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Some Refinement of Holder's and Its Reverse Inequality

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Abstract

Holder's inequality, its refinement, and reverse have received considerable attention in the theory of mathematical analysis and differential equations. In this paper, we give some refinements of Holder's inequality and its reverse using a simple analytical technique of algebra and calculus. Our results show many results related to holder's inequality as special cases of the inequalities presented.

Keywords

Young's Inequality, Kittaneh-Manasrah's Inequality, Integrable Function, Holder's Cauchy-Schwarz Inequality

1. Introduction

Holder's inequality is a fundamental inequality in mathematical analysis that generalizes the Cauchy-Schwarz inequality to multiple sequences and different exponents. It is used in many areas of mathematics such as probability theory, functional analysis, and differential equations [1]. The inequality has been refined and reversed in many ways over the years [2]. For example, the reverse Holder inequality is used to deal with square (or higher-power) roots of expressions in inequalities since those can be eliminated through successive multiplication [3]. Both the holder's inequality and Cauchy play an important role in many areas of mathematics [1]. Several authors have studied and obtained the generalization, refinement, sharpening, variation, and application of this inequality in the literature. A family of inequalities concerning inner products of vectors and functions began with Cauchy [4]. The extension and generalizations later led to inequalities of Schwarz, Minkowski, and Holder. Inequalities appear frequently in algebra, geometry, and analysis; they are powerful mathematical tools that appear across different areas of mathematics, helping mathematicians and scientists describe relationships, establish limits and bounds, and solve a wide variety of problems [2]. Many researchers have worked on generalization of Holder, its reverse, and refinement (see for example [1]-[19]).

At the heart of Holder's inequality lies a remarkable mathematical relationship. Given real number p, q, and r such that $1 < p, q, r \le \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, and measurable functions f and g defined on a measurable space, Holder's inequality can be succinctly stated as follows:

$$\int \left| f(x) \cdot g(x) \right| \mathrm{d}x \le \left(\int \left| f(x) \right|^p \mathrm{d}x \right)^{\frac{1}{p}} \cdot \left(\int \left| g(x) \right|^q \mathrm{d}x \right)^{\frac{1}{q}} \tag{1}$$

Holder's inequality has significant implications in various branches of mathematics and analysis, including functional analysis, probability theory, and partial differential equations [1]. It is particularly useful in proving convergence properties of sequences of functions, estimating norms of integral operators, and establishing relationships between different function spaces. Holder's inequality is a crucial concept in mathematics, providing a connection between norms, integrals, and inner products of functions and vectors. Refinements of Holder's inequality involve adjusting the exponents or introducing additional terms to obtain more accurate upper bounds for specific situations [3] [5]. These refinements are valuable when dealing with particular types of functions or when extra information about the functions is available. By tailoring the inequality, refinements yield sharper estimates and reveal nuanced relationships between functions [5] [6] [7]. On the other hand, reverses of Holder's inequality focus on establishing lower bounds for the given expression [1] [5] [8]. While the original inequality provides an upper bound, a reverse inequality gives insight into the minimum possible value. Reverses contribute to proving the optimality of Holder's inequality and understanding the tightness of the bounds it establishes. They're especially useful when trying to characterize scenarios in which functions are interdependent in specific ways.

The aim of this paper is achieved through the following objectives: 1) to use algebraic and calculus techniques to improve upper bounds by refining Holders inequality; 2) to explore lower bounds through the reverse of Holders' inequalities refinement. The study is of great importance in Mathematical analysis, information theory, theory of elasticity, and others. In order to prove the main results, we need the following lemma.

2. Lemmas

The following two lemmas will be needed throughout the proof of our theorems.

Lemma 2.1 Let $a, b \ge 1$ and $\lambda \in (0,1)$ we have

$$s\left(\sqrt{a}-\sqrt{b}\right)^2+A(\lambda)\log^2\left(\frac{a}{b}\right)\le \lambda a+(1-\lambda)b-a^{\lambda}b^{1-\lambda}$$
$$\le (1-s)\left(\sqrt{a}-\sqrt{b}\right)^2+B(\lambda)\log^2\left(\frac{a}{b}\right),$$

where $s=\min\{\lambda,1-\lambda\}$, $A(\lambda)=\frac{\lambda(1-\lambda)}{2}-\frac{s}{4}$ and $B(\lambda)=\frac{\lambda(1-\lambda)}{2}-\frac{1-s}{4}$.

