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Lagrangian Formulation of Fractional Nonholonomic Constrained Damping Systems

Ola A. Jarab'ah

Department of Applied Physics, Tafila Technical University, Tafila, Jordan
Email: oasj85@yahoo.com

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Abstract

Fractional Euler Lagrange equations for fractional nonholonomic constrained damping systems have been presented. The equations of motion are obtained using fractional Euler Lagrange equations in a similar manner to the usual technique. The results of fractional method reduce to those obtained from classical method when $\mu \to 0$ and $\alpha, \beta \to 1$ are equal unity only. This work is discussed using illustrative example.

Keywords

Euler-Lagrange Equations, Nonholonomic Constraints, Generalized Momenta

1. Introduction

Nonholonomic mechanics refers to the mechanical systems that are subject to constraints on the velocities. This mechanics is very active area in classical mechanics.

The studying of mechanical systems with nonholonomic constraints has a long history in classical mechanics [1] [2] [3]. In these references nonholonomic mechanical systems are described within the variational framework by Euler Lagrange equations with extra terms corresponding to the constraint forces. Also nonholonomic constraints have been intensively presented by researchers [4]-[14].

The Euler Lagrange formulates the basis of Lagrangian or Hamiltonian mechanics [15]. The main role of Lagrangian mechanics is that the given equations are characterized with only one scalar function the Lagrangian L, or the Hamiltonian H [16], but in classical mechanics there are some methods that describe nonconservative systems in such formalism. The method presented by Rayleigh, he introduces a function R (called Rayleigh's dissipation function).

The role of fractional derivative has been growing rapidly during the last few years because of its active area in science and engineering [17] [18] [19]. Riewe has used the fractional derivatives to develop a formalism which can be used for both conservative and nonconservative systems [20] [21].

The classical calculus of variations was extended by Agrawal [22] for systems containing Riemann-Liouville fractional derivatives. The resulting equations are found to be similar to those for variational problems containing integral order derivatives. In other words, the results of fractional calculus of variations reduce to those obtained from traditional fractional calculus of variations when the derivative of fractional order replaced by integral order. Recently, Euler Lagrange equations for holonomic constrained systems with regular Lagrangian have been presented by Hasan [23] using the fractional variationl problems. More recently, the fractional Euler Lagrange equations are used by Jarab'ah [24] [25] to obtain the equations of motion for first order irregular Lagrangian with holonomic constraints and second order Lagrangian for nonconservative systems. In this paper, damping systems with fractional nonholonomic constraints will discuss as a continuation of the previous work [26].

This paper is organized as follows: In Section 2, fractional derivatives formulation is discussed. In Section 3, formulation of fractional lagrangian for nonholonomic constraints is explained. In Section 4, one illustrative example is studied in detail. The work closes with some concluding remarks in Section 5.

2. Fractional Derivatives Formulation

The left Riemann-Liouville fractional derivative written as [27] [28]:

$$ {}_aD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \int_a^x (x-\tau)^{n-\alpha-1} f(\tau)\,\mathrm{d}\tau \tag{1} $$

Which is defined as the LRLFD, and the right Riemann-Liouville fractional derivative written as:

$$ {}_xD_b^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \int_x^b (\tau-x)^{n-\alpha-1} f(\tau)\,\mathrm{d}\tau \tag{2} $$

Which is defined as the RRLFD.

Where Γ represents the Euler's gamma function and α is the order of the derivative such that $n-1 \le \alpha < n$, and is not equal to zero. If α is an integer, these derivatives are written as:

$$ {}_aD_x^\alpha f(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^\alpha f(x) \tag{3} $$

and

$$ {}_xD_b^\alpha f(x) = \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^\alpha f(x) \tag{4} $$

The fractional operator, ${}_aD_x^\alpha f(x)$ can be written as [29].

$$ {}_aD_x^{\alpha} = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\, {}_aD_x^{\alpha-n} \tag{5} $$

where,

$$ \alpha = 1, 2, \cdots $$

Using that

$$ D_t^1 = \frac{\mathrm{d}}{\mathrm{d}t} \tag{6} $$

$$ D_t^0 = 1 \tag{7} $$

Thus, if $\alpha = \beta = 1$, we find that:

$$ {}_tD_b^{\alpha} = -\frac{\mathrm{d}}{\mathrm{d}t} \tag{8} $$

and

$$ {}_aD_t^{\alpha} = \frac{\mathrm{d}}{\mathrm{d}t} \tag{9} $$

Theorem: Let f and g be two continuous functions on $[a,b]$. Then, for all $x \in [a,b]$, the following properties hold:

1) For

$$ m > 0,\quad {}_aD_x^m = \left[f(x) + g(x) \right] = {}_aD_x^m f(x) + {}_aD_x^m g(x) \tag{10} $$

2) For

$$ m \ge n \ge 0,\quad {}_aD_x^m \left({}_aD_x^{-n} f(x) \right) = {}_aD_x^{m-n} f(x) \tag{11} $$

3) For

$$ m > 0,\quad {}_aD_x^m \left({}_aD_x^{-m} f(x) \right) = f(x) \tag{12} $$

4) For

$$ m > 0,\quad \int_a^b \left({}_aD_x^m f(x) \right) g(x)\,\mathrm{d}x = \int_a^b f(x) \left({}_xD_b^m g(x) \right) \mathrm{d}x \tag{13} $$

3. Formulation of Fractional Lagrangian for Nonholonomic Constraints

The nonholonomic constraints are time independent and linear in the velocities:

$$ f_i = f_i\left(q_j, \dot{q}_j \right) = 0 \tag{14} $$

And the Lagrangian containing a fractional derivative takes the following form:

$$ L_0 = L_0\left({}_aD_t^{\alpha-1} q,\, {}_tD_b^{\beta-1} q,\, {}_aD_t^{\alpha} q,\, {}_tD_b^{\beta} q, t \right) \tag{15} $$

