Advances in Pure Mathematics, 2023, 13, 504-536
https://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

Duality between Bessel Functions and Chebyshev Polynomials in Expansions of Functions

Alfred Wünsche

Humboldt-Universität Berlin, Institut für Physik, Nichtklassische Strahlung (MPG), Berlin, Germany
Email: alfred.wuensche@physik.hu-berlin.de

How to cite this paper: Wünsche, A. (2023) Duality between Bessel Functions and Chebyshev Polynomials in Expansions of Functions. *Advances in Pure Mathematics*, **13**, 504-536.
https://doi.org/10.4236/apm.2023.138034

Received: June 27, 2023
Accepted: August 27, 2023
Published: August 30, 2023

Abstract

In expansions of arbitrary functions in Bessel functions or Spherical Bessel functions, a dual partner set of polynomials play a role. For the Bessel functions, these are the Chebyshev polynomials of first kind and for the Spherical Bessel functions the Legendre polynomials. These two sets of functions appear in many formulas of the expansion and in the completeness and (bi)-orthogonality relations. The analogy to expansions of functions in Taylor series and in moment series and to expansions in Hermite functions is elaborated. Besides other special expansion, we find the expansion of Bessel functions in Spherical Bessel functions and their inversion and of Chebyshev polynomials of first kind in Legendre polynomials and their inversion. For the operators which generate the Spherical Bessel functions from a basic Spherical Bessel function, the normally ordered (or disentangled) form is found.

Keywords

Spherical Bessel Functions, Chebyshev Polynomials, Legendre Polynomials, Hermite Polynomials, Derivatives of Delta Functions, Normally and Anti-Normally Ordered Operators

1. Introduction

Expansion of functions in complete sets of functions plays an outstanding role to find approximations of functions in physics and mathematics. The best known expansions of functions are surely the Taylor series and the Fourier series. The topics of present article are expansions in Bessel functions $\mathrm{J}_n(z)$ and in Spherical Bessel functions $\mathrm{j}_n(z)$ and problems connected with them. About the Bessel (or Cylinder) functions very much is known, a huge number of series and integrals and other relations and we have to say what is new in present paper.

One basic idea was that for expansions in series of functions we have as a rule two sets of involved functions which satisfy a mutual orthogonality and a completeness relation. In case of Hermite polynomials and Hermite functions, these are the same sets of functions. For the Taylor series, these are the monomials $x^n,(n=0,1,\cdots)$ together with the n-th derivatives of the delta function $\delta^{(n)}(x),(n=0,1,\cdots)$ and we have a bi-orthogonality between them that is lesser known. In case of Bessel functions and modified Bessel functions, these are two known expansions. The first is a known formula for the expansion of an exponential function e^{ixz} in a series of Bessel functions $\mathrm{J}_n(z)$ with the Chebyshev polynomials of first kind $\mathrm{T}_n(x)$ as coefficients that, in principle, is known but rarely represented in this way. The second concerns the known expansion of the same exponential function e^{ixz} in a series of Spherical Bessel functions $\mathrm{j}_n(z)\equiv\sqrt{\frac{\pi}{2z}}\mathrm{J}_{n+\frac{1}{2}}(z)$ with Legendre polynomials $\mathrm{P}_n(x)$ as coefficients. Thus the partner for bi-orthogonality in case of Bessel functions $\mathrm{J}_n(z)$ is the Chebyshev polynomials of first kind $\mathrm{T}_n(x)$ and in case of Spherical Bessel functions $\mathrm{j}_n(z)$ the Legendre polynomials $\mathrm{P}_n(x)$ (or, more exactly, their Fourier transforms). This to elaborate was an aim of present article. Besides this we derive some interesting relations between these two mentioned kinds of polynomials and the representation of the monomials z^n as series in Bessel and Spherical Bessel functions, which seems to be new. Furthermore, we give a formula for the normal ordering of the operators $\left(\frac{1}{z}\frac{\partial}{\partial z}\right)^n$ and $\left(\frac{\partial}{\partial z}\frac{1}{z}\right)^n$ which are related to the Spherical Bessel functions.

As main sources about the Bessel and modified Bessel functions, we used the monographs or corresponding chapters of Watson [1], Whittaker and Watson [2], Bateman and Erdélyi [3], Korenyev [4], Arfken [5], and the comprehensive lexicographic articles of Olver [6] (chap. 9) and of Antosiewicz [7] (chap. 10) in [8] and of Olver and Maximon [9] (chap. 10) in [10] and the tables of Gradsteyn and Ryzhik [11] and of Prudnikov, Brychkov and Marichev [12]. Partially, the same sources [3] and additionally the articles of Hochstrasser [13] and of Koornwinder, Wong, Koekoek and Swarttouw [14] we used for the Chebyshev and Legendre, Gegenbauer and Jacobi polynomials.

The expansions which we consider are included in that which is called Neumann expansions (e.g., [2] [4] [9]). There exist also other series of other kinds using Bessel functions. In the Schlömilch series, only one Bessel function is used but with arguments of this Bessel function in equal steps, in Fourier-Bessel series the zeros of only one Bessel function are used and, furthermore, Bessel-Dini expansions and Lommel expansions [4].

In Section 2, we discuss the mentioned bi-orthogonality (or duality) of power functions and derivatives of the Delta function. In Sections 3-6, we develop the duality between Bessel functions and Chebyshev polynomials of first kind and in

Sections 7-10, the duality between Spherical Bessel functions and Legendre polynomials. In Section 11, we present in short form the analogies to expansion of functions in series of Hermite functions or Hermite polynomials where in this case we have a self-duality of the Hermite functions.

2. Taylor Series as Example for a Bi-Orthonormal Expansion

We illustrate in this Section for a known expansion (likely the best one) of a function into a Taylor series by manipulations of the representations what we intend to make in analogous form for some other less known series expansions. Therefore, it may be favorable to read this Section before studying the next Sections with the proper new approaches to the topics.

The Taylor series of a continuous and infinitely continuously differentiable function $f(x)$ defined on the whole real axis $-\infty < x < +\infty$ is

$$f(x) = \sum_{n=0}^{+\infty} \frac{x^n}{n!} f^{(n)}(0). \tag{2.1}$$

Using the delta function $\delta(x)$ in the sense of the theory of generalized functions (e.g., [15] [16]) Equation (2.1) can be written

$$\begin{aligned} f(x) &= \sum_{n=0}^{+\infty} \frac{x^n}{n!} \int_{-\infty}^{+\infty} \mathrm{d}y \, \delta(y) f^{(n)}(y) = \sum_{n=0}^{+\infty} \frac{x^n}{n!} \left(\int_{-\infty}^{+\infty} \mathrm{d}y (-1)^n \delta^{(n)}(y) f(y) \right) \\ &= \int_{-\infty}^{+\infty} \mathrm{d}y \left(\sum_{n=0}^{+\infty} x^n \frac{(-1)^n}{n!} \delta^{(n)}(y) \right) f(y) \equiv \int_{-\infty}^{+\infty} \mathrm{d}y \, \delta(x-y) f(y). \end{aligned} \tag{2.2}$$

From this follows the operator identity applicable to functions with subsequent integration

$$\sum_{n=0}^{+\infty} x^n \delta(y) \frac{1}{n!} \frac{\partial^n}{\partial y^n} = \sum_{n=0}^{+\infty} x^n \frac{(-1)^n}{n!} \delta^{(n)}(y) = \delta(x-y). \tag{2.3}$$

If we apply this identity on both sides to a function $f(y)$ on the right-hand side and integrate the expression over the whole real axis y we obtain the Taylor series (2.1). Relations of the kind (2.3) are called completeness relations, here for the space D of continuous and continuously differentiable functions. It states that the functions $x^n, (n = 0, 1, 2, \cdots)$ are complete for the expansion of arbitrary function in the spaces D and thus form a possible basis but leaves open whether or not this basis is over-complete.