Lemma 2.2 Let $0<a,b\le 1$, and $\lambda\in(0,1)$ we have

$$s\left(\sqrt{a}-\sqrt{b}\right)^2+A(\lambda)ab\log^2\left(\frac{a}{b}\right)\le \lambda a+(1-\lambda)b-a^{\lambda}b^{1-\lambda}$$
$$\le (1-s)\left(\sqrt{a}-\sqrt{b}\right)^2+B(\lambda)ab\log^2\left(\frac{a}{b}\right),$$

where $s, A(\lambda), B(\lambda)$ are given in lemma 2.1.

3. Main Results

Theorem 2.1. Let $1<p<\infty$, $1<q<\infty$, $1\le r<\infty$, with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. If f and g are two positive functions which admit integral on $[a, b]$ for which there exist $\int_a^b f^p(x)\mathrm{d}x$ and $\int_a^b g^q(x)\mathrm{d}x$ finite with $\int_a^b f^p(x)\mathrm{d}x>0$, $\int_a^b g^q(x)\mathrm{d}x$ and

$$1<\frac{f^p(x)}{\int_a^b f^p(x)\mathrm{d}x}\frac{\int_a^b g^q(x)\mathrm{d}x}{g^q(x)}\le M,\quad \forall x\in[a,b].$$

Then, we have

$$1-\frac{\int_a^b f(x)\left(g(x)\right)^{q\left(1-\frac{1}{p}\right)}\mathrm{d}x}{\left(\int_a^b f^p(x)\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_a^b g^q(x)\mathrm{d}x\right)^{1-\frac{1}{p}}}$$
$$\le \frac{2}{\min\left\{\frac{1}{p},\frac{p-1}{p}\right\}}\left[1-\frac{\int_a^b f^{\frac{p}{2}}(x)g^{\frac{q}{2}}(x)\mathrm{d}x}{\left(\int_a^b f^p(x)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_a^b g^q(x)\mathrm{d}x\right)^{\frac{1}{2}}}\right] \tag{2}$$
$$+\left(\frac{p-1}{2p^2}-\frac{1}{4\min\left\{\frac{1}{p},\frac{p-1}{p}\right\}}\right)\log^2(M)$$

Proof: From lemma 2.1, let $b=1$, $\lambda=\frac{1}{p}$, $1-\lambda=1-\frac{1}{p}$, we have

$$\frac{1}{p}a+\left(1-\frac{1}{p}\right)-a^{\frac{1}{p}}$$
$$\le\left(1-\frac{1}{\max\left\{\frac{1}{p},\frac{p-1}{p}\right\}}\right)\left(\sqrt{a}-1\right)^2+\left[\frac{p-1}{2p^2}-\frac{1}{4}\left(1-\frac{1}{\max\left\{\frac{1}{p},\frac{p-1}{p}\right\}}\right)\right]\log^2 M \tag{3}$$

Substituting $a=\frac{f^{p}(x)}{\int_a^b f^{p}(x)\mathrm{d}x}\frac{\int_a^b g^{q}(x)\mathrm{d}x}{g^{q}(x)}>1$ into (3), we get

$$\frac{1}{p}\frac{f^{p}(x)}{\int_a^b f^{p}(x)\mathrm{d}x}\frac{\int_a^b g^{q}(x)\mathrm{d}x}{g^{q}(x)}+\left(1-\frac{1}{p}\right)-\left(\frac{f^{p}(x)}{\int_a^b f^{p}(x)\mathrm{d}x}\frac{\int_a^b g^{q}(x)\mathrm{d}x}{g^{q}(x)}\right)^{\frac{1}{p}}$$

$$\leq\left(1-\frac{1}{\max\left\{\frac{1}{p},\frac{p-1}{p}\right\}}\right)\left[\frac{f^{p}(x)}{\int_a^b f^{p}(x)\mathrm{d}x}\frac{\int_a^b g^{q}(x)\mathrm{d}x}{g^{q}(x)}-2\frac{f^{\frac{p}{2}}(x)}{\left(\int_a^b f^{p}(x)\mathrm{d}x\right)^{\frac{1}{2}}}\frac{\left(\int_a^b g^{q}(x)\mathrm{d}x\right)^{\frac{1}{2}}}{g^{\frac{q}{2}}(x)}+1\right] \tag{4}$$

$$+\left[\frac{p-1}{2p^{2}}-\frac{1}{4}\left(1-\frac{1}{\max\left\{\frac{1}{p},\frac{p-1}{p}\right\}}\right)\right]\log^{2}(M)$$

