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A Key to Solving the Angle Trisection Problem

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Abstract

This paper describes the methodology (or approach) that was key to the solution of the angle trisection problem published earlier in article entitled, "***A Procedure For Trisecting An Acute Angle***." It was an approach that required first, designing a working model of a trisector mechanism, second, studying the motion of key elements of the mechanism and third, applying the fundamental principles of kinematics to arrive at the desired results. In presenting these results, since there was no requirement to provide a detailed analysis of the final construction, this information was not included. However, now that the publication is out, it is considered appropriate as well as instructive to explain more fully the mechanism analysis of the trisector in graphical detail, as covered in Section 3 of this paper, that formed the basis of the long sought solution to the age-old *Angle Trisection Problem.*

Keywords

Archimedes' Construction, College Geometry, College Mathematics, Angle Trisection, Trisector, Famous Problems in Mathematics, History of Mathematics, Mechanism Analysis, Kinematics, Geometer's Sketch Pad

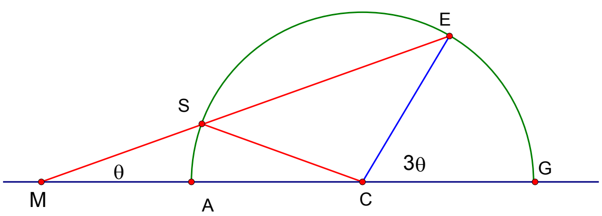
1. Introduction

This paper is a sequel to its forerunner entitled, "***A Procedure For Trisecting an Acute Angle***," published earlier [1] where it presented a construction capable of dividing any arbitrary acute angle into three exactly equal parts using a straightedge and compass only [1] [2] [3] [4]. In presenting this construction, there was no need to describe the methodology for its development, since it was not required. However, given the fact that a) *The Angle Trisection* (one of the three unsolvable problems of Geometry, the other two being *the squaring of a circle* and *the doubling of a cube*), has been for centuries one of the most intriguing challenges for mathematicians, and b) nowhere else in the literature [2] [3] [4] [5] has the principles of kinematics [1] [6] [7] [8] been applied in tackling this problem, it is considered appropriate as well as instructive, to document the kinematic details [6] that formed the basis of said construction, so that readers who are interested in this background would be better informed.

The application of kinematics principles [1] [6] [7] [8] for solving the trisection problem was an idea prompted by an assertion that "while the angle trisection could not be achieved **using an unmarked straightedge and compass**, yet a mechanism can be built to perform this task perfectly" [6] [8] [9]. Hence, performing a motion analysis on an actual trisector (See **Figure A1**) seemed a logical approach for seeking to gain an insight to the trisection problem [1] [6] [7] [8].

2. Theory

As stated in the reference article [1], the procedure is based on the well-known **Archimedes' Construction** [2] represented in the diagram below that illustrates the **geometric requirements** that must be met in order to arrive at **an exact trisection**, and the general theorem relating to arcs and angles.



Let $\angle$ECG (or $3\angle\theta$) be the required angle to be trisected. With center at C and radius CE describe a semicircle. Given that a line from point E can be drawn to cut the semicircle at S and intersect the extended side GC at some point M such that the distance SM is equal to the radius SC, then from the general theorem relating to arcs and angles,

$$\angle EMG = 1/2(\angle ECG - \angle SCM) \quad (1)$$

$$2\angle EMG + \angle SCM = \angle ECG \quad (2)$$

Since ΔCSM is an isosceles Δ

$$\angle SCM = \angle EMG = \angle\theta \quad (3)$$

Therefore,

$$3\angle EMG = \angle ECG \text{ or } 3\angle\theta = \angle ECG \text{ or } \angle EMA = 1/3\angle ECG. \quad (4)$$

TO SUMMARIZE:

Geometric Requirements for EXACT TRISECTION are:

1) Segments SM, SC, EC, and CG are all equal, and 2) $\angle$SMA = $\angle$SCA,

ONCE these requirements are met,

THEN, the EXACT trisection of the given angle ($\angle$E'CG) is achieved or $\angle$EMA = 1/3$\angle$ECG.

NOTE also that, except for the given angle $\angle$ECG being an acute angle, there are no other restrictions on the measure of this angle.

Therefore, the measure of ∠ECG can be any real number (or Arbitrary).