If we multiply (2.3) by the set of basis functions y^m and integrate this then over y we find

$$\begin{aligned} x^m &= \int_{-\infty}^{+\infty} \mathrm{d}y \, \delta(x-y) y^m = \int_{-\infty}^{+\infty} \mathrm{d}y \left(\sum_{n=0}^{+\infty} x^n \frac{(-1)^n}{n!} \delta^{(n)}(y) \right) y^m \\ &= \sum_{n=0}^{+\infty} x^n \left(\int_{-\infty}^{+\infty} \mathrm{d}y \frac{(-1)^n}{n!} \delta^{(n)}(y) y^m \right) \equiv \sum_{n=0}^{+\infty} x^n \delta_{n,m}, \end{aligned} \tag{2.4}$$

where we changed the order of integration and summation under the assumption of absolute convergence of the series. From (2.4) we conclude

$$\int_{-\infty}^{+\infty} \mathrm{d}y \frac{(-1)^n}{n!} \delta^{(n)}(y) y^m = \delta_{n,m}. \tag{2.5}$$

This is a mutual orthonormality relation of the set of functions $x^m, (m = 0,1,2,\cdots)$ to the set of functions $\frac{(-1)^n}{n!} \delta^{(n)}(x), (n = 0,1,2,\cdots)$ where it does not play a role for our illustration that the second set of functions are Generalized functions. It is called a bi-orthonormality since both sets of functions are not identical. The derivation is only true if the set of functions $x^n, (n = 0,1,2,\cdots)$ is not over-complete since in other case at least one the functions can be expressed by a superposition of the others. This can be shorter written with introduction of a scalar product definition $(f, g) \equiv \int_{-\infty}^{+\infty} \mathrm{d}x f(x) g(x)$ [1].

Due to symmetry $\delta(x-y) = \delta(y-x)$ one may derive from (2.3) also another interesting relation in the following way. We multiply both sides with a function $f(y)$ and integrate this then over y and find

$$\begin{aligned} f(x) &= \int_{-\infty}^{+\infty} \mathrm{d}y f(y) \delta(y-x) = \int_{-\infty}^{+\infty} \mathrm{d}y f(y) \sum_{n=0}^{+\infty} y^n \frac{(-1)^n}{n!} \delta^{(n)}(x) \\ &= \sum_{n=0}^{+\infty} \left(\frac{1}{n!} \int_{-\infty}^{+\infty} \mathrm{d}y f(y) y^n \right) (-1)^n \delta^{(n)}(x). \end{aligned} \tag{2.6}$$

This is an expansion of the functions $f(x)$ into a series of the delta function $\delta(x)$ and its derivatives which we write

$$f(x) = \sum_{n=0}^{+\infty} (-1)^n f_n(0) \delta^{(n)}(x), \tag{2.7}$$

with the abbreviation

$$f_n(0) \equiv \frac{1}{n!} \int_{-\infty}^{+\infty} \mathrm{d}y f(y) y^n. \tag{2.8}$$

We call such expansions moment series and applied them already to the derivation of generalized boundary conditions in electrodynamics of continuous media [18] [19].

In generalization to an arbitrary fixed reference point $x = x_0$ of the expansion the Taylor series (2.1) becomes

$$f(x) = \sum_{n=0}^{+\infty} \frac{(x-x_0)^n}{n!} f^{(n)}(x_0). \tag{2.9}$$

In these more general cases of the expansion (2.9) we find instead of (2.3) the more general completeness relation

$$\sum_{n=0}^{+\infty} (x-x_0)^n \frac{(-1)^n}{n!} \delta^{(n)}(y-x_0) = \delta(x-y), \tag{2.10}$$

and the bi-orthonormality

[1]Quantum theory has solved this elegantly in form of Dirac's notations by states and co-states (ket's $|x\rangle$ and bra's $\langle y|$) where $\langle y|x\rangle$ stands for the scalar product and $|x\rangle\langle y|$ for a dyadic product. In form of the coherent states $|\alpha\rangle$ quantum mechanics provides at once a famous set of over-complete and not mutually orthogonal states [17].

$$\int_{-\infty}^{+\infty}\mathrm{d}x\frac{(-1)^n}{n!}\delta^{(n)}(x-x_0)(x-x_0)^m=\delta_{n,m}. \tag{2.11}$$

The analogue to the moment series for a general reference point $x=x_0$ is

$$f(x)=\sum_{n=0}^{\infty}f_n(x_0)(-1)^n\delta^{(n)}(x-x_0), \tag{2.12}$$

with the moments

$$f_n(x_0)\equiv\frac{1}{n!}\int_{-\infty}^{+\infty}\mathrm{d}x\,f(x)(x-x_0)^n. \tag{2.13}$$

3. Duality between Bessel Functions and Fourier Transforms of Chebyshev Polynomials of First Kind

The starting point of the following considerations is the Generating function for Bessel functions of first kind $\mathrm{J}_n(z)$ with integer n (e.g., Watson [1] (II.2.1 (2.1) and Whittaker Watson [2]))

$$\exp\left(\frac{z}{2}\left(t-\frac{1}{t}\right)\right)=\sum_{n=-\infty}^{+\infty}t^n\mathrm{J}_n(z). \tag{3.1}$$

This is a Laurent series and from substitution $t\to\frac{1}{t}$ (or equivalently $n\to-n$ in t^n) follows

$$\mathrm{J}_{-n}(z)=(-1)^n\mathrm{J}_n(z). \tag{3.2}$$

Sometimes it is more convenient to use instead of $\mathrm{J}_n(z)$ the modified Bessel functions or Bessel functions of imaginary argument $\mathrm{I}_n(z)$ with the following connections to $\mathrm{J}_n(z)$

$$\mathrm{I}_n(z)\equiv(-\mathrm{i})^n\mathrm{J}_n(\mathrm{i}z),\quad\Leftrightarrow\quad\mathrm{J}_n(\mathrm{i}z)\equiv\mathrm{i}^n\mathrm{I}_n(z). \tag{3.3}$$

From the invariance of the left-hand side of (3.1) with respect to the simultaneous substitutions $z\to-z,t\to-t$ follows

$$\mathrm{J}_n(-z)=(-1)^n\mathrm{J}_n(z)=\mathrm{J}_{-n}(z),\quad\mathrm{I}_n(-z)=\mathrm{I}_n(z)=(-1)^n\mathrm{I}_{-n}(z). \tag{3.4}$$

The relation (3.1) together with (3.2) and (3.4) can be taken as basis for the construction of the whole theory of Bessel functions of integer (and by interpolation of non-integer) indices that according to [1] [2] goes back to Schlömilch and such an approach is sometimes used as, for example, by the here cited authors also in more recent time.

By the substitution

$$t=\mathrm{e}^{\mathrm{i}\theta},\quad\leftrightarrow\quad t^{-1}=\mathrm{e}^{-\mathrm{i}\theta}, \tag{3.5}$$

relation (3.1) makes the transition into

$$\exp(\mathrm{i}z\sin(\theta))=\sum_{n=-\infty}^{+\infty}\mathrm{J}_n(z)\mathrm{e}^{\mathrm{i}n\theta}=\mathrm{J}_0(z)+\sum_{n=1}^{+\infty}\mathrm{J}_n(z)\left(\mathrm{e}^{\mathrm{i}n\theta}+(-1)^n\mathrm{e}^{-\mathrm{i}n\theta}\right), \tag{3.6}$$

and by the substitution

$$t\equiv\mathrm{i}\mathrm{e}^{\mathrm{i}\theta},\quad\leftrightarrow\quad t^{-1}=-\mathrm{i}\mathrm{e}^{-\mathrm{i}\theta}, \tag{3.7}$$

the transition into

$$\exp\left(\mathrm{i}z\cos(\theta)\right)=\sum_{n=-\infty}^{+\infty}\mathrm{i}^{n}\,\mathrm{J}_{n}(z)\,\mathrm{e}^{\mathrm{i}n\theta}=\mathrm{J}_{0}(z)+\sum_{n=1}^{+\infty}\mathrm{i}^{n}\,\mathrm{J}_{n}(z)\left(\mathrm{e}^{\mathrm{i}n\theta}+\mathrm{e}^{-\mathrm{i}n\theta}\right). \quad (3.8)$$

Both relations (3.6) and (3.8) possess two aspects. First, the exponents on the left-hand side are periodic functions of the variable θ and the right-hand sides are Fourier series of the periodic functions with the Bessel functions as coefficients. The other aspect is that they provide expansions for two special functions in series of Bessel functions $\mathrm{J}_{n}(z)$.