Simplifying (4) completely, we have

$$\frac{1}{p}\frac{f^{p}(x)}{\int_a^b f^{p}(x)\mathrm{d}x}+\frac{g^{q}(x)}{\int_a^b g^{q}(x)\mathrm{d}x}-\frac{1}{p}\frac{g^{q}(x)}{\int_a^b g^{q}(x)\mathrm{d}x}-\frac{f(x)}{\left(\int_a^b f^{p}(x)\mathrm{d}x\right)^{1-\frac{1}{p}}}$$

$$\leq\left(1-\frac{1}{\max\left\{\frac{1}{p},\frac{p-1}{p}\right\}}\right)\left[\frac{f^{p}(x)}{\int_a^b f^{p}(x)\mathrm{d}x}-2\frac{f^{\frac{p}{2}}(x)g^{\frac{q}{2}}(x)}{\left(\int_a^b f^{p}(x)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_a^b g^{q}(x)\mathrm{d}x\right)^{\frac{1}{2}}}+\frac{g^{q}(x)}{\int_a^b g^{q}(x)\mathrm{d}x}\right] \tag{5}$$

$$+\left[\frac{p-1}{2p^{2}}-\frac{1}{4}\left(1-\frac{1}{\max\left\{\frac{1}{p},\frac{p-1}{p}\right\}}\right)\right]\log^{2}(M)\frac{g^{q}(x)}{\int_a^b g^{q}(x)\mathrm{d}x}$$

By integrating inequality (5), we obtain

$$\frac{1}{p}+1-\frac{1}{p}-\frac{\int_a^b f(x)\left(g(x)\right)^{2\left(1-\frac{1}{p}\right)}\mathrm{d}x}{\left(\int_a^b f^{p}(x)\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_a^b g^{q}(x)\mathrm{d}x\right)^{1-\frac{1}{p}}}$$

$$\leq\left(1-\frac{1}{\max\left\{\frac{1}{p},\frac{p-1}{p}\right\}}\right)\left[2-2\frac{\int_a^b f^{\frac{p}{2}}(x)g^{\frac{q}{2}}(x)\mathrm{d}x}{\left(\int_a^b f^{p}(x)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_a^b g^{q}(x)\mathrm{d}x\right)}\right] \tag{6}$$

$$+\left[\frac{p-1}{2p^{2}}-\frac{1}{4}\left(1-\frac{1}{\max\left\{\frac{1}{p},\frac{p-1}{p}\right\}}\right)\right]\log^{2}(M)$$

Using the fact that, $1-\frac{1}{\max\left\{\frac{1}{p},\frac{p-1}{p}\right\}}=\frac{1}{\min\left\{\frac{1}{p},\frac{p-1}{p}\right\}}$, in (6) to get

$$1-\frac{\int_a^b f(x)(g(x))^{q\left(1-\frac{1}{p}\right)}\mathrm{d}x}{\left(\int_a^b f^p(x)\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_a^b g^q(x)\mathrm{d}x\right)^{1-\frac{1}{p}}}$$

$$\le\frac{2}{\min\left\{\frac{1}{p},\frac{p-1}{p}\right\}}\left[1-\frac{\int_a^b f^{\frac{p}{2}}(x)g^{\frac{q}{2}}(x)\mathrm{d}x}{\left(\int_a^b f^p(x)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_a^b g^q(x)\mathrm{d}x\right)^{\frac{1}{2}}}\right] \tag{7}$$

$$+\left(\frac{p-1}{2p^2}-\frac{1}{4\min\left\{\frac{1}{p},\frac{p-1}{p}\right\}}\right)\log^2(M)$$

This completes the proof.

Theorem 2.2: Let $1<p<\infty$, $1<q<\infty$, $1\le r<\infty$, with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. If *f* and *g* are two positive functions which admits integral on [*a*, *b*] for which there exit $\int_a^b f^p(x)\mathrm{d}x$ and $\int_a^b g^q(x)\mathrm{d}x$ finite with $\int_a^b f^p(x)\mathrm{d}x>0$, $\int_a^b g^q(x)\mathrm{d}x>0$ and

$$m<\frac{f^p(x)}{\int_a^b f^p(x)\mathrm{d}x}\frac{\int_a^b g^q(x)\mathrm{d}x}{g^q(x)}<1,\quad\forall x\in[a,b].$$

Then we have

$$1-\frac{\int_a^b f(x)(g(x))^{q\left(1-\frac{1}{p}\right)}\mathrm{d}x}{\left(\int_a^b f^p(x)\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_a^b g^q(x)\mathrm{d}x\right)^{1-\frac{1}{p}}}$$

$$\le\frac{2}{\min\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\left[1-\frac{\int_a^b f^{\frac{p}{2}}(x)g^{\frac{q}{2}}(x)\mathrm{d}x}{\left(\int_a^b f^p(x)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_a^b g^q(x)\mathrm{d}x\right)^{\frac{1}{2}}}\right] \tag{8}$$

$$+\left(\frac{q-1}{2q^2}-\frac{1}{4\min\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\log^2\left(\frac{1}{m}\right)$$

