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Generalized Series of Bernoulli Type

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Abstract

The problem of evaluating an infinite series whose successive terms are reciprocal squares of the natural numbers was posed without a solution being offered in the middle of the seventeenth century. In the modern era, it is part of the theory of the Riemann zeta-function, specifically $\zeta(2)$. Jakob Bernoulli attempted to solve it by considering other more tractable series which were superficially similar and which he hoped could be algebraically manipulated to yield a solution to the difficult series. This approach was eventually unsuccessful, however, Bernoulli did produce an early monograph on summation of series. It remained for Bernoulli's student and countryman Leonhard Euler to ultimately determine the sum to be $\frac{\pi^2}{6}$. We characterize a class of series based on generalizing Bernoulli's original work by adding two additional parameters to the summations. We also develop a recursion formula that allows summation of any member of the class.

Keywords

Bernoulli, Series, Convergence, Sum, Recursion Formula, Zeta-Function, Sine, Maclaurin Series, Infinite Product

1. Introduction

We explore power series of the type $S(k,m)=\sum_{n=1}^{\infty}\frac{n^m}{k^n}$, where the parameters $k>1$ are real and $m\geq 1$ is an integer. These sums were studied by Jakob Bernoulli in the late seventeenth century in a more limited context. Although a general formula is sparsely available in the literature for these sums, it does not appear to be supported with proof or even attribution. Our purpose is to put a formula for $S(k,m)$ on a more robust footing. To that end we derive a general recursion formula for $S(k,m)$ and use it to produce a list of these sums for small integer values of m and arbitrary values of k. In addition, we explicitly reprise and extend Bernoulli's original results where $k=2$. Finally, we explore several interesting series whose sums would not be easily obtained without the recursion method that we have developed.

2. Historical Background

Jakob Bernoulli [1] [2] [3] published *Tractatus de Seriebus Infinitis* (Tract on Infinite Series) in 1689 in which he considered (among others) the series $\sum_{n=1}^{\infty}\frac{n}{2^n}$ and $\sum_{n=1}^{\infty}\frac{n^2}{2^n}$ in the course of attempting to evaluate $\sum_{n=1}^{\infty}\frac{1}{n^2}$. Bernoulli evaluated the first two series, however he was unsuccessful in his quest to sum the third, being moved to appeal to his fellow mathematical correspondents that he would be grateful for any assistance. Evaluating $\sum_{n=1}^{\infty}\frac{1}{n^2}$ was a challenge proposed by the Italian mathematician Pietro Mengoli in 1650. Ultimately, Bernoulli's protegé Leonhard Euler determined $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$ in 1734. This problem became known as the Basel problem after the hometown of Euler and the Bernoulli clan. Interestingly, Euler's proof was not totally sound by modern standards. He assumed that the sine, as represented by a Maclaurin series, could be expressed as an infinite product of linear terms involving the zeroes of the sine. Weierstrass showed that Euler's bold intuition was, in fact, correct nearly a century later. In any case, Euler presented his result in 1735 at the St. Petersburg Academy of Sciences and it launched his reputation as one of the leading mathematicians not only of the eighteenth century but all time.

3. Purpose

With regard to generalized power series [3] [4] [5] [6] [7] of Bernoulli type, although a formula for $S(k,m)=\sum_{n=1}^{\infty}\frac{n^m}{k^n}$ exists sparingly in the literature [8] [9] [10], it does not appear to be supported by any detailed justification or even attribution. In ([8], Sec. 0.232(3)), $S(k,m)$ is given as

$\frac{1}{(k-1)^{m+1}}\sum_{i=1}^{m}\left[\frac{1}{k^{m-i}}\sum_{j=0}^{i}\frac{(-1)^j(m+1)!(i-j)^m}{j!(m+1-j)!}\right]$ with integer $m\geq 1$ and $k\neq 1$

[sic] (obviously $k>1$ is necessary to ensure convergence). Our purpose is to furnish a robust justification for a formula for $S(k,m)$. This is achieved by deriving a transparent recursion formula for $S(k,m)$ with $k>1$ and $m\in\mathbb{N}$. Using this recursion we provide calculation-friendly expressions for any $k>1$ and $1\leq m<10$, we extend Bernoulli's original results for $k=2$, and we finally consider some series that would be difficult to sum without our method.

4. Basic Argument

First note that $S(k,m)$ converges [1] [2] [5] [6] [7] for all $k>1$ and $m\geq 1$.