3. Trisector Design and Analysis

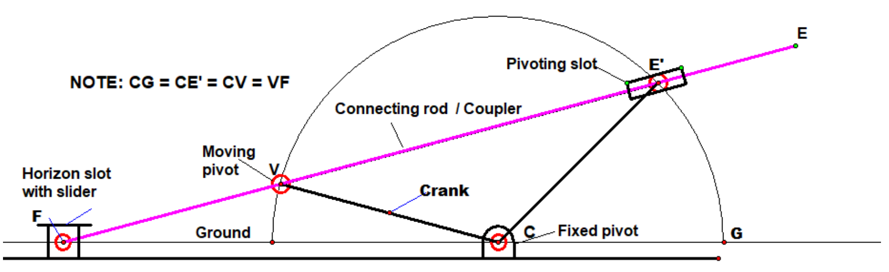
The trisector mechanism [6] illustrated in **Figure 1(a)** is modeled after the Archimedes construction [2] discussed above. This is a compound mechanism [6] [7] [8] consisting of a slider-crank linkage CVF [6] [7] [8] and a sliding-coupler linkage CVE [6] [7] [8], where both linkages share a common crank CV and a common connecting rod E'F. Also link section VF and cranks CV and CE are all equal in length.

Mechanism operating as a slider-crank [6] [7] [8] (**Figure 1(b)**):

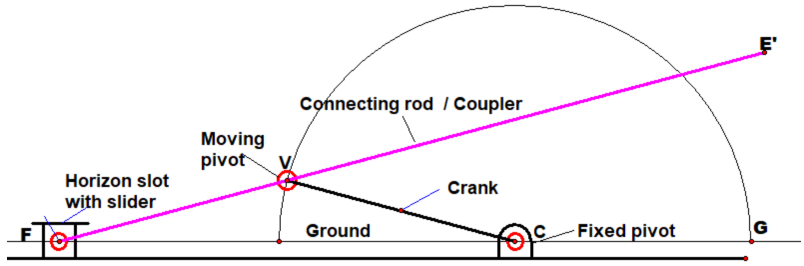
In this operating mode, as crank CV is rotated in one direction or the other, between the 180˚, and 90˚, positions, the connecting rod E'F undergoes combined motion, where sliding occurs only at the end F, as the rod is constrained to move within the fixed horizontal slot, while both sliding and rotation occur at the other end E', where the rod moves within the pivoting slot. Meanwhile, the angle that the connection rod E'F makes with the horizontal slot maintains a constant relationship that is 1/3 of the angle formed by link CE and said slot. Or, ∠E'FC = 1/3∠E'CG.

Mechanism operating as a sliding-coupler [6] [7] [8] (**Figure 1(c)**, and **Figure 1(d)**).

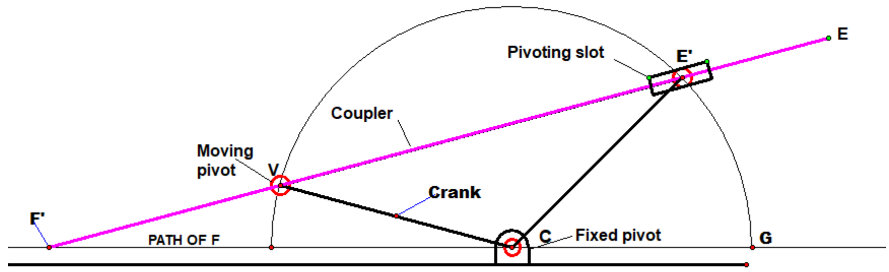
For this operating mode, we assume link CE' is held at a fixed angular position (*i.e.* the angle to be trisected) and the connecting rod E'F is disconnected from the horizontal slider and renamed E'F'. Therefore, as crank CV is rotated in one direction or the other, the mechanism then behaves like a sliding-coupler, where E'F', acting like a coupler, undergoes sliding and rotation at the pivoting slot end E' and pure rotation at the free end F', since F' is not constrained as before to move within the horizontal slot. In this mode, it can be seen that the path of F' (See **Figure 1(d)**} is actually a smooth circular path that intercepts the horizontal path that it would normally describe when constrained within the horizontal slot. This point of interception is **a unique point**, as it locates the vertex of **the required trisection angle (∠E'MG or ∠E'MA = 1/3∠E'CG)**, formed by the connecting rod E'M and the horizontal slot to comply with the **Archimedes' Construction [2]**. See **Figure 1(e)** and note **the identical angular relationship** between this figure and **Archimedes' Construction** [2] shown in Section 2 on THEORY.