Proof: From lemma 2.2 let $\lambda=\frac{1}{q}$, $1-\lambda=1-\frac{1}{q}$ and $a=1$ we get

$$\frac{1}{q}+\left(1-\frac{1}{q}\right)b-b^{\frac{1}{p}}$$

$$\le\left[1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right]\left(1-\sqrt{b}\right)^2+\left[\frac{q-1}{2q^2}-\frac{1}{4}\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\right]\log^2\left(\frac{1}{m}\right) \tag{9}$$

Substituting $b=\frac{f^{p}(x)}{\int_a^b f^{p}(x)\mathrm{d}x}\frac{\int_a^b g^{q}(x)\mathrm{d}x}{g^{q}(x)}>1$ into (9) we get

$$\frac{1}{q}+\left(1-\frac{1}{q}\right)\left(\frac{f^{p}(x)}{\int_a^b f^{p}(x)\mathrm{d}x}\cdot\frac{\int_a^b g^{q}(x)\mathrm{d}x}{g^{q}(x)}\right)-\left(\frac{f^{p}(x)}{\int_a^b f^{p}(x)\mathrm{d}x}\cdot\frac{\int_a^b g^{q}(x)\mathrm{d}x}{g^{q}(x)}\right)^{\frac{1}{p}}$$
$$\le\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\left[1-\left(\frac{f^{p}(x)}{\int_a^b f^{p}(x)\mathrm{d}x}\cdot\frac{\int_a^b g^{q}(x)\mathrm{d}x}{g^{q}(c)}\right)^{\frac{1}{2}}\right]^{2} \tag{10}$$
$$+\left[\frac{q-1}{2q^{2}}-\frac{1}{4}\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\right]\log^{2}\left(\frac{1}{m}\right)$$

Simplifying Equation (10) to obtain

$$\frac{1}{q}\frac{g^{q}(x)}{\int_a^b g^{q}(x)\mathrm{d}x}+\frac{f^{p}(x)}{\int_a^b f^{p}(x)\mathrm{d}x}-\frac{1}{q}\frac{f^{p}(x)}{\int_a^b f^{p}(x)\mathrm{d}x}-\frac{f(x)}{\left(\int_a^b f^{p}(x)\mathrm{d}x\right)^{\frac{1}{p}}}\cdot\frac{\{g(x)\}^{q\left(1-\frac{1}{p}\right)}}{\left(\int_a^b g^{q}(x)\mathrm{d}x\right)^{1-\frac{1}{p}}}$$
$$\le\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\left[\frac{g^{q}(x)}{\int_a^b g^{q}(x)\mathrm{d}x}-\frac{2f^{\frac{p}{2}}(x)g^{\frac{q}{2}}(x)}{\left(\int_a^b f^{p}(x)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_a^b g^{q}(x)\mathrm{d}x\right)^{\frac{1}{2}}}+\frac{f^{p}(x)}{\int_a^b f^{p}(x)\mathrm{d}x}\right] \tag{11}$$
$$+\frac{g^{q}(x)}{\int_a^b g^{q}(x)\mathrm{d}x}\left[\frac{q-1}{2q^{2}}-\frac{1}{4}\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\right]\log^{2}\left(\frac{1}{m}\right)$$

On integrating inequality (11), then (11) becomes

$$\frac{1}{q}+1-\frac{1}{q}-\frac{\int_a^b f(x)\left[g(x)\right]^{2\left(1-\frac{1}{p}\right)}\mathrm{d}x}{\left(\int_a^b f^{p}(x)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_a^b g^{q}(x)\mathrm{d}x\right)^{1-\frac{1}{p}}}$$
$$\le\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\left[1-\frac{2f^{\frac{p}{2}}(x)g^{\frac{q}{2}}(x)\mathrm{d}x}{\left(\int_a^b f^{p}(x)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_a^b g^{q}(x)\mathrm{d}x\right)^{\frac{1}{2}}}+1\right] \tag{12}$$
$$+\left[\frac{q-1}{2q^{2}}-\frac{1}{4}\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\right]\log^{2}\left(\frac{1}{m}\right)$$

Using the fact that, $1-\frac{1}{\max\left[\frac{1}{q},\frac{q-1}{q}\right]}=\frac{1}{\min\left\{\frac{1}{q},\frac{q-1}{q}\right\}}$, in (12) we have

$$1-\frac{\int_a^b f(x)(g(x))^{q\left(1-\frac{1}{p}\right)}\mathrm{d}x}{\left(\int_a^b f^p(x)\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_a^b g^q(x)\mathrm{d}x\right)^{1-\frac{1}{p}}}$$

$$\le\frac{2}{\min\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\left[1-\frac{f^{\frac{p}{2}}(x)g^{\frac{q}{2}}(x)}{\left(\int_a^b f^p(x)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_a^b g^q(x)\mathrm{d}x\right)^{\frac{1}{2}}}\right] \tag{13}$$

$$+\left[\frac{q-1}{2q^2}-\frac{1}{4\min\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right]\log^2\left(\frac{1}{m}\right)$$

This completes the proof.