Under the assumption that z and θ are real variables and taking into account the symmetry relations (3.2) we find the following separation of (3.8) in Real and Imaginary part

$$\begin{aligned}\cos\left(z\cos(\theta)\right)&=\sum_{m=-\infty}^{+\infty}(-1)^{m}\,\mathrm{J}_{2m}(z)\cos(2m\theta)\\&=\mathrm{J}_{0}(z)+2\sum_{m=1}^{+\infty}(-1)^{m}\,\mathrm{J}_{2m}(z)\cos(2m\theta),\\\sin\left(z\cos(\theta)\right)&=\sum_{m=-\infty}^{+\infty}(-1)^{m}\,\mathrm{J}_{2m+1}(z)\cos\left((2m+1)\theta\right)\\&=2\sum_{m=0}^{+\infty}(-1)^{m}\,\mathrm{J}_{2m+1}(z)\cos\left((2m+1)\theta\right),\end{aligned} \quad (3.9)$$

and similar relations with $\sin(\theta)$ in the argument by the substitution $\cos(\theta)\rightarrow\cos\left(\theta-\frac{\pi}{2}\right)=\sin(\theta)$ These formulae which are known as Jacobi expansions (e.g., Korenyev [4]) played an important role in the physics of the radio for the spectrum of phase modulation of carrier signals and also for frequency modulation since the frequency is the derivative of the phase with respect to the time. In **Appendix A** we collect for convenience the cases of modulation of Trigonometric and Hyperbolic functions by Trigonometric or Hyperbolic functions.

The differential equation for general Bessel functions $\mathrm{J}_{\nu}(z)$ with real or even complex ν is

$$0=\left\{\frac{\partial^{2}}{\partial z^{2}}+\frac{1}{z}\frac{\partial}{\partial z}+1-\frac{\nu^{2}}{z^{2}}\right\}\mathrm{J}_{\nu}(z). \quad (3.10)$$

A second linearly independent solution to the operator of this differential equation is for integer $\nu=n$ mostly denoted by $\mathrm{Y}_{n}(z)$ but we do not need it in the following and do not discuss it. The essential part of the operator of this differential equation appears if we write the two-dimensional Laplace operator $\nabla^{2}\equiv\frac{\partial^{2}}{\partial x^{2}}+\frac{\partial^{2}}{\partial y^{2}}$ in polar coordinates (r,φ) in combination with an angular part (see **Appendix B**, in particular, case $N=2$ in (B.9)). Further terms to the operator of the differential equation for the Bessel functions result if we consider instead the more general wave equation (with Fourier-transformed time) which is the Helmholtz equation. In **Appendix B** we derive for the N-dimensional

Laplace operator the radial and the angular parts, the last in coordinate-invariant way and extend it to the *N*-dimensional wave-equation operator.

In the following we use the Chebyshev polynomials of first kind $\mathrm{T}_n(x)$ and of second kind $\mathrm{U}_n(x)$ with the explicit representations[2]

$$\mathrm{T}_n(x)=\begin{cases}1, & n=0,\\ \frac{n}{2}\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k(n-1-k)!}{k!(n-2k)!}(2x)^{n-2k}, & n\neq 0,\end{cases}$$

$$\mathrm{U}_n(x)=\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k(n-k)!}{k!(n-2k)!}(2x)^{n-2k}, \tag{3.11}$$

with the symmetries

$$\mathrm{T}_n(x)=(-1)^n\,\mathrm{T}_n(-x)=\mathrm{T}_{-n}(x),$$

$$\mathrm{U}_n(x)=(-1)^n\,\mathrm{U}_n(-x)=-\mathrm{U}_{-n-2}(x). \tag{3.12}$$

Thus the full sets of Chebyshev polynomials with non-negative and negative indices are over-complete and can be reduced to a complete but not over-complete set. One of the basic possible representations of the Chebyshev polynomials by the Hypergeometric Function ${}_2\mathrm{F}_1(a,b;c;z)$ and, more specially, by the Jacobi polynomials $\mathrm{P}_n^{(\alpha,\beta)}(x)$ is

$$\mathrm{T}_n(x)={}_2\mathrm{F}_1\left(-n,n;\frac{1}{2};\frac{1-x}{2}\right)=\frac{2^{2n}n!^2}{(2n)!}\mathrm{P}_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x),$$

$$\mathrm{U}_n(x)=(n+1)\,{}_2\mathrm{F}_1\left(-n,n+2;\frac{3}{2};\frac{1-x}{2}\right)=\frac{2^{2n+1}(n+1)!^2}{(2n+2)!}\mathrm{P}_n^{\left(\frac{1}{2},\frac{1}{2}\right)}(x), \tag{3.13}$$

whereas the Bessel functions $\mathrm{J}_\nu(z)$ for arbitrary real and, moreover, complex ν are special cases of the Degenerate (or Confluent) Hypergeometric function ${}_1\mathrm{F}_1(a;c;z)$ according to

$$\begin{aligned}\mathrm{J}_\nu(z)&=\frac{1}{\nu!}\left(\frac{z}{2}\right)^\nu \mathrm{e}^{-\mathrm{i}z}\,{}_1\mathrm{F}_1\left(\frac{1}{2}+\nu;1+2\nu;\mathrm{i}2z\right)\\ &=\frac{1}{\nu!}\left(\frac{z}{2}\right)^\nu \mathrm{e}^{\mathrm{i}z}\,{}_1\mathrm{F}_1\left(\frac{1}{2}+\nu;1+2\nu;-\mathrm{i}2z\right)=\left(\frac{z}{2}\right)^\nu\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+\nu)!}\left(\frac{z}{2}\right)^{2k}.\end{aligned} \tag{3.14}$$

The well-known recurrence and differentiation relations are

$$\mathrm{J}_{\nu+1}(z)+\mathrm{J}_{\nu-1}(z)=2\frac{\nu}{z}\mathrm{J}_\nu(z),$$

$$\mathrm{J}_{\nu+1}(z)-\mathrm{J}_{\nu-1}(z)=-2\frac{\partial}{\partial z}\mathrm{J}_\nu(z). \tag{3.15}$$

The Taylor series in case of integer $\nu=n$ is

$$\mathrm{J}_n(z)=\left(\frac{z}{2}\right)^n\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+n)!}\left(\frac{z}{2}\right)^{2k}. \tag{3.16}$$

[2]The general formula (3.11) for $\mathrm{T}_n(x)$ is undetermined for $n=0$ of kind $0\cdot\infty$ but, by common definition, $\mathrm{T}_0(x)\equiv 1$ is fixed.