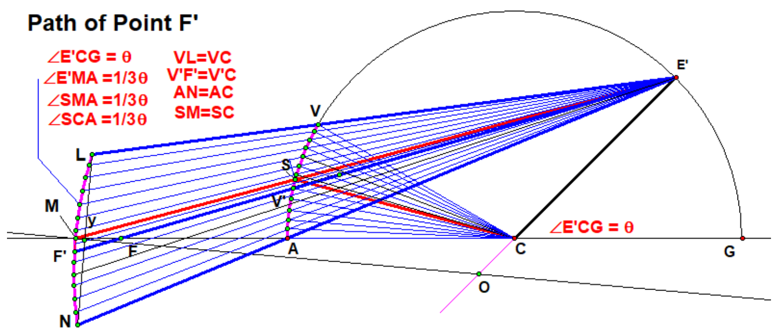
(a)



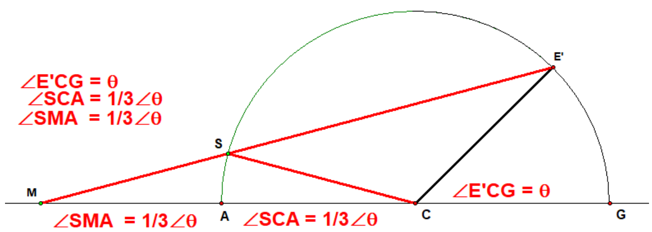
(b)



(c)



(d)



(e)

Figure 1. (a) Trisector mechanism; (b) Trisector mechanism in slider-crank mode; (c) Trisector mechanism in sliding-coupler mode; (d) Path of F' for angle θ trisection; (e) Resultant angles for angle θ trisection.

4. Procedure

To illustrate the procedure, we consider the 30˚ angle that has been "proven" to be not trisectable to represent **a typical acute angle**, and the 45˚ angle that is *known* to be trisectable as a benchmark angle, since this angle is known to be trisectable. Then, let it be required to *develop a construction for dividing each of these angles into three exactly equal parts, using an unmarked straightedge and* *compass only*. The constructions for these angles are given in the following Figure 2(a) to Figure 3(c).

STEP 1 See Figure 2(a)

1) Using CG as the base, erect a perpendicular CC' at C.

2) With center at C and any convenient radius, describe a semicircle from point G cutting perpendicular CC' at E, and terminating at A on GC (extended).

3) Using CE as a base, form an equilateral triangle CEV, where V is the vertex.

4) Extend segment EV to meet GC (extended) at a point F.

STEP 2 See Figure 2(b) and Figure 3(a)

1) Using point E as center and CE as radius, describe an arc cutting the semicircle in STEP 1 at point E' to form the given angle $\angle$E'CG = 30˚ for Figure 2(b) and $\angle$E'CG = 45˚ for Figure 3(a).

2) Construct a ray from E' through point V and locate point L on said ray such that VL = VC.

3) Construct segment E'F, cutting the semicircle at V'.

4) Join V' to C with segment V'C and extend segment V'F to V'F' such that V'F' = V'C.

5) Construct a ray from E' through point A and locate point N on said ray such that AN = AC.

STEP 3 See Figure 2(c) and Figure 3(b)

1) Join L to N with segment NL.

2) Locate midpoint Y of segment NL.

3) At midpoint Y, construct a perpendicular line to segment NL meeting line E'C (ext'd) at point O.

4) With center at O and radius ON describe an arc <NF'L> from N through F' to L, cutting the baseline GF (extended) at a point M.

Note that center O for circular path of F' could also be found using points F' and L.

5) Join E' to M with E'M cutting AV at a point S to form the required trisection angle, $\angle$E'MG, and making $\angle$E'MA = 1/3$\angle$E'CG, in compliance with the **Archimedes' Construction** [2].

STEP 4

Join S to C with a segment SC to complete the construction, which makes segment SM = SC. See Figure 2(d) and Figure 3(c), and note **the identical angular relationship** between this figure and **Archimedes' Construction** [2] shown in Section 2 on THEORY.