Theorem 2.3. Let $1<p<\infty$, $1<q<\infty$, $1\le r<\infty$, with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. If f and g are two positive functions which admit integral on [a, b] for which there exist $\int_a^b f^p(x)\mathrm{d}x$ and $\int_a^b g^q(x)\mathrm{d}x$ finite with $\int_a^b f^p(x)\mathrm{d}x>0$, and $\int_a^b g^q(x)\mathrm{d}x>0$

$$m<\frac{f^p(x)}{\int_a^b f^p(x)\mathrm{d}x}\frac{\int_a^b g^q(x)\mathrm{d}x}{g^q(x)}<1$$

$$1-\frac{\int_a^b f(x)(g(x))^{q\left(1-\frac{1}{p}\right)}\mathrm{d}x}{\left(\int_a^b f^p(x)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_a^b g^q(x)\mathrm{d}x\right)^{1-\frac{1}{p}}}$$

$$\le\frac{2}{\min\left(\frac{1}{q},\frac{q-1}{q}\right)}\left[1-\frac{2f^{\frac{p}{2}}(x)g^{\frac{q}{2}}(x)\mathrm{d}x}{\left(\int_a^b f^p(x)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_a^b g^q(x)\mathrm{d}x\right)^{\frac{1}{2}}}\right] \tag{14}$$

$$+\left(\frac{q-1}{2q^2}-\frac{1}{4\min\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\log^2\left(\frac{1}{m}\right)$$

Proof: From lemma 2.2 let $\lambda=\frac{1}{q}$, $1-\lambda=1-\frac{1}{q}$ and $a=1$ we have

$$\frac{1}{q}+\left(1-\frac{1}{q}\right)b-b^{\frac{1}{p}}$$

$$\le\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\left(1-\sqrt{b}\right)^2+\left(\frac{q-1}{2q^2}-\frac{1}{4}\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\right)\log^2\left(\frac{1}{m}\right)$$

Substituting $b=\frac{f^p(x)}{\int_a^b f^p(x)\mathrm{d}x}\frac{\int_a^b g^q(x)\mathrm{d}x}{g^q(x)}>1$ into (14) we get

$$
\begin{aligned}
&\frac{1}{q}+\left(1-\frac{1}{q}\right)\frac{f^{p}(x)}{\int_a^p f^{p}(x)\mathrm{d}x}\frac{\int_a^b g^{q}(x)\mathrm{d}x}{g^{q}(x)}-\left(\frac{f^{p}(x)}{\int_a^b f^{p}(x)\mathrm{d}x}\frac{\int_a^b g^{q}(x)\mathrm{d}x}{g^{q}(x)}\right)^{\frac{1}{p}}\\
&\le\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\left(1-\left(\frac{f^{p}(x)}{\int_a^b f^{p}(x)\mathrm{d}x}\frac{\int_a^b g^{q}(x)\mathrm{d}x}{g^{q}(x)}\right)^{\frac{1}{2}}\right)^{2}\\
&\quad+\left(\frac{q-1}{2q^{2}}-\frac{1}{4}\left(1-\frac{1}{\max\left\{\frac{1}{p},\frac{p-1}{p}\right\}}\right)\right)\log^{2}\left(\frac{1}{m}\right)
\end{aligned}
\tag{15}
$$

Simplifying (15) completely to have

$$
\begin{aligned}
&\left(\frac{1}{q}\frac{g^{q}(x)}{\int_a^b g^{q}(x)\mathrm{d}x}+\frac{f^{p}(x)}{\int_a^b f^{p}(x)\mathrm{d}x}-\frac{1}{q}\frac{f^{p}(x)}{\int_a^b f^{p}(x)\mathrm{d}x}-\frac{f(x)}{\left(\int_a^b f^{p}(x)\mathrm{d}x\right)^{\frac{1}{p}}}\frac{\left(g(x)\right)^{q\left(1-\frac{1}{p}\right)}}{\left(\int_a^b g^{q}(x)\mathrm{d}x\right)^{1-\frac{1}{p}}}\right)\\
&\le\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\left[\frac{g^{q}(x)}{\int_a^b g^{q}(x)\mathrm{d}x}-\frac{2f^{\frac{p}{2}}(x)g^{\frac{q}{2}}(x)}{\left(\int_a^b f^{p}(x)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_a^b g^{q}(x)\mathrm{d}x\right)^{\frac{1}{2}}}+\frac{f^{p}(x)}{\int_a^b f^{p}(x)\mathrm{d}x}\right]\\
&\quad+\frac{g^{q}(x)}{\int_a^b g^{q}(x)\mathrm{d}x}\left(\frac{q-1}{2q^{2}}-\frac{1}{4}\left(1-\frac{1}{\max\left(\frac{1}{q},\frac{q-1}{q}\right)}\right)\right)\log^{2}\left(\frac{1}{m}\right)
\end{aligned}
\tag{16}
$$