The motion of a nonholonomic system will be determined by using of the Euler Lagrange equation and constraints. The fractional Euler Lagrange equation in fractional form is given by:

$$ \frac{\partial L}{\partial q} + {}_tD_b^{\alpha} \frac{\partial L}{\partial\, {}_aD_t^{\alpha} q} + {}_aD_t^{\beta} \frac{\partial L}{\partial\, {}_tD_b^{\beta} q} + \lambda \frac{\partial f}{\partial\, {}_aD_t^{\alpha} q} + \lambda \frac{\partial f}{\partial\, {}_tD_b^{\beta} q} = 0 \tag{16} $$

where $L = L_0\left({}_aD_t^{\alpha-1}q, {}_tD_b^{\beta-1}q, {}_aD_t^{\alpha}q, {}_tD_b^{\beta}q\right)e^{\mu t}$, which represents the damping case through the $e^{\mu t}$ factor and μ is called the damping factor and λ is called Lagrange multiplier.

The generalized momenta can be obtained from:

$$p_\alpha = \frac{\partial L}{\partial\, {}_aD_t^{\alpha}q} \tag{17}$$

and

$$p_\beta = \frac{\partial L}{\partial\, {}_tD_b^{\beta}q} \tag{18}$$

4. Illustrative Example

-The Sliding of a Balanced Skate.

Let us consider as an illustrating example the problem of a balanced skate on horizontal ice. We assume that length, time and mass are equal to one, so that the Lagrangian would take the following form [30]:

$$L_0 = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \tag{19}$$

In the presence of damping process $e^{\mu t}$, and using fractional derivatives the Lagrangian in Equation (19) becomes.

$$L = \frac{1}{2}\left[\left({}_0D_t^{\alpha}x\right)^2 + \left({}_0D_t^{\alpha}y\right)^2 + \left({}_0D_t^{\alpha}z\right)^2\right]e^{\mu t} \tag{20}$$

The nonholonomic constraint equation is:

$$f = \dot{x}\sin z - \dot{y}\cos z = 0 \tag{21}$$

In fractional form Equation (21) takes this form:

$$f = \left({}_0D_t^{\alpha}x\right)\sin z - \left({}_0D_t^{\alpha}y\right)\cos z = 0 \tag{22}$$

using the following Euler Lagrange equation

$$\frac{\partial L}{\partial q} + {}_tD_b^{\alpha}\frac{\partial L}{\partial\, {}_aD_t^{\alpha}q} + {}_aD_t^{\beta}\frac{\partial L}{\partial\, {}_tD_b^{\beta}q} + \lambda\frac{\partial f}{\partial\, {}_aD_t^{\alpha}q} + \lambda\frac{\partial f}{\partial\, {}_tD_b^{\beta}q} = 0 \tag{23}$$

The corresponding Euler Lagrange equations are

$${}_tD_b^{\alpha}\left(e^{\mu t}\, {}_0D_t^{\alpha}x\right) + \lambda\left(\sin z\right) = 0 \tag{24}$$

and

$${}_tD_b^{\alpha}\left(e^{\mu t}\, {}_0D_t^{\alpha}y\right) + \lambda\left(-\cos z\right) = 0 \tag{25}$$

also

$${}_tD_b^{\alpha}\left(e^{\mu t}\, {}_0D_t^{\alpha}z\right) = 0 \tag{26}$$

Using Equation (24) and Equation (25), the Lagrange multiplier is

$$\lambda = \frac{{}_tD_b^{\alpha}\left(e^{\mu t}\, {}_0D_t^{\alpha}y\right) + {}_tD_b^{\alpha}\left(e^{\mu t}\, {}_0D_t^{\alpha}x\right)}{\cos z - \sin z} \tag{27}$$

From Equation (17), the conjugate momenta are:

$$p_x = \frac{\partial L}{\partial\, {}_0D_t^\alpha x} = e^{\mu t}\left({}_0D_t^\alpha x\right) \tag{28}$$

$$p_y = \frac{\partial L}{\partial\, {}_0D_t^\alpha y} = e^{\mu t}\left({}_0D_t^\alpha y\right) \tag{29}$$

$$p_z = \frac{\partial L}{\partial\, {}_0D_t^\alpha z} = e^{\mu t}\left({}_0D_t^\alpha z\right) \tag{30}$$

If $\mu \to 0$ and $\alpha \to 1$.

The acceleration takes the following form

$$\ddot{x} = \lambda \sin z \tag{31}$$

$$\ddot{y} = -\lambda \cos z \tag{32}$$

$$\ddot{z} = 0 \tag{33}$$

And the Lagrange multiplier becomes

$$\lambda = \frac{\ddot{x} + \ddot{y}}{\sin z - \cos z} \tag{34}$$

Finally, the conjugate momenta are

$$p_x = \left({}_0D_t^\alpha x\right) = \dot{x} \tag{35}$$

$$p_y = \left({}_0D_t^\alpha y\right) = \dot{y} \tag{36}$$

$$p_z = \left({}_0D_t^\alpha z\right) = \dot{z} \tag{37}$$

which are in exact agreement with that obtained by classical method.

5. Conclusion

In this work nonholonomic constraints are studied for damping systems using fractional Lagrangian. From this Lagrangian we can find the equations of motion, the Lagrange multiplier λ and the generalized momenta. The results of fractional technique reduce to those obtained from classical technique when $\mu \to 0$ and $\alpha, \beta \to 1$ are equal unity.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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