DECLARATION:

While the Geometer's Sketch Pad [10] *was employed in this procedure, the use of the software is not a violation of the straightedge and compass rule, since its sole purpose was for* (1) *the layout of lines and arcs* (*with precision but no measurements taken, except at the end for final results*), *and* (2) *for color coding, which is appropriate for an effective graphical presentation.*

Otherwise, the construction can easily be hand drawn.

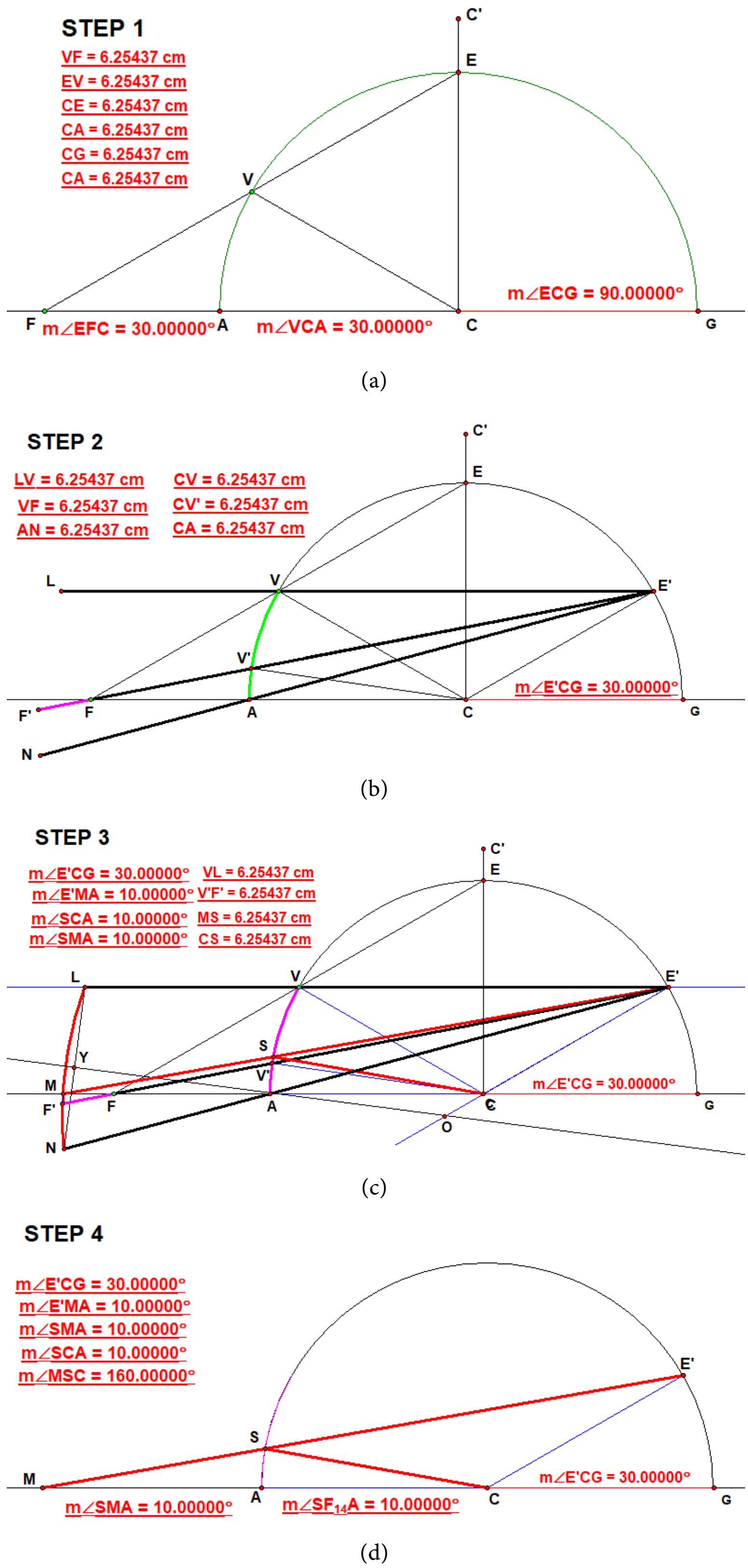


Figure 2. (a) Archimedes' construction for 90˚ trisection; (b) 30X10 STEP 2 Construction; (c) Composite construction for 30˚ angle trisection; (d) Resultant angles for 30˚ angle trisection.