This follows from the ratio test since the ratio of successive terms is $\frac{1}{k}\left(\frac{n+1}{n}\right)^m$ and $\lim_{n\to\infty}\frac{1}{k}\left(\frac{n+1}{n}\right)^m=\frac{1}{k}<1$. To illustrate the basic approach to evaluating these sums, we take the simple case of $S(k,1)=\sum_{n=1}^{\infty}\frac{n}{k^n}$. We can write $\sum_{n=1}^{\infty}\frac{n}{k^n}-\sum_{n=1}^{\infty}\frac{n-1}{k^n}=\sum_{n=1}^{\infty}\frac{1}{k^n}$. Then re-indexing with $j=n-1$, we have $\sum_{n=1}^{\infty}\frac{n-1}{k^n}=\sum_{j=1}^{\infty}\frac{j}{k^{j+1}}=\frac{1}{k}\sum_{j=1}^{\infty}\frac{j}{k^j}$. After reverting $j\to n$ we obtain $\sum_{n=1}^{\infty}\frac{n}{k^n}-\frac{1}{k}\sum_{n=1}^{\infty}\frac{n}{k^n}=\sum_{n=1}^{\infty}\frac{1}{k^n}=\frac{1/k}{1-1/k}=\frac{1}{k-1}$ upon summing the geometric series. Now we have $\left(1-\frac{1}{k}\right)\sum_{n=1}^{\infty}\frac{n}{k^n}=\frac{1}{k-1}$, and we conclude $S(k,1)=\frac{k}{(k-1)^2}$. So Bernoulli's historical sum $\sum_{n=1}^{\infty}\frac{n}{2^n}=S(2,1)=2$.

For integer $m>1$ the situation is more complicated since we must expand $(n-1)^m$ to apply our method. We remark that if we tried to use non-integer values of m, say, for example, $m=\frac{3}{2}$, we would immediately have to contend with an infinite binomial expansion $(n-1)^m=n^m\left(1-\frac{1}{n}\right)^m=n^{\frac{3}{2}}\left[1+\sum_{j=1}^{\infty}\frac{(-1)^j\left(\frac{3}{2}\right)_{(j)}}{j!}\left(\frac{1}{n}\right)^j\right]$ where $\left(\frac{3}{2}\right)_{(j)}$ indicates the j^{th} lower factorial of $\frac{3}{2}$. We elect to err on the side of simplicity (discretion) and insist on integer $m>1$. There is no corresponding limitation on k, however, as long as $k>1$.

Tackling the case of $S(k,2)=\sum_{n=1}^{\infty}\frac{n^2}{k^n}$ we can write $\sum_{n=1}^{\infty}\frac{n^2}{k^n}-\sum_{n=1}^{\infty}\frac{(n-1)^2}{k^n}=\sum_{n=1}^{\infty}\frac{2n-1}{k^n}$. Then re-indexing with $j=n-1$ as in the first example, we have $\sum_{j=1}^{\infty}\frac{j^2}{k^{j+1}}=\frac{1}{k}\sum_{j=1}^{\infty}\frac{j^2}{k^j}$ and after reversion $j\to n$ we get $\frac{1}{k}\sum_{n=1}^{\infty}\frac{n^2}{k^n}$. Combining everything, $\sum_{n=1}^{\infty}\frac{n^2}{k^n}-\frac{1}{k}\sum_{n=1}^{\infty}\frac{n^2}{k^n}=\sum_{n=1}^{\infty}\frac{2n-1}{k^n}$, or $\left(1-\frac{1}{k}\right)\sum_{n=1}^{\infty}\frac{n^2}{k^n}=\sum_{n=1}^{\infty}\frac{2n-1}{k^n}$. Recall $\sum_{n=1}^{\infty}\frac{2n-1}{k^n}=2S(k,1)-\frac{1}{k-1}$ after we substitute our earlier result and again sum the geometric series $\sum_{n=1}^{\infty}\frac{1}{k^n}$. Finally, $S(k,2)=\frac{k}{k-1}\left[2S(k,1)-\frac{1}{k-1}\right]=\frac{2kS(k,1)}{k-1}-S(k,1)$, so

$$S(k,2)=\frac{k+1}{k-1}S(k,1)=\frac{k+1}{k-1}\cdot\frac{k}{(k-1)^2}=\frac{k(k+1)}{(k-1)^3}.$$

As additional perspective for the general case, consider $S(k,3)$. Following the outline for our $S(k,2)$ argument,
$S(k,3)=\sum_{n=1}^{\infty}\frac{n^3}{k^n}-\sum_{n=1}^{\infty}\frac{(n-1)^3}{k^n}=\sum_{n=1}^{\infty}\frac{3n^2-3n+1}{k^n}$. The re-indexing, substitution and geometric series summation steps give
$\left(1-\frac{1}{k}\right)\sum_{n=1}^{\infty}\frac{n^3}{k^n}=3S(k,2)-3S(k,1)+\frac{1}{k-1}$. Finally,
$S(k,3)=\frac{k}{k-1}\left[3S(k,2)-3S(k,1)+\frac{1}{k-1}\right]$. Using the values for $S(k,2)$ and $S(k,1)$ from above, $S(k,3)=\frac{k}{k-1}\left[\frac{3k(k+1)}{(k-1)^3}-\frac{3k}{(k-1)^2}+\frac{1}{k-1}\right]=\frac{k(k^2+4k+1)}{(k-1)^4}$.
A pattern is emerging.

