 + 1^2\right)/2 = 5$ and $\left(3^2 - 1^2\right)/2 = 4$.

Thus $3^2 + 4^2 = 5^2$ (FT or PT).

Here are examples with prime numbers (A) which is always odd (**Table 1**).

For the cases of non-prime numbers and product of odd numbers, **Table 2** shows the illustration to generate F Ts.

Any set of three (A, B, C) when multiplied with square of any number, then it generates another FT.

Fermat conjecture cannot be limited to three number (A, B, C) it can be expanded to any numbers of family. Illustration is given below

$$A^2 + B^2 = C^2$$

we like to extend the relation to

$$A^2 + B^2 + D^2 = E^2$$

Table 1. Generation of FT from prime numbers.

Serial no	Value of A (primes)	Value of B	Value of C	Final expression
1	7	24	25	$7^2 + 24^2 = 25^2$
2	11	60	61	$11^2 + 60^2 = 61^2$
3	13	84	85	$13^2 + 84^2 = 85^2$
4	17	144	145	$17^2 + 144^2 = 145^2$
5	19	181	180	$19^2 + 180^2 = 181^2$
6	23	264	265	$23^2 + 264^2 = 265^2$
7	29	420	421	$29^2 + 420^2 = 421^2$
8	31	480	481	$31^2 + 480^2 = 481^2$

Table 2. Generation of FT from products of odd numbers.

Serial no	Product of odd numbers (A)	Value of B	Value of C	Final expression
1	$21 = 3 \times 7$	20	29	$21^2 + 20^2 = 29^2$
2	$15 = 3 \times 5$	8	17	$8^2 + 15^2 = 17^2$
3	$27 = 9 \times 3$	36	45	$27^2 + 36^2 = 45^2$
4	$39 = 3 \times 13$	80	89	$39^2 + 80^2 = 89^2$
5	$45 = 15 \times 3$	78	87	$45^2 + 78^2 = 87^2$
6	$69 = 23 \times 3$	260	269	$69^2 + 260^2 = 269^2$
7	$747 = 83 \times 9$	3404	3485	$747^2 + 3404^2 = 3485^2$
8	$891 = 99 \times 9$	4860	4941	$891^2 + 4860^2 = 4941^2$

where all are integers. This is a family of 4 numbers Fermat Quartet (A, B, D, E) where all are natural numbers.

As per Fermat triplet (if there is no common factor) A or B will be even and other one will be odd so that C will be odd number.

$A^2 + B^2 = C^2$ as per Equation (1).

Equation (1) will be incorporated by another term say D generating Equation (5)

Thus

$$A^2 + B^2 + D^2 = E^2 \to C^2 + D^2 = E^2 \tag{6}$$

And

$$C^2 = E^2 - D^2 = (E - D)(E + D) \tag{7}$$

C^2 may be expressed as

$$C^2 = F \times G \tag{8}$$

where F and G both must be odd as C is odd.

Then as per Equation (7) it can be expressed as $F = E - D$ and $G = E + D$.

So, $E = (G + F)/2$ and $D = (G - F)/2$.

Thus, Equation (5)

$$A^2 + B^2 + D^2 = E^2 \quad \text{or} \quad A^2 + B^2 + \left[(G - H)/2\right]^2 = \left[(G + H)/2\right]^2 .$$

This expression generate Fermat quartet.

This principle may be used any number of terms for the Fermat expression.

As example

$$7^2 + 24^2 = 25^2 \quad \text{(Three members family)} \tag{9}$$

Let me introduce the family of 4 members.

Thus $7^2 + 24^2 + C^2 = D^2$ using Equation (9).

We get $25^2 = D^2 - C^2$ or $625 = (D - C)(D + C)$, $625 = 5 \times 125 = 25 \times 25$. Last term is not acceptable as both factors have same values because ($D - C$) and ($D + C$) should have different values, so ($D - C$) = 5 and ($D + C$) = 125.

Thus $D=(5+125)/2=65$ and $C=(125-5)/2=60$.

Henceforth $7^2+24^2+C^2=D^2$.

So, $7^2+24^2+60^2=65^2$ (Fermat quartet) as mentioned in Equation (5).

Again, for example of five members family (Fermat Quintet, it is expressed as

$$7^2+24^2+60^2+E^2=F^2$$

Or $7^2+24^2+60^2=F^2-E^2=(F-E)(F+E)$.

Or $4225=(F-E)(F+E)$.

Now $4225=5\times845=25\times169=13\times325=65\times65$.

Last factor is not acceptable as both the terms are equal.

Let second one is chosen (25×169).

So, $(F-E)=25$ and $(F+E)=169$.

Thus $F=(25+169)/2=97$ and $E=(169-25)/2=72$.

Thus

$$7^2+24^2+60^2+72^2=97^2 \quad (\textit{Fermat quintet})$$

If first factor is chosen as 5×845.

Then $F=(845+5)/2=425$ and $E=(845-5)/2=420$.

Thus

$$7^2+24^2+60^2+420^2=425^2 \quad (\textit{Fermat quintet})$$

With this principle it can generate Fermat family of any number, so

$$A^n+B^n+C^n+D^n+\cdots=X^n$$

is possible $n=2$ and A, B, C etc and are natural numbers.