On integrating Equation (16) with respect to *x*, we get

$$
\begin{aligned}
&\frac{1}{q}+1-\frac{1}{q}-\frac{\int_a^b f(x)\left(g(x)\right)^{q\left(1-\frac{1}{p}\right)}\mathrm{d}x}{\left(\int_a^b f^{p}(x)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_a^b g^{q}(x)\mathrm{d}x\right)^{1-\frac{1}{p}}}\\
&\le\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\left[1-\frac{2f^{\frac{p}{2}}(x)g^{\frac{q}{2}}(x)\mathrm{d}x}{\left(\int_a^b f^{p}(x)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_a^b g^{q}(x)\mathrm{d}x\right)^{\frac{1}{2}}}+1\right]\\
&\quad+\left(\frac{q-1}{2q^{2}}-\frac{1}{4}\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\right)\log^{2}\left(\frac{1}{m}\right)
\end{aligned}
\tag{17}
$$

Using the fact that, $1-\frac{1}{\max\left[\frac{1}{q},\frac{q-1}{q}\right]}=\frac{1}{\min\left\{\frac{1}{q},\frac{q-1}{q}\right\}}$, in (17) we have

$$1-\frac{\int_a^b f(x)\left(g(x)\right)^{q\left(1-\frac{1}{p}\right)}\mathrm{d}x}{\left(\int_a^b f^p(x)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_a^b g^q(x)\mathrm{d}x\right)^{1-\frac{1}{p}}}$$

$$\leq \frac{2}{\min\left(\frac{1}{q},\frac{q-1}{q}\right)}\left[1-\frac{2f^{\frac{p}{2}}(x)g^{\frac{q}{2}}(x)\mathrm{d}x}{\left(\int_a^b f^p(x)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_a^b g^q(x)\mathrm{d}x\right)^{\frac{1}{2}}}\right] \tag{18}$$

$$+\left(\frac{q-1}{2q^2}-\frac{1}{4\min\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\log^2\left(\frac{1}{m}\right)$$

This completes the proof.

Theorem 2.4. Let $1<p<\infty$, $1<q<\infty$, $1\leq r<\infty$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. If f and g are two positive functions which admit integral on $[a, b]$ for which there exist $\int_a^b f^p(x)\mathrm{d}x$ and $\int_a^b g^q(x)\mathrm{d}x$ are finite with $\int_a^b f^p(x)\mathrm{d}x>0$, $\int_a^b g^q(x)\mathrm{d}x>0$, then

$$m<k\frac{f^p(x)}{\int_a^b f^p(x)\mathrm{d}x}\frac{\int_a^b g^q(x)\mathrm{d}x}{g^q(x)}<1,\quad x\in[a,b],k>0$$

Proof: Taking lemma 2.1, let $a=1$, $\lambda=\frac{1}{q}$, $1-\lambda=1-\frac{1}{q}$ we have

$$\frac{1}{q}+\left(1-\frac{1}{q}\right)b-b^{\frac{1}{p}}$$

$$\leq\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{1}\right\}}\right)\left(1-\sqrt{b}\right)^2+\left(\frac{q-1}{2q^2}-\frac{1}{4}\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\right)\log^2\left(\frac{1}{m}\right) \tag{19}$$

Substituting $b=\frac{f^p(x)}{\int_a^b f^p(x)\mathrm{d}x}\frac{\int_a^b g^q(x)\mathrm{d}x}{g^q(x)}>1$ into (19) we get

$$\frac{1}{q}+\left(1-\frac{1}{q}\right)\frac{kf^p(x)}{\int_a^b f^p(x)\mathrm{d}x}\frac{\int_a^b g^q(x)\mathrm{d}x}{g^q(x)}-\left(\frac{kf^p(x)}{\int_a^b f^p(x)\mathrm{d}x}\frac{\int_a^b g^q(x)\mathrm{d}x}{g^q(x)}\right)^{\frac{1}{p}}$$

$$\leq\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\left(1-\left(\frac{kf^p(x)}{\int_a^b f^p(x)\mathrm{d}x}\frac{\int_a^b g^q(x)\mathrm{d}x}{g^q(x)}\right)^{\frac{1}{2}}\right)^2 \tag{20}$$

$$+\left(\frac{q-1}{2q^2}-\frac{1}{4}\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\right)\log^2\left(\frac{1}{m}\right)$$