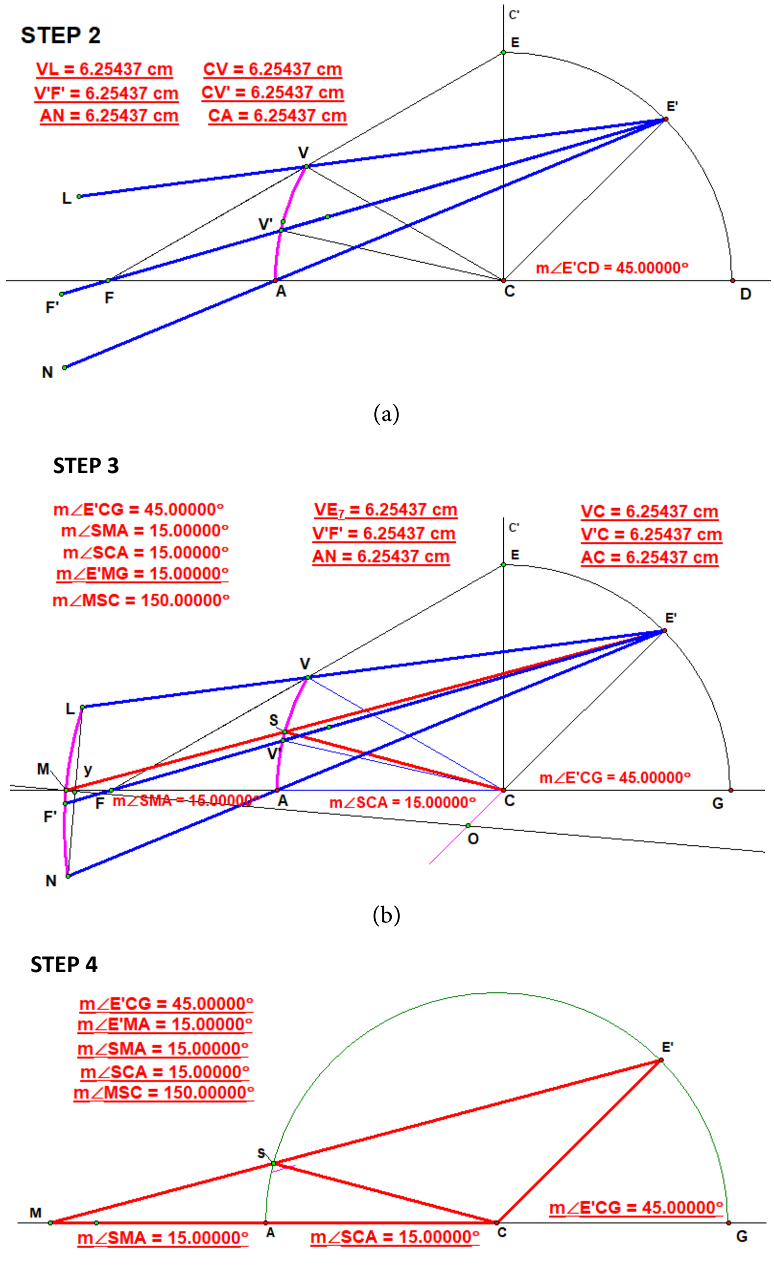


Figure 3. (a) 30X10 STEP 2 construction; (b) Composite construction for 45˚ trisection; (c) Resultant angles for 45˚ trisection.

5. Proof

Referring to Figure 2(d) and Figure 3(c), and applying the general theorem relating to arcs and angles (See Section 2 on THEORY of this paper), we get

$$\angle E'MG = 1/2(\angle E'CG - \angle SCM) \text{ or } \angle E'MA = 1/2(\angle E'CG - \angle SCM)$$

$$2\angle E'MA = \angle E'CG - \angle SCM$$

$$2\angle E'MA + \angle SCM = \angle E'CG$$

Since $\angle SCM = \angle E'MA$

Then 3∠E’MA = ∠E’CG.