On application shake this relation can generate a new class mathematical formalism for application of Fermat theorem. Most of the time the application of basic mathematics come late and it is expected that this extension will introduce a new class of cryptography which is under study.

2. Conclusions

Any products of odd numbers or prime number (p) (which is product of p and 1) can generate family of Fermat last theorem. Fermat last theorem was expressed with three natural numbers.

It has been shown that $A^n+B^n+C^n+D^n+\cdots=X^n$ is possible for $n=2$ where A, B, C etc are natural numbers. It is an extension of Fermat relation.

Acknowledgements

This article is our tribute to Pierre de Fermat, a great mathematician of world. I express my thank to Dr Arindam Pramanik for preparing the manuscript and to Indian Statistical Institute, Kolkata, India for academic help We also recognize GLA Univesrsity, Mathura, India for technical help.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Singh, S. (1997) Fermat's Last Theorem: The Story of a Riddle that Confounded the World's Greatest Minds for 358 Years.

[2] Wiles, A. (1995) Modular Elliptic Curves and Fermat's Last Theorem. *Annals of Mathematics,* **141**, 443-551.

[3] Hilbert, D. (1897) Die Theorie der algebraischen Zahlkörper: Bericht, erstattet der Deutschen Mathematiker-Vereinigung. Georg Reimer, Berlin.

[4] de Bessy, F. (1676) Traité des Triangles Rectangles en Nombres. Hachette Livre-BNF, Paris.

[5] Euler, L. and Euler, L. (1917) *Leonhardi Euleri Opera omnia: Series 1. Opera mathematica. Commentationes arithmeticae*. Teubner, Stuttgart.

[6] Kausler, C.F. (1802) Nova demonstratio theorematis nec summam, nec differentiam duorum cuborum cubum esse posse. *Acta Academiae Scientiarum Imperialis Petropolitanae*, **13**, 245-253.

[7] Barlow, P. (1811) An Elementary Investigation of the Theory of Numbers: With Its Application to the Indeterminate and Diophantine Analysis, the Analytical and Geometrical Division of the Circle, and Several Other Curious Algebraical and Arithmetical Problems. J. Johnson and Company, Hong Kong.

[8] Legendre, A.M. (1830) Théorie des nombres. Vol 2, Firmin Didot Frères, Paris.

[9] Betrand, J.L.F. (1851) Traité élémentaire d'algèbre. L. Hachette et cie, Paris.

[10] Lebesgue, V.A. (1853) Resolution des Équations biquadratiques z2 = x4±2my4, z2 = 2mx4y4, 2mz2 = x4±y4. *Journal de Mathématiques Pures et Appliquée*, **18**, 73-86. https://doi.org/10.2307/2118559

[11] Tafelmacher, A. (1893) Sobre la ecuacion x4 + y4 = z4. 307 p. https://revistapsicologia.uchile.cl/index.php/ANUC/article/view/20645

[12] Gambioli, D. (1901) Memoria bibliografica sull'ultimo teorema di Fermat. 145-192.

[13] Bang, A. (1905) Nyt Bevis for at Ligningen (I) x 4—Z 4 = y 4 ikke kan have rationale Løsninger. *Nyt Tidsskrift for Matematik*, **16**, 35-36.

[14] Sommer, J. (1907) Vorlesungen über zahlentheorie: Einfuhrung in die theorie der algebraischen zahlkorper. BG Teubner, Cambridge.

[15] Nutzhorn, F. (1912) Den ubestemte Ligning x 4 + y 4 = z 4. *Nyt Tidsskrift for Matematik*, **23**, 33-38.

[16] Carmichael, R.D. (1913) On the Impossibility of Certain Diophantine Equations and Systems of Equations. *The American Mathematical Monthly*, **20**, 213-221. https://doi.org/10.1080/00029890.1913.11997962

[17] Hancock, H. (1925) The Foundations of the Theory of Algebraic Numbers. *Science*, **61**, 5-10. https://doi.org/10.1126/science.61.1566.5

[18] Grant, M. and Perella, M. (1999) 83.25 Descending to the Irrational. *The Mathematical Gazette*, **83**, 263-267. https://doi.org/10.2307/3619054

[19] Barbara, R. (2007) 91.33 Fermat's Last Theorem in the Case n = 4. *The Mathematical Gazette*, **91**, 260-262. https://doi.org/10.1017/S002555720018163X

[20] Alexanderson, G. (2012) About the Cover: Sophie Germain and a Problem in Number Theory. *Bulletin of the American Mathematical Society*, **49**, 327-331. https://doi.org/10.1090/S0273-0979-2012-01370-8

[21] Mazur, B. and Weil, A. (1977) Ernst Edward Kummer, Collected Papers. Springer Verlog, Berlin, Hiedelberrg, New York.

[22] Kummer, E.E. and Weil, A. (1975) Collected Papers II Function Theory. Geometry and Miscellaneous. Springer, Berlin, Hiedelberrg, New York.

[23] Buhler, J., Crandall, R., Ernvall, R. and Metsänkylä, T. (1993) Irregular Primes and Cyclotomic Invariants to Four Million. *Mathematics of Computation*, **61**, 151-153. https://doi.org/10.1090/S0025-5718-1993-1197511-5