Simplifying Equation (20) completely to have

$$\left[\frac{1}{q}\frac{g^q(x)}{\int_a^b g^q(x)\mathrm{d}x}+\frac{kf^p(x)}{\int_a^b f^p(x)\mathrm{d}x}-\frac{1}{q}\frac{f^p(x)}{\int_a^b f^p(x)\mathrm{d}x}-\frac{\sqrt[p]{k}f(x)}{\left(\int_a^b f^p(x)\mathrm{d}x\right)^{\frac{1}{p}}}\frac{(g(x))^{q\left(1-\frac{1}{p}\right)}}{\left(\int_a^b g^q(x)\mathrm{d}x\right)^{1-\frac{1}{p}}}\right]$$

$$\le\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\left(\frac{g^q(x)}{\int_a^b g^q(x)\mathrm{d}x}-\frac{2\sqrt{k}f^{\frac{p}{2}}(x)}{\left(\int_a^b f^p(x)\mathrm{d}x\right)^{\frac{1}{2}}}\frac{g^{\frac{q}{2}}(x)}{\left(\int_a^b g^q(x)\mathrm{d}x\right)^{\frac{1}{2}}}+\frac{g^q(x)}{\int_a^b g^q(x)\mathrm{d}x}\right) \tag{21}$$

$$+\frac{g^q(x)}{\int_a^b g^q(x)\mathrm{d}x}\left[\frac{q-1}{2q^2}-\frac{1}{4}\left(1-\frac{1}{\max\left(\frac{1}{q},\frac{q-1}{q}\right)}\right)\right]\log^2\left(\frac{1}{m}\right)$$

On integrating Equation (21) with respect to x, we have

$$\frac{1}{q}+k-\frac{1}{q}-\frac{\sqrt[p]{k}f(x)}{\left(\int_a^b f^p(x)\mathrm{d}x\right)^{\frac{1}{p}}}\frac{(g(x))^{q\left(1-\frac{1}{p}\right)}\mathrm{d}x}{\left(\int_a^b g^q(x)\mathrm{d}x\right)^{1-\frac{1}{p}}}$$

$$\le\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\left[1-\frac{2\sqrt{k}f^{\frac{p}{2}}(x)g^{\frac{q}{2}}(x)}{\left(\int_a^b f^p(x)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_a^b g^q(x)\mathrm{d}x\right)^{\frac{1}{2}}}+1\right] \tag{22}$$

$$+\left(\frac{q-1}{2q^2}-\frac{1}{4}\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\right)\log^2\left(\frac{1}{m}\right)$$

Using the fact that $1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}=\frac{1}{\min\left\{\frac{1}{q},\frac{q-1}{q}\right\}}$ in Equation (22) we have

$$k-\frac{\sqrt[p]{k}f(x)}{\left(\int_a^b f^p(x)\mathrm{d}x\right)^{\frac{1}{p}}}\frac{(g(x))^{q\left(1-\frac{1}{p}\right)}\mathrm{d}x}{\left(\int_a^b g^q(x)\mathrm{d}x\right)^{1-\frac{1}{p}}}$$

$$\le\frac{2}{\min\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\left[1-\frac{\sqrt{k}f^{\frac{p}{2}}(x)g^{\frac{q}{2}}(x)\mathrm{d}x}{\left(\int_a^b f^p(x)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_a^b g^q(x)\mathrm{d}x\right)^{\frac{1}{2}}}\right] \tag{23}$$

$$+\left(\frac{q-1}{2q^2}-\frac{1}{\min\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\log^2\left(\frac{1}{m}\right)$$

This completes the proof.

Theorem 2.5. Let $1<p<\infty$, $1<q<\infty$, $1\le r<\infty$, with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. If f and g are two positive function $f\in L^p$, $g\in L^q$ with $\|f\|_p>0$, $\|g\|>0$ for which there exist

$$1<\frac{f^p}{\|f\|_p^p}\frac{\|g\|_q^q}{g^q}\le M,\quad \forall x\in[a,b], M>0$$

Proof: Taking in theorem 2.2, $b=1$, $\lambda=\frac{1}{p}$, $1-\lambda=1-\frac{1}{p}$, we will obtain

$$\frac{1}{p}a+\left(1-\frac{1}{p}\right)-a^{\frac{1}{p}}$$
$$\le\left(1-\frac{1}{\max\left(\frac{1}{p},\frac{p-1}{p}\right)}\right)\left(\sqrt{a}-1\right)^2+\left(\frac{p-1}{2p^2}-\frac{1}{4}\left(1-\frac{1}{\max\left(\frac{1}{p},\frac{p-1}{p}\right)}\right)\right)\log M \tag{24}$$