As is well-known (e.g. [3]) the Chebyshev polynomials satisfy the identities

$$\mathrm{T}_n(\cos(\theta)) = \cos(n\theta), \quad \mathrm{U}_n(\cos(\theta)) = \frac{\sin((n+1)\theta)}{\sin(\theta)}, \tag{3.17}$$

and similar relations by substitution $\cos(\theta) \to \mathrm{ch}(t), \theta \to \mathrm{i}t$. Using now the identities (3.17) the expansions (3.9) can be written

$$\cos(z\cos(\theta)) = \mathrm{J}_0(z)\underbrace{\mathrm{T}_0(\cos(\theta))}_{=1} + 2\sum_{m=1}^{+\infty}(-1)^m \mathrm{J}_{2m}(z)\mathrm{T}_{2m}(\cos(\theta)),$$

$$\sin(z\cos(\theta)) = 2\sum_{m=0}^{+\infty}(-1)^m \mathrm{J}_{2m+1}(z)\mathrm{T}_{2m+1}(\cos(\theta)). \tag{3.18}$$

In particular, introducing the abbreviation

$$x \equiv \cos(\theta), \tag{3.19}$$

relations (3.18) can be represented as follows (we change now the order of functions with variables z and x)

$$\cos(xz) = \mathrm{J}_0(z) + 2\sum_{m=1}^{+\infty}(-1)^m \mathrm{T}_{2m}(x)\mathrm{J}_{2m}(z),$$

$$\sin(xz) = 2\sum_{m=0}^{+\infty}(-1)^m \mathrm{T}_{2m+1}(x)\mathrm{J}_{2m+1}(z). \tag{3.20}$$

In the more compact form of (3.8) the expansions (3.20) can be written (observe: $\mathrm{i}^{-n}\mathrm{T}_{-n}(x) = (-1)^n \mathrm{i}^n \mathrm{T}_n(x)$)

$$\mathrm{e}^{\mathrm{i}xz} = \sum_{n=-\infty}^{+\infty} \mathrm{i}^n \mathrm{T}_n(x)\mathrm{J}_n(z) = \mathrm{J}_0(z) + 2\sum_{n=1}^{+\infty}\mathrm{i}^n \mathrm{T}_n(x)\mathrm{J}_n(z). \tag{3.21}$$

Later, we use mainly the identities (3.20) and (3.21).

It is well-known that the Chebyshev polynomials belong to the Classical Orthogonal Polynomials due to the orthogonality relations with weight functions $\frac{2}{\pi}\frac{1}{\sqrt{1-x^2}}$ and $\frac{2}{\pi}\sqrt{1-x^2}$, respectively (e.g., [3])

$$\frac{2}{\pi}\int_{-1}^{+1}\mathrm{d}x\frac{\mathrm{T}_m(x)\mathrm{T}_n(x)}{\sqrt{1-x^2}} = \frac{2}{\pi}\int_0^{\pi}\mathrm{d}\theta\,\mathrm{T}_m(\cos(\theta))\mathrm{T}_n(\cos(\theta)) = \begin{cases} 2\delta_{m0}\delta_{n0}, & n=0, \\ \delta_{mn}, & n \neq 0, \end{cases}$$

$$\frac{2}{\pi}\int_{-1}^{+1}\mathrm{d}x\sqrt{1-x^2}\,\mathrm{U}_m(x)\mathrm{U}_n(x) = \frac{2}{\pi}\int_0^{\pi}\mathrm{d}\theta\sin^2(\theta)\mathrm{U}_m(\cos(\theta))\mathrm{U}_n(\cos(\theta)) = \delta_{mn}. \tag{3.22}$$

However, this is an orthogonality over the interval from −1 to +1 (or 0 to π in variable θ) but not over the whole real axis or semi-axis. In the identities (3.20) and (3.21) the Chebyshev polynomials of first kind $\mathrm{T}_n(u)$ play a fully other role and are in a certain sense dual to the Bessel functions $\mathrm{J}_n(z)$ over the whole real axis. Furthermore, the whole set of Bessel functions $\mathrm{J}_n(z)$ with non-negative and negative indices is also over-complete in the same way as the Chebyshev polynomials $\mathrm{T}_n(x)$.

The identities (3.20) and (3.21) possess still another remarkable aspect. The multiplicative variables x and z on the right-hand sides are separated in different functions of the expansions on the right-hand sides. In this form it is simple to generalize these expansions by an arbitrary real or even complex parameter α and obtains from (3.20)

$$\cos(xz)=\cos\left(\alpha x\frac{z}{\alpha}\right)=\mathrm{J}_0\left(\frac{z}{\alpha}\right)+2\sum_{m=1}^{+\infty}(-1)^m\,\mathrm{T}_{2m}(\alpha x)\mathrm{J}_{2m}\left(\frac{z}{\alpha}\right),$$
$$\sin(xz)=\sin\left(\alpha x\frac{z}{\alpha}\right)=2\sum_{m=0}^{+\infty}(-1)^m\,\mathrm{T}_{2m+1}(\alpha x)\mathrm{J}_{2m+1}\left(\frac{z}{\alpha}\right), \tag{3.23}$$

and from (3.21)

$$\begin{aligned}\mathrm{e}^{\mathrm{i}xz}&=\exp\left(\mathrm{i}\alpha x\frac{z}{\alpha}\right)=\sum_{n=-\infty}^{+\infty}\mathrm{i}^n\,\mathrm{T}_n(\alpha x)\mathrm{J}_n\left(\frac{z}{\alpha}\right)\\&=\underbrace{\mathrm{T}_0(\alpha x)}_{=1}\mathrm{J}_0\left(\frac{z}{\alpha}\right)+2\sum_{n=1}^{+\infty}\mathrm{i}^n\,\mathrm{T}_n(\alpha x)\mathrm{J}_n\left(\frac{z}{\alpha}\right).\end{aligned} \tag{3.24}$$

The expansions (3.23) and (3.24) and former analogous expansions converge for arbitrary real and complex variables x and z and therefore the free parameter α in its full generality can be also chosen as a complex number.

It seems that the combination of Bessel functions and of Chebyshev polynomials of first kind here still in the special expansions (3.24) is new and in the next two Sections we show how this can be used to derive more general expansions with an arbitrary free parameter α into Bessel functions as well as into Fourier transforms of Chebyshev polynomials of first kind.

4. Expansions of Functions into a Series of Bessel Functions of Integer Indices

In this Section we derive an expansion of a general continuous function via its Fourier decomposition into a series of Bessel functions. We define the Fourier transform $\tilde{f}(y)$ of a function $f(x)$ as follows

$$\tilde{f}(y)=\int_{-\infty}^{+\infty}\mathrm{d}x\,f(x)\mathrm{e}^{-\mathrm{i}yx},\quad\Leftrightarrow\quad f(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}y\,\tilde{f}(y)\mathrm{e}^{\mathrm{i}xy}. \tag{4.1}$$

Then, if we apply the identities (3.23) or (3.24) we find

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}y\,\tilde{f}(u)\mathrm{e}^{\mathrm{i}xy}=\frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}\mathrm{i}^n\left(\int_{-\infty}^{+\infty}\mathrm{d}y\,\tilde{f}(y)\mathrm{T}_n(\alpha y)\right)\mathrm{J}_n\left(\frac{x}{\alpha}\right), \tag{4.2}$$

or split in real and imaginary part

$$\begin{aligned}f(x)=\frac{1}{2\pi}\Bigg\{&\sum_{m=-\infty}^{+\infty}(-1)^m\left(\int_{-\infty}^{+\infty}\mathrm{d}y\,\tilde{f}(y)\mathrm{T}_{2m}(\alpha y)\right)\mathrm{J}_{2m}\left(\frac{x}{\alpha}\right)\\&+\mathrm{i}\sum_{m=-\infty}^{+\infty}(-1)^m\left(\int_{-\infty}^{+\infty}\mathrm{d}y\,\tilde{f}(u)\mathrm{T}_{2m+1}(\alpha y)\right)\mathrm{J}_{2m+1}\left(\frac{x}{\alpha}\right)\Bigg\}.\end{aligned} \tag{4.3}$$

This is at once a decomposition of $f(x)$ into an even and odd part. Expansions of functions $f(x)$ in series of Bessel functions become possible if the Fourier transform $\tilde{f}(y)$ of the function $f(x)$ is known.

