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Canonical Treatment of Elliptical Motion

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Abstract

The constrained motion of a particle on an elliptical path is studied using Hamiltonian mechanics through Poisson bracket and Lagrangian mechanics through Euler Lagrange equation using non-natural Lagrangian. We calculate the generalized momentum p_θ and we find that this quantity is not conserved and the conjugate coordinate θ is not a cyclic coordinate.

Keywords

Hamilton Jacobi Equation, Generalized Momentum, Elliptical Motion, Cyclic Coordinate

1. Introduction

In classical mechanics, the Hamilton Jacobi equation used to integrate a Hamiltonian system of differential equations using canonical method [1] [2] [3] [4].

The Hamilton Jacobi formalism has been developed by [5] [6], through this formalism, the action function was formulated using Hamilton Jacobi equation, then the equations of motion were obtained, this function helps one to obtain the conjugate momentum.

The Hamilton Jacobi formalism with the canonical method for second order singular Lagrangians was developed using Caratheodory's equivalence Lagrangian method by Pimentel and Teixeira [7]. In this approach, the equations of motion for the canonical variables of singular second order systems were obtained as total differential equations in many variables, and the Hamilton Jacobi partial differential equations for second order singular systems were investigated. The Hamilton Jacobi equation to higher order singular Lagrangians was presented by [8].

Recently, the Hamilton Jacobi partial differential equations have been studied for systems containing fractional derivatives using the canonical method [9] [10]. More recently, a powerful approach, the canonical method, has been developed for dissipative systems [11]. In this approach, the equations of motion are written as total differential equations and the formulation leads to a set of Hamilton Jacobi partial differential equations which are familiar to regular systems.

An elliptical orbit can be defined as the oval shaped. For example, the planets revolving around the sun in the solar system follow elliptical orbits [12] [13]. Thus, most of the objects in outer space follow elliptical orbit. The size of an ellipse is measured by the major and minor axis. The major axis measures the longest distance across the ellipse while the minor axis measures the shortest. The point at which the planet is closest to the earth in an elliptical orbit is called the perigee. The point at which the planet is furthest from the earth in an elliptical orbit is called the apogee.

Then the particle moving on an ellipse as a constrained system is presented by [14], where the motion is analyzed using Hamilton Jacobi equation using natural Lagrangian and Dirac's approach. In this work we will study the motion of a particle on an elliptical path using non-natural Lagrangian.

The paper is organized as. In Section 2, we discuss the motion of a constrained particle on an ellipse within the framework of the Hamiltonian mechanic and Lagrangian mechanics treatment. In Section 3, we present a conclusion.

2. Hamiltonian Treatment for the Motion of a Particle on an Elliptical Path

In this section we will present a particle moving on an ellipse as a constrained system by considering the motion in the horizontal xy-plane, the Lagrangian that describes the motion of a particle on an ellipse is expressed as [14]:

$$L = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right) \tag{1}$$

using the non-natural Lagrangian Equation (1) becomes:

$$L = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right) \mathrm{e}^{\lambda t} \tag{2}$$

The constraint equation is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{3}$$

using the parametric equations,

$$x = a \cos\theta$$
$$y = b \sin\theta$$

Substituting these parametric equations into Equation (2) we have:

$$L = \frac{1}{2} m \dot{\theta}^2 \left(a^2 \sin^2\theta + b^2 \cos^2\theta \right) \mathrm{e}^{\lambda t} \tag{4}$$

- The Hamiltonian Mechanics:

In this section we will use Hamilton Jacobi equation.

From the Lagrangian that formulated in Equation (4), we can find the generalized momentum as follows:

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m\dot{\theta}\left(a^2 \sin^2\theta + b^2 \cos^2\theta\right)\mathrm{e}^{\lambda t} \tag{5}$$

Then,

$$\dot{\theta} = \frac{\mathrm{e}^{-\lambda t} p_\theta}{m\left(a^2 \sin^2\theta + b^2 \cos^2\theta\right)} \tag{6}$$

Squaring of Equation (6)

$$\dot{\theta}^2 = \frac{\mathrm{e}^{-2\lambda t} p_\theta^2}{m^2\left(a^2 \sin^2\theta + b^2 \cos^2\theta\right)^2} \tag{7}$$