Putting $a=\frac{f^p}{\|f\|_p^p}\cdot\frac{\|g\|_q^q}{g^q}>1$ we will have

$$\frac{1}{p}\cdot\frac{f^p}{\|f\|}\cdot\frac{\|g\|_q^q}{g^q}+\left(1-\frac{1}{p}\right)-\left(\frac{f^p}{\|f\|_p^p}\cdot\frac{\|g\|_q^q}{g^q}\right)^{\frac{1}{p}}$$
$$\le\left(1-\frac{1}{\max\left(\frac{1}{p},\frac{p-1}{p}\right)}\right)\left[\left(\frac{f^p}{\|f\|_p^p}\cdot\frac{\|g\|_q^q}{g^q}\right)^{\frac{1}{2}}-1\right]^2 \tag{25}$$
$$+\left(\frac{p-1}{2p^2}-\frac{1}{4}\left(1-\frac{1}{\max\left(\frac{1}{p},\frac{p-1}{p}\right)}\right)\right)\log^2(M)$$

Simplifying (25) completely we get

$$\left(\frac{1}{p}\frac{f^p}{\|f\|_p^p}+\frac{g^q}{\|g\|_q^q}-\frac{1}{p}\frac{g^q}{\|g\|_q^q}-\frac{f}{\|f\|_p}\cdot\frac{g^{q\left(1-\frac{1}{p}\right)}}{\left(\|g\|_q^q\right)^{1-\frac{1}{p}}}\right)$$
$$\le\left(1-\frac{1}{\max\left(\frac{1}{p},\frac{p-1}{p}\right)}\right)\left(\frac{f^p}{\|f\|_p^p}-\frac{2f^{\frac{p}{2}}g^{\frac{q}{2}}}{\|f\|_p^{\frac{p}{2}}\|g\|_g^{\frac{q}{2}}}+\frac{g^q}{\|g\|_q^q}\right) \tag{26}$$
$$+\left[\frac{p-1}{2p^2}-\frac{1}{4}\left(1-\frac{1}{\max\left(\frac{1}{p},\frac{p-1}{p}\right)}\right)\right]\log^2(M)$$

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On Integrating both sides we have;

$$\frac{1}{p}+1-\frac{1}{p}-\frac{f\,g^{q\left(1-\frac{1}{p}\right)}}{\|f\|_p\left(\|g\|_q^q\right)^{1-\frac{1}{p}}}$$
$$\le\left(1-\frac{1}{\max\left(\frac{1}{p},\frac{p-1}{p}\right)}\right)\left[1-\frac{2\int_\Omega f^{\frac{p}{2}}g^{\frac{q}{2}}\mathrm{d}\mu}{\|f\|_p^{\frac{p}{2}}\|g\|_g^{\frac{q}{2}}}+1\right] \tag{27}$$
$$+\left(\frac{p-1}{2p^2}-\frac{1}{4}\left(1-\frac{1}{\max\left(\frac{1}{p},\frac{p-1}{p}\right)}\right)\right)\log^2(M)$$

Using the fact that $1-\frac{1}{\max\left\{\frac{1}{p},\frac{p-1}{p}\right\}}=\frac{1}{\min\left\{\frac{1}{p},\frac{p-1}{p}\right\}}$ in Equation (27) we have

$$1-\frac{f\,g^{q\left(1-\frac{1}{p}\right)}}{\|f\|_p\left(\|g\|_q^q\right)^{1-\frac{1}{p}}}$$
$$\le\frac{2}{\min\left(\frac{1}{p},\frac{p-1}{p}\right)}\left[1-\frac{\int_\Omega f^{\frac{p}{2}}g^{\frac{q}{2}}\mathrm{d}\mu}{\|f\|_p^{\frac{p}{2}}\|g\|_g^{\frac{q}{2}}}\right]+\left(\frac{p-1}{2p^2}-\frac{1}{4\min\left(\frac{1}{p},\frac{p-1}{p}\right)}\right)\log^2(M)$$

This completes the proof.

The refinement of Hoders' inequality explain the fact that $1/\infty$ means zero. In the above proof, if $p=\infty$, it means that $\|f\|_\infty$ is equivalent to essential supremum of $|f|$; also, in the Holder's inequality, $0\times\infty$ and $\infty\times 0$ means 0. The above mathematical analysis finds application in both algebra and calculus in the area of mathematics.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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