5. General Recursion

Now for the general case. For $S(k,m)$ we observe that
$n^m-(n-1)^m=\sum_{r=1}^{m}(-1)^{r-1}\binom{m}{r}n^{m-r}$. Note that the highest power of n has been stripped from the sum and the parity of the binomial coefficients has been adjusted. Following the prior pattern we have
$S(k,m)=\left(1-\frac{1}{k}\right)\sum_{n=1}^{\infty}\frac{n^m}{k^n}=\sum_{n=1}^{\infty}\frac{1}{k^n}\sum_{r=1}^{m}(-1)^{r-1}\binom{m}{r}n^{m-r}$. For a given index r, we may note that
$\sum_{n=1}^{\infty}\frac{1}{k^n}(-1)^{r-1}\binom{m}{r}n^{m-r}=(-1)^{r-1}\binom{m}{r}\sum_{n=1}^{\infty}\frac{n^{m-r}}{k^n}=(-1)^{r-1}\binom{m}{r}S(k,m-r)$. The general recursion formula for $S(k,m)$ can then be written
$\frac{k}{k-1}\sum_{r=1}^{m}(-1)^{r-1}\binom{m}{r}S(k,m-r)$. This sum can be reduced to the rational expression $S(k,m)=\frac{P_m(k)}{(k-1)^{m+1}}$, where $P_m(k)$ is a polynomial of degree m. Moreover, if m is odd, $P_m(k)=kQ(k)$ where $Q(x)$ is a monic, symmetric polynomial of degree $m-1$ irreducible over $\mathbb{Q}$ and if m is even $P_m(k)=k(k+1)R(k)$ where $R(k)$ is similarly a monic, symmetric polynomial of degree $m-2$ also irreducible over $\mathbb{Q}$. For $m\in[1..9]$ and including the geometric series $\sum_{n=1}^{\infty}\frac{1}{k^n}$ [in our notation $S(k,0)$] for completeness, we have:

1) $S(k,0)=\frac{1}{k-1}$
2) $S(k,1)=\frac{k}{(k-1)^2}$
3) $S(k,2)=\frac{k(k+1)}{(k-1)^3}$

4) $S(k,3)=\dfrac{k(k^2+4k+1)}{(k-1)^4}$

5) $S(k,4)=\dfrac{k(k+1)(k^2+10k+1)}{(k-1)^5}$

6) $S(k,5)=\dfrac{k(k^4+26k^3+66k^2+26k+1)}{(k-1)^6}$

7) $S(k,6)=\dfrac{k(k+1)(k^4+56k^3+246k^2+56k+1)}{(k-1)^7}$

8) $S(k,7)=\dfrac{k(k^6+120k^5+1191k^4+2416k^3+1191k^2+120k+1)}{(k-1)^8}$

9) $S(k,8)=\dfrac{k(k+1)(k^6+246k^5+4047k^4+11572k^3+4047k^2+246k+1)}{(k-1)^9}$

10) $S(k,9)=\dfrac{k(k^8+502k^7+14608k^6+88234k^5+156190k^4+88234k^3+14608k^2+502k+1)}{(k-1)^{10}}$.

For the irreducibility of $Q(k)$ and $R(k)$ over $\mathbb{Q}$ in the above formulas, the Rational Root Test [11] establishes that the only possible rational roots are ±1. Certainly 1 is ruled out by the positivity of all the coefficients and −1 is ruled out with a simple arithmetic check.

6. Bernoulli's Original Series Extended

Let us recap the series Bernoulli originally studied and extend the list according to our formulas above.

1) $S(2,1)=2$

2) $S(2,2)=6$

3) $S(2,3)=26$

4) $S(2,4)=150$

5) $S(2,5)=1082$

6) $S(2,6)=9366$

7) $S(2,7)=94586$

8) $S(2,8)=1091670$

9) $S(2,9)=14174522$

7. Epilogue

We can also determine the non-obvious sums of several interesting series [5] [6] [7]:

1) $\sum_{n=1}^{\infty} e^{2\ln n-n}=\sum_{n=1}^{\infty}\dfrac{n^2}{e^n}=S(e,2)=\dfrac{e(e+1)}{(e-1)^3}$

2) $\sum_{n=1}^{\infty} e^{\ln n-n\ln\pi}=S(\pi,1)=\dfrac{\pi}{(\pi-1)^2}$

3) $\sum_{n=1}^{\infty}\frac{n^5}{5^n}=S(5,5)=\frac{k\left(k^4+26k^3+66k^2+26k+1\right)}{(k-1)^6}=\frac{28280}{4096}=6.904$

4) $\sum_{n=2}^{\infty}\frac{n^2-n}{k^n}=\frac{2k}{(k-1)^3}$

5) $\sum_{n=1}^{\infty}\frac{(n-1)n(n+1)}{k^n}=\frac{6k^2}{(k-1)^4}$

6) $\sum_{n=1}^{\infty}\frac{2n-1}{k^n}=\frac{1}{k}+\frac{3}{k^2}+\frac{5}{k^3}+\frac{7}{k^4}+\frac{9}{k^5}+\cdots=\frac{k+1}{(k-1)^2}$

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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