Using the standard form of the Hamiltonian:

$$H_0 = p\dot{\theta} - L \tag{8}$$

Inserting Equation (4) and Equation (6) into Equation (8), our Hamiltonian is:

$$H = \frac{\mathrm{e}^{-\lambda t} p_\theta^2}{m\left(a^2 \sin^2\theta + b^2 \cos^2\theta\right)} - \frac{1}{2m}\frac{\mathrm{e}^{-\lambda t} p_\theta^2}{\left(a^2 \sin^2\theta + b^2 \cos^2\theta\right)} \tag{9}$$

Finally,

$$H = \frac{\mathrm{e}^{-\lambda t} p_\theta^2}{2m\left(a^2 \sin^2\theta + b^2 \cos^2\theta\right)} \tag{10}$$

The Poisson bracket of two functions, A, B, with respect to the canonical variables q, p is written as [3]:

$$\left[A, B\right]_{q,p} = \sum_{i=1}^{N} \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} \tag{11}$$

where A and B are functions of the generalized coordinates q_i and the generalized momenta p_i. The total time derivative of some function of the canonical variables and time, $A(q, p, t)$, using Poisson bracket and Hamilton's equations of motion is written as:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \sum_{i=1}^{N}\left(\frac{\partial A}{\partial q_i}\frac{\mathrm{d}q_i}{\mathrm{d}t} + \frac{\partial A}{\partial p_i}\frac{\mathrm{d}p_i}{\mathrm{d}t}\right) = \sum_{i=1}^{N}\left(\frac{\partial A}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial H}{\partial q_i}\right) \tag{12}$$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \left[A, H\right] = \dot{A}. \tag{13}$$

So that,

$$\dot{p}_\theta = \left[p_\theta, H\right] = \frac{1}{2}m\dot{\theta}^2\mathrm{e}^{\lambda t}\left(a^2 - b^2\right)\sin 2\theta \tag{14}$$

This means that; p_θ is not a constant of motion (is not conserved) and θ is not a cyclic coordinate.

- Lagrangian mechanics:

In this section we will use Euler Lagrange equation in the following form [3]:

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \tag{15}$$

Then using θ coordinate, Euler Lagrange equation becomes:

$$\frac{\partial L}{\partial \theta}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right)=0 \tag{16}$$

Remembering that,

$$p_\theta=\frac{\partial L}{\partial \dot{\theta}} \tag{17}$$

Substituting of Equation (17) into Equation (16) we find;

$$\frac{\partial L}{\partial \theta}-\frac{\mathrm{d}}{\mathrm{d}t}\left(p_\theta\right)=0 \tag{18}$$

Thus,

$$\frac{\partial L}{\partial \theta}-\dot{p}_\theta=0 \tag{19}$$

which means;

$$\frac{\partial L}{\partial \theta}=\dot{p}_\theta \tag{20}$$

Makin use of Equation (4) we get,

$$\frac{\partial L}{\partial \theta}=\frac{1}{2}m\dot{\theta}^2\mathrm{e}^{\lambda t}\left(a^2-b^2\right)\sin 2\theta=\dot{p}_\theta \tag{21}$$

Thus,

$$\frac{1}{2}m\dot{\theta}^2\mathrm{e}^{\lambda t}\left(a^2-b^2\right)\sin 2\theta=\dot{p}_\theta \tag{22}$$

This result obtained from Equation (22) using Euler Lagrange equation is in exact agreement with the result that obtained from Equation (14) using Poisson bracket then, we find that; $\dot{p}_\theta \neq 0$.

So that we can say; for non-natural Lagrangian the generalized momentum p_θ conjugate to the coordinate θ is not conserved (is not a constant of motion); because θ is not a cyclic coordinate.

3. Conclusion

In this paper the constrained motion of a particle on an elliptical path is studied using Lagrangian mechanics through Euler Lagrange mechanics, using non-natural Lagrangian. Then, we calculate the generalized momentum p_θ and we find that this quantity is not conserved and the conjugate coordinate θ is not a cyclic coordinate. For natural Lagrangian which means at the limits $\lambda \to 0$, the same result is obtained using Hamiltonian mechanics in Ref. [14] using Poisson bracket.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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