Using the symmetry of the Chebyshev polynomials of first kind and of the Bessel functions

$$\mathrm{T}_{-n}(x)=\mathrm{T}_n(x),\quad \mathrm{J}_{-n}(z)=(-1)^n\,\mathrm{J}_n(z), \tag{4.4}$$

expansions (4.2) or (4.3) can be represented

$$
\begin{aligned}
f(x)=\frac{1}{2\pi}&\left\{\left(\int_{-\infty}^{+\infty}\mathrm{d}y\,\tilde{f}(u)\mathrm{T}_0(\alpha y)\right)\mathrm{J}_0\left(\frac{x}{\alpha}\right)\right.\\
&+2\sum_{m=1}^{+\infty}(-1)^m\left(\int_{-\infty}^{+\infty}\mathrm{d}y\,\tilde{f}(y)\mathrm{T}_{2m}(\alpha y)\right)\mathrm{J}_{2m}\left(\frac{x}{\alpha}\right)\\
&\left.+\mathrm{i}2\sum_{m=0}^{+\infty}(-1)^m\left(\int_{-\infty}^{+\infty}\mathrm{d}y\,\tilde{f}(u)\mathrm{T}_{2m+1}(\alpha y)\right)\mathrm{J}_{2m+1}\left(\frac{x}{\alpha}\right)\right\}.
\end{aligned}
\tag{4.5}
$$

Formulae (4.2), (4.3) and (4.5) are the expansion of a function $f(x)$ into a series of Bessel functions of integer indices with a free parameter α of the following principal form

$$
\begin{aligned}
f(x)&=\sum_{n=-\infty}^{+\infty}c_n(\alpha)\mathrm{J}_n\left(\frac{x}{\alpha}\right)\\
&=c_0(\alpha)\mathrm{J}_0\left(\frac{x}{\alpha}\right)+\sum_{n=1}^{+\infty}\left(c_{2m}(\alpha)+c_{-2m}(\alpha)\right)\mathrm{J}_{2m}\left(\frac{x}{\alpha}\right)\\
&\quad+\sum_{m=0}^{+\infty}\left(c_{2m+1}(\alpha)-c_{-2m-1}(\alpha)\right)\mathrm{J}_{2m+1}\left(\frac{x}{\alpha}\right),
\end{aligned}
\tag{4.6}
$$

that means with relations for the coefficients in front of the Bessel functions with negative and positive indices

$$
\begin{aligned}
f(x)\equiv{}&c_0(\alpha)\mathrm{J}_0\left(\frac{x}{\alpha}\right)+2\sum_{n=1}^{+\infty}c_{2m}(\alpha)\mathrm{J}_{2m}\left(\frac{x}{\alpha}\right)\\
&+2\sum_{m=0}^{+\infty}c_{2m+1}(\alpha)\mathrm{J}_{2m+1}\left(\frac{x}{\alpha}\right),\quad\left(c_{-n}(\alpha)\equiv(-1)^n c_n(\alpha)\right),
\end{aligned}
\tag{4.7}
$$

with the coefficients which can be represented by the unique formula

$$
c_n(\alpha)=\frac{\mathrm{i}^n}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}y\,\tilde{f}(y)\mathrm{T}_n(\alpha y),\quad(n=0,1,2,\cdots).
\tag{4.8}
$$

This formula can be applied if we know the Fourier transform $\tilde{f}(y)$ of the function $f(x)$.

Changing the order of integrations the formula (4.8) for the coefficients can be also represented in the following way

$$
\begin{aligned}
c_n(\alpha)&=\frac{\mathrm{i}^n}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}y\,\mathrm{T}_n(\alpha y)\int_{-\infty}^{+\infty}\mathrm{d}x\,f(x)\mathrm{e}^{-\mathrm{i}yx}\\
&=\frac{\mathrm{i}^n}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}x\,f(x)\left(\int_{-\infty}^{+\infty}\mathrm{d}y\,\mathrm{T}_n(\alpha y)\mathrm{e}^{-\mathrm{i}yx}\right)\\
&=\frac{\mathrm{i}^n}{2\pi|\alpha|}\int_{-\infty}^{+\infty}\mathrm{d}x\,f(x)\tilde{\mathrm{T}}_n\left(\frac{x}{\alpha}\right),
\end{aligned}
\tag{4.9}
$$

where $\tilde{\mathrm{T}}_n(y)$ is the Fourier transform of $\mathrm{T}_n(x)$ in the sense of (4.1). The Fourier transform of a polynomial, contrary to the Fourier transform of an arbitrary function, is in every case a finite superposition of the delta function and of their derivatives and thus always a Generalized function.

Using the well-known expression of the Chebyshev polynomials of first kind $\mathrm{T}_n(x)$ in powers of x given in (3.11) we find its Fourier transform which similar to all polynomials is a singular function which can be expressed by a superposition of derivatives of the delta function as follows

$$
\begin{aligned}
\tilde{\mathrm{T}}_n(y) &= \frac{n}{2}\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k(n-1-k)!2^{n-2k}}{k!(n-2k)!}\int_{-\infty}^{+\infty}\mathrm{d}x\,x^{n-2k}\mathrm{e}^{-\mathrm{i}yx} \\
&= \frac{n}{2}\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k(n-1-k)!}{k!(n-2k)!}\left(\mathrm{i}2\frac{\partial}{\partial y}\right)^{n-2k}\int_{-\infty}^{+\infty}\mathrm{d}x\mathrm{e}^{-\mathrm{i}yx} \\
&= 2\pi\mathrm{T}_n\left(\mathrm{i}\frac{\partial}{\partial y}\right)\delta(y).
\end{aligned}
\tag{4.10}
$$

From this follows

$$
\tilde{\mathrm{T}}_n\left(\frac{y}{\alpha}\right) = 2\pi\mathrm{T}_n\left(\mathrm{i}\alpha\frac{\partial}{\partial y}\right)\delta\left(\frac{y}{\alpha}\right) = 2\pi|\alpha|\mathrm{T}_n\left(\mathrm{i}\alpha\frac{\partial}{\partial y}\right)\delta(y).
\tag{4.11}
$$

Inserting this into the formula (4.9) for the coefficients and changing the integration variable $x \to y$ we find

$$
\begin{aligned}
c_n(\alpha) &= \mathrm{i}^n\int_{-\infty}^{+\infty}\mathrm{d}y\,f(y)\left(\mathrm{T}_n\left(\mathrm{i}\alpha\frac{\partial}{\partial y}\right)\delta(y)\right) \\
&= \mathrm{i}^n\int_{-\infty}^{+\infty}\mathrm{d}y\,\delta(y)\left(\mathrm{T}_n\left(-\mathrm{i}\alpha\frac{\partial}{\partial y}\right)f(y)\right) \\
&= \mathrm{i}^n\left\{\mathrm{T}_n\left(-\mathrm{i}\alpha\frac{\partial}{\partial y}\right)f(y)\right\}_{y=0},
\end{aligned}
\tag{4.12}
$$

and explicitly using the form of the Chebyshev polynomials $\mathrm{T}_n(x)$ in (3.11)

$$
c_n(\alpha) = \frac{n}{2}\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(n-1-k)!}{k!(n-2k)!}(2\alpha)^{n-2k}f^{(n-2k)}(0).
\tag{4.13}
$$

The transition from the first to the second line in (4.12) is in the sense of the theory of generalized functions.

In Section 6 we consider special cases of expansions in Bessel functions. First, we try to derive completeness and (bi)-orthogonality relation for the expansions in Bessel functions.