Therefore

For the 30˚ angle: ∠E’MA = 1/3∠E’CG = 1/3(30˚) = 10.00000˚ (QED}

For the 45˚ benchmark angle: ∠E’MA = 1/3∠E’CG = 1/3(45˚) = 15.00000˚ (QED).

6. Summary

This paper has described the approach that was key to solving the angle trisection problem in the reference article [1]. The approach was first, to design a trisector mechanism [6], modeled after the well-known **Archimedes’ Construction [2]**, second, to study the motion of the key elements of this mechanism, and third, to apply fundamental principles of kinematics (*i.e.* displacement analysis) [6] [7] [8], to arrive at the desired result. The procedure when applied to the mechanism at the 30˚ angle that has been “proven” to be not-trisectable, and the benchmark angle 45˚ that is known to be trisectable, each produced a construction having an **identical angular relationship** with **Archimedes’ Construction** [2] in which the required trisection angle was found to be one-third of their respective given angles. For example, the trisection angle (*i.e.* ∠EMA = 1/3 ∠ECG) for the 30˚ angle was 10.00000˚, and the same for the benchmark angle 45˚ was 15.00000˚, as shown in **Figure 2(d)** and **Figure 3(c)** and Section 4 on Proof. Based on said **identical angular relationship** and **the numerical results** (*i.e.* to five decimal places), which represent the highest degree of accuracy and precision attainable by The Geometer’s Sketch Pad software [10], one can only conclude that the geometric requirements for arriving at an ***exact*** trisection of the 30˚ angle (which has been “proven” to be not-trisectable) have been met,

Furthermore, the construction, by demonstrating its capability of trisecting both trisectable (*i.e.* 45˚) and the “proven” non-trisectable (*i.e.* 30˚), it has also demonstrated its validity for trisecting any arbitrary acute angle. Therefore, one can only conclude that the long sought solution to the age-old Angle Trisection Problem has been finally accomplished, notwithstanding the theoretical proofs of Wantzel, Dudley, and others [2] [3] [4] [5] [11]-[16].

Note that, as stated in the reference article [1], the use of a trisector model [6] was *only* to study and gain an understanding of the motion of key elements. Therefore, it is not a violation of the unmarked straightedge and compass rule.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix

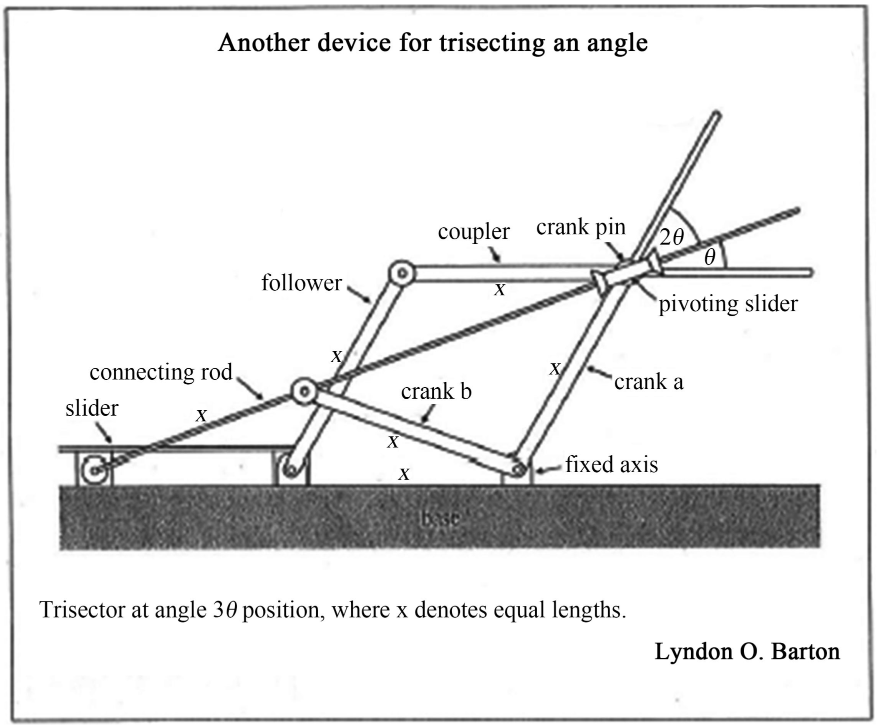

Figure A1. Angle trisector model.