5. Completeness and Orthogonality Relations between Bessel Functions and Fourier Transforms of Chebyshev Polynomials of First Kind

By combination of the formal expansion (4.6) with the form of the coefficients (4.12) we find the relation

$$
\begin{aligned}
f(x) &= \sum_{n=-\infty}^{+\infty}\mathrm{J}_n\left(\frac{x}{\alpha}\right)\int_{-\infty}^{+\infty}\mathrm{d}y\,f(y)\mathrm{i}^n\mathrm{T}_n\left(\mathrm{i}\alpha\frac{\partial}{\partial y}\right)\delta(y) \\
&= \int_{-\infty}^{+\infty}\mathrm{d}y\,f(y)\sum_{n=-\infty}^{+\infty}\mathrm{J}_n\left(\frac{x}{\alpha}\right)\mathrm{i}^n\mathrm{T}_n\left(\mathrm{i}\alpha\frac{\partial}{\partial y}\right)\delta(y) \\
&= \int_{-\infty}^{+\infty}\mathrm{d}y\,f(y)\delta(x-y).
\end{aligned}
\tag{5.1}
$$

Therefore, it should be

$$\sum_{n=-\infty}^{+\infty} \mathrm{J}_n\left(\frac{x}{\alpha}\right) \mathrm{i}^n \mathrm{T}_n\left(\mathrm{i}\alpha\frac{\partial}{\partial y}\right) = \exp\left(-x\frac{\partial}{\partial y}\right), \tag{5.2}$$

where the operator on the right-hand side is the displacement operator for the shift of the variable y of an arbitrary function $f(y)$ from y to $y \to y - x$. Indeed, if we substitute in (3.24) $z \to x, x \to \mathrm{i}\frac{\partial}{\partial y}$ we obtain the identity (5.2). From this identity follows then immediately

$$\sum_{n=-\infty}^{+\infty} \mathrm{J}_n\left(\frac{x}{\alpha}\right) \mathrm{i}^n \mathrm{T}_n\left(\mathrm{i}\alpha\frac{\partial}{\partial y}\right)\delta(y) = \delta(x-y). \tag{5.3}$$

This is a kind of completeness relation for expansion of continuous functions defined over the whole real axis and is in analogy to (2.3). However, it is over-complete since the Bessel functions $\mathrm{J}_n(z)$ with negative indices n are related to that with corresponding positive indices and in analogous way the Chebyshev polynomials of first kind by (4.4). Excluding this over-completeness we find from (5.3)

$$\left\{\mathrm{J}_0\left(\frac{x}{\alpha}\right) + 2\sum_{n=1}^{+\infty} \mathrm{J}_n\left(\frac{x}{\alpha}\right) \mathrm{i}^n \mathrm{T}_n\left(\mathrm{i}\alpha\frac{\partial}{\partial y}\right)\right\}\delta(y) = \delta(x-y). \tag{5.4}$$

It is no more an over-completeness since from the expansion of the Bessel functions $\mathrm{J}_n\left(\frac{x}{\alpha}\right), (n = 0,1,2,\cdots)$ exactly one begins with the powers proportional to $x^n, (n = 0,1,2,\cdots)$.

We now multiply the completeness relation (5.4) with the Bessel function $\mathrm{J}_m\left(\frac{y}{\alpha}\right)$ and integrate the obtained relation over the variable y and find

$$\begin{aligned} \mathrm{J}_m\left(\frac{x}{\alpha}\right) &= \int_{-\infty}^{+\infty} \mathrm{d}y\,\delta(x-y)\mathrm{J}_m\left(\frac{y}{\alpha}\right) \\ &= \int_{-\infty}^{+\infty} \mathrm{d}y\,\mathrm{J}_m\left(\frac{y}{\alpha}\right)\left\{\mathrm{J}_0\left(\frac{x}{\alpha}\right) + 2\sum_{n=1}^{+\infty} \mathrm{J}_n\left(\frac{x}{\alpha}\right) \mathrm{i}^n \mathrm{T}_n\left(\mathrm{i}\alpha\frac{\partial}{\partial y}\right)\right\}\delta(y) \\ &= \mathrm{J}_0\left(\frac{x}{\alpha}\right)\underbrace{\mathrm{J}_m(0)}_{=\delta_{m,0}} + 2\sum_{n=1}^{+\infty} \mathrm{J}_n\left(\frac{x}{\alpha}\right)\int_{-\infty}^{+\infty} \mathrm{d}y\,\mathrm{J}_m\left(\frac{y}{\alpha}\right) \mathrm{i}^n \mathrm{T}_n\left(\mathrm{i}\alpha\frac{\partial}{\partial y}\right)\delta(y). \end{aligned} \tag{5.5}$$

From this relation we conclude the orthonormality relation

$$2\int_{-\infty}^{+\infty} \mathrm{d}y\,\mathrm{J}_m\left(\frac{y}{\alpha}\right) \mathrm{i}^n \mathrm{T}_n\left(\mathrm{i}\alpha\frac{\partial}{\partial y}\right)\delta(y) = \begin{cases} 2\delta_{m0}\delta_{n0}, & n = 0, \\ \delta_{mn}, & n \neq 0. \end{cases} \tag{5.6}$$

The left-hand side of this identity can be also represented according to the rules of differentiations of the delta function in the following way

$$\begin{aligned} 2\left\{\mathrm{i}^n \mathrm{T}_n\left(-\mathrm{i}\alpha\frac{\partial}{\partial y}\right)\mathrm{J}_m\left(\frac{y}{\alpha}\right)\right\}_{y=0} &= 2\int_{-\infty}^{+\infty} \mathrm{d}y\,\delta(y)\mathrm{i}^n \mathrm{T}_n\left(-\mathrm{i}\alpha\frac{\partial}{\partial y}\right)\mathrm{J}_m\left(\frac{y}{\alpha}\right) \\ &= \begin{cases} 2\delta_{m0}\delta_{n0}, & n = 0, \\ \delta_{mn}, & n \neq 0, \end{cases} \end{aligned} \tag{5.7}$$

with the right-hand side similar to the right-hand side of (3.22). It is not very difficult to check this formula numerically for low values n and, at least, larger values m by computer[3].

With the formulae (5.3) and (5.7) one may make also expansions of continuous and continuously differentiable functions in terms of the generalized functions $\mathrm{i}^n \mathrm{T}_n\left(\mathrm{i}\alpha \frac{\partial}{\partial x}\right)\delta(x)$ in analogy to the moment series (2.7) with the coefficients in form of the moments (2.8). However, in present case the coefficients become mixed moments with the Bessel functions as coefficients and it is not to see whether or not such expansions will get some importance in future. We mention yet another feature of the identities (5.6) and (5.7). The integration goes here over the full interval from $-\infty$ to $+\infty$ and there is no way to shorten this interval to that from 0 to $+\infty$.

6. Examples of Expansions in Bessel Functions

We consider now examples for expansion of functions in series over Bessel functions with integer indices.

First we discuss the expansion of the monomials z^n into Bessel functions. For this purpose we may use the formula (3.24) where we make on both sides an expansion in powers of x^n for which the second line is best appropriate. If we insert the explicit form of the Chebyshev polynomials $\mathrm{T}_n(x)$ given in (3.11) then we find

$$
\begin{aligned}
\mathrm{e}^{\mathrm{i}xz} &= 1+\sum_{n=1}^{\infty}\frac{(\mathrm{i}x)^n}{n!}z^n \\
&= \mathrm{J}_0\left(\frac{z}{\alpha}\right)+2\sum_{m=1}^{\infty}\mathrm{i}^m\frac{m}{2}\sum_{k=0}^{\left[\frac{m}{2}\right]}\frac{(-1)^k(m-1-k)!}{k!(m-2k)!}(2\alpha x)^{m-2k}\mathrm{J}_m\left(\frac{z}{\alpha}\right) \qquad (6.1)\\
&= \mathrm{J}_0\left(\frac{z}{\alpha}\right)+2\sum_{k=0}^{\infty}\mathrm{J}_{2k}\left(\frac{z}{\alpha}\right)+\sum_{n=1}^{\infty}\frac{(\mathrm{i}2\alpha x)^n}{n!}\sum_{k=0}^{\infty}\frac{(n-1+k)!}{k!}(n+2k)\mathrm{J}_{n+2k}\left(\frac{z}{\alpha}\right).
\end{aligned}
$$

Collecting on both sides the terms proportional to x^n for the expansions of z^n in series of Bessel functions follows a relation equivalent to

$$
z^n = \begin{cases} \mathrm{J}_0(z)+2\sum_{k=0}^{\infty}\mathrm{J}_{2k}(z)=1, & n=0, \\ 2^n\sum_{k=0}^{\infty}\frac{(n-1+k)!}{k!}(n+2k)\mathrm{J}_{n+2k}(z), & n\neq 0. \end{cases} \qquad (6.2)
$$

The involvement of the free parameter α is here trivial and can be obtained by the substitution $z \to \frac{z}{\alpha}$. A result for this kind of relation one finds in Watson [1] (p. 138) which he refers to Gegenbauer. It agrees with our result almost but not fully concerning the case $n=0$ in (6.2).

[3]However, it is more difficult to check the expressions with the delta function directly by computer since up to now this function is not programmed in "Mathematica" and one has to use for this purpose approximations such as by the Gauss bell function and then have to make a limiting transition to the delta function but this we did not make.

Expansions for arbitrary functions with a Taylor series can be used to establish their expansions into series of Bessel functions. In this approach one has then to collect the sum terms proportional to $\mathrm{J}_n(z)$ and to reorder the sums in corresponding way.

Using the duality between Bessel functions $\mathrm{J}_n(z)$ and Chebyshev polynomials of first kind $\mathrm{T}_n(x)$ in (3.24) one may also obtain the expansion of power functions x^n into Chebyshev polynomials. In analogy to (6.1) one has[4]

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}xz} &= 1+\sum_{n=1}^{\infty}\frac{(\mathrm{i}z)^n}{n!}x^n \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!^2}\left(\frac{z}{2\alpha}\right)^{2k}\mathrm{T}_0(\alpha x)+2\sum_{m=1}^{\infty}\mathrm{i}^m\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(m+k)!}\left(\frac{z}{2\alpha}\right)^{m+2k}\mathrm{T}_m(\alpha x) \\
&= \mathrm{T}_0(\alpha x)+\sum_{n=1}^{\infty}\left(\frac{\mathrm{i}z}{2\alpha}\right)^n\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\mathrm{T}_{n-2k}(\alpha x).
\end{aligned} \tag{6.3}$$

Collecting on both sides the terms in front of z^n one finds ([13] and, more generally, [14])

$$x^n=\begin{cases}\mathrm{T}_0(x)=1, & n=0,\\ \dfrac{1}{2^n}\displaystyle\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\mathrm{T}_{n-2k}(x), & n\neq 0,\end{cases} \tag{6.4}$$

where the involvement of the free parameter α such as in the case (6.2) is trivial and can be reached by the substitution $x\to\alpha x$.

From Section 7 on we deal with Bessel functions with semi-integer indices and closely related Spherical Bessel functions with some different behavior from that for integer indices.

7. Expansions of Functions into Series of Spherical Bessel functions

Up to now we discussed expansions of functions into series over Bessel functions $\mathrm{J}_\nu(z)$ with integer indices $\nu=n,(n=0,1,2,\cdots)$. Now we will consider expansions of functions into series over Bessel functions $\mathrm{J}_\nu(z)$ with semi-integer indices $\nu=n+\frac{1}{2},(n=0,1,2,\cdots)$ modified by an additional z-dependent factor. This leads us as dual partner to the Legendre polynomials that is not expressed in the title of the article.

The general Taylor series of Bessel functions $\mathrm{J}_\nu(z)$ with arbitrary complex ν was given in (3.14). From this formula follows for semi-integer indices $\nu=n+\frac{1}{2}$

$$\mathrm{J}_{n+\frac{1}{2}}(z)=\left(\frac{z}{2}\right)^{n+\frac{1}{2}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\left(k+n+\frac{1}{2}\right)!}\left(\frac{z}{2}\right)^{2k}. \tag{7.1}$$

[4]On the second line of the following identity we have only Chebyshev polynomials with non-negative indices. For an easier representation we have in third line also Chebyshev polynomials with negative indices that means an over-complete set of Chebyshev polynomials where we used the relation $\mathrm{T}_n(x)=\mathrm{T}_{-n}(x)$.

These functions are not entire functions because they possess a branch point at $z=0$ due to factor $\left(\frac{z}{2}\right)^{\frac{1}{2}}$. We remove this factor by the definition

$$\mathrm{j}_n(z) \equiv \sqrt{\frac{\pi}{2z}} \mathrm{J}_{n+\frac{1}{2}}(z), \tag{7.2}$$

with the expansion into a Taylor series

$$\mathrm{j}_n(z) = \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{1}{2}\right)!}{k!\left(k+n+\frac{1}{2}\right)!} \left(\frac{z}{2}\right)^{2k}, \quad \mathrm{j}_n(z) = (-1)^n \mathrm{j}_n(-z). \tag{7.3}$$

and arrived with the definition (7.2) at a sequence of functions $\mathrm{j}_n(z), (n = 0, \pm 1, \pm 2, \cdots)$ which are called the Spherical Bessel functions. They are the three-dimensional analogue to the Bessel functions for dimension $N=2$ concerning the splitting of the Laplace or wave-equation operator in Radius and Spherical coordinates (see **Appendix B**, in particular, the case $N=3$ in (B.9)). In comparison to (B.9) the definition of the Spherical Bessel functions contains an additional common factor $\sqrt{\frac{\pi}{2}}$ for all these functions to get a more convenient normalization. For $z<0$ the Bessel functions $\mathrm{J}_{n+\frac{1}{2}}(z)$ as well as the root $\frac{1}{\sqrt{z}}$ become imaginary but their coupling makes them real-valued on the whole real axis. The Spherical Bessel functions appear if we write down the three-dimensional wave equation or Helmholtz equation $\left(\nabla^2 + \kappa^2\right)\psi(\mathrm{r}) = 0$ in Spherical coordinates as separated solutions for the radial part (e.g., [5]) (**Appendix B**). The solutions for the angular part are then the Spherical harmonics.

From the general recurrence and differentiation relations for Bessel functions (3.15) follows for the Spherical Bessel functions

$$\begin{aligned} \mathrm{j}_{n+1}(z) + \mathrm{j}_{n-1}(z) &= \frac{2n+1}{z} \mathrm{j}_n(z), \\ \mathrm{j}_{n+1}(z) - \mathrm{j}_{n-1}(z) &= -\left(2\frac{\partial}{\partial z} + \frac{1}{z}\right) \mathrm{j}_n(z), \end{aligned} \tag{7.4}$$

and by linear combinations for the raising and lowering relations for the indices of the Spherical Bessel functions

$$\begin{aligned} \mathrm{j}_{n+1}(z) &= \left(\frac{n}{z} - \frac{\partial}{\partial z}\right) \mathrm{j}_n(z) = -z^n \frac{\partial}{\partial z} \frac{1}{z^n} \mathrm{j}_n(z), \\ \mathrm{j}_{n-1}(z) &= \left(\frac{n+1}{z} + \frac{\partial}{\partial z}\right) \mathrm{j}_n(z) = \frac{1}{z^{n+1}} \frac{\partial}{\partial z} z^{n+1} \mathrm{j}_n(z). \end{aligned} \tag{7.5}$$

The differential equation for Spherical Bessel functions $\mathrm{j}_n(z)$ is (see (B.9), case $N=3$)

$$0 = \left\{ \frac{\partial^2}{\partial z^2} + \frac{2}{z} \frac{\partial}{\partial z} + 1 - \frac{n(n+1)}{z^2} \right\} \mathrm{j}_n(z). \tag{7.6}$$

For $n=0$ and $n=-1$ we find from the Taylor series (7.3)

$$\mathrm{j}_0(z)=\frac{\sin(z)}{z}=-\left(\frac{1}{z}+\frac{\partial}{\partial z}\right)\mathrm{j}_{-1}(z),$$

$$\mathrm{j}_{-1}(z)=\frac{\cos(z)}{z}=\left(\frac{1}{z}+\frac{\partial}{\partial z}\right)\mathrm{j}_0(z). \tag{7.7}$$

Using the raising and lowering relations (7.5) for the indices one finds the Spherical Bessel functions $\mathrm{j}_n(z)$ with arbitrary integer indices n in the following way

$$\mathrm{j}_n(z)=(-z)^n\left(\frac{1}{z}\frac{\partial}{\partial z}\right)^n\frac{\sin(z)}{z},\quad \mathrm{j}_{-n-1}(z)=z^n\left(\frac{1}{z}\frac{\partial}{\partial z}\right)^n\frac{\cos(z)}{z}, \tag{7.8}$$

that can be proved by complete induction. The Spherical Bessel functions $\mathrm{j}_{-n-1}(z),(n=0,1,2,\cdots)$ are singular at $z=0$. A greater number of explicit relations for the Spherical Bessel functions are given in **Appendix C**. In this **Appendix** we give also formulae for normal and anti-normal ordering of the operators $\left(\frac{1}{z}\frac{\partial}{\partial z}\right)^n$ and $\left(\frac{\partial}{\partial z}\frac{1}{z}\right)^n$.

The Spherical Bessel functions possess a property of orthogonality on the whole real axis which, apparently, is exceptional within the sequences of modified Bessel function (B.9) as non-singular solutions of the N-dimensional Radial part of the Helmholtz equation corresponding to dimension $N=3$ (e.g., [5], p. 629, 11.174; [12], p. 211)

$$\int_{-\infty}^{+\infty}\mathrm{d}z\,\mathrm{j}_m(z)\,\mathrm{j}_n(z)=\frac{\pi}{m+n+1}\delta_{mn}. \tag{7.9}$$

This means that an appropriate continuous function on the real axis can be expanded into Spherical Bessel functions according to

$$f(x)=\sum_{n=0}^{\infty}c_n\mathrm{j}_n(z), \tag{7.10}$$

with the coefficients c_n determined by

$$c_n=\frac{2n+1}{\pi}\int_{-\infty}^{+\infty}\mathrm{d}z\,f(z)\,\mathrm{j}_n(z). \tag{7.11}$$

We mention that the orthogonality of the Spherical Bessel function (7.9) is not true for restriction of the integral to one of the semi-axes, for example from $0\le z<+\infty$. On the other side as radial part of solutions of the three-dimensional Laplace or Helmholtz equation the Spherical Bessel functions are in most applications restricted to the positive semi-axis.

In the next Section we show that the dual partner in expansion of functions into Spherical Bessel functions are the Legendre polynomials.

8. Legendre Polynomials as Dual Partner to Spherical Bessel Functions in Expansion of Functions

Starting point for the derivation of functions into series of Spherical Bessel functions are here again special expansions for $\cos(xz)$ and $\sin(xz)$ in analogy to (3.23) as follows

$$\begin{aligned}\cos(xz)&=\cos\left(\alpha x\frac{z}{\alpha}\right)=\sum_{m=0}^{\infty}(-1)^m(4m+1)\mathrm{P}_{2m}(\alpha x)\,\mathrm{j}_{2m}\left(\frac{z}{\alpha}\right),\\ \sin(xz)&=\sin\left(\alpha x\frac{z}{\alpha}\right)=\sum_{m=0}^{\infty}(-1)^m(4m+3)\mathrm{P}_{2m+1}(\alpha x)\,\mathrm{j}_{2m+1}\left(\frac{z}{\alpha}\right),\end{aligned}\tag{8.1}$$

and both these formulae can be combined to

$$\mathrm{e}^{\mathrm{i}xz}=\exp\left(\mathrm{i}\alpha x\frac{z}{\alpha}\right)=\sum_{n=0}^{\infty}\mathrm{i}^n(2n+1)\mathrm{P}_n(\alpha x)\,\mathrm{j}_n\left(\frac{z}{\alpha}\right),\tag{8.2}$$

where $\mathrm{P}_n(x),(n=0,1,2,\cdots)$ are the Legendre polynomials and where α is a free eligible parameter. The expansion (8.2) is a known relation and one may find it, for example, in the encyclopedic articles [7] [9] and can prove it, for example, using the expansion (7.10) with the formula for the coefficients (7.11). It is the analogue of the expansion of the Exponential function (3.24) in Chebyshev polynomials of first kind with Bessel functions with integer indices as coefficients.

The Legendre polynomials possess the explicit form (observe $\left(-\frac{1}{2}\right)!=\sqrt{\pi}$)

$$\mathrm{P}_n(x)=\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k\left(n-k-\frac{1}{2}\right)!}{k!(n-2k)!\left(-\frac{1}{2}\right)!}(2x)^{n-2k},\quad \mathrm{P}_n(x)=(-1)^n\mathrm{P}_n(-x)=\mathrm{P}_{-n-1}(x).\tag{8.3}$$

Similar to the Chebyshev polynomials the Legendre polynomials are a special case of the Jacobi polynomials $\mathrm{P}_n^{(\alpha,\beta)}(x)$ with equal upper indices $\alpha=\beta$ (Ultraspherical polynomials). They are related to Gegenbauer polynomials $\mathrm{C}_n^{\nu}(x)$ by

$$\mathrm{P}_n^{(\alpha,\alpha)}(x)=\frac{(n+\alpha)!(2\alpha)!}{(n+2\alpha)!\alpha!}\mathrm{C}_n^{\alpha+\frac{1}{2}}(x),\tag{8.4}$$

and altogether are a special case of the Hypergeometric function ${}_2\mathrm{F}_1(a,b;c;z)$, in particular

$$\mathrm{P}_n(x)={}_2\mathrm{F}_1\left(-n,n+1;1;\frac{1-x}{2}\right)=\mathrm{P}_n^{(0,0)}(x).\tag{8.5}$$

The set of Legendre polynomials is complete for non-negative indices n and is separated from the set of Legendre polynomials with negative indices which are also complete (see **Appendix D** for comparison with Chebyshev and Gegenbauer polynomials). The set of Spherical Bessel functions is also complete for non-negative indices but for negative indices it forms a new series which is equivalent to a second solution of the differential equation.

The Legendre polynomials belong to the Classical Orthogonal Polynomials and are orthogonal (and thus self-dual) in the interval $-1\le x\le 1$ according to

$$\int_{-1}^{+1}\mathrm{d}x\,\mathrm{P}_m(x)\mathrm{P}_n(x)=\int_0^{\pi}\mathrm{d}\theta\sin(\theta)\mathrm{P}_m(\cos(\theta))\mathrm{P}_n(\cos(\theta))=\frac{2}{m+n+1}\delta_{mn},\tag{8.6}$$

but they are not orthogonal on the whole real axis or semi-axis and are here applied in a different sense.

The Chebyshev polynomials of first kind and the Legendre polynomials are related by the identities

$$\mathrm{P}_n(x)=\frac{1}{\pi}\sum_{k=0}^{n}\frac{\left(k-\frac{1}{2}\right)!\left(n-k-\frac{1}{2}\right)!}{k!(n-k)!}\mathrm{T}_{n-2k}(x),$$

$$\mathrm{T}_n(x)=\begin{cases}\mathrm{P}_0(x)=1, & n=0,\\ -\frac{n}{8}\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{\left(k-\frac{3}{2}\right)!(n-k-1)!}{k!\left(n-k+\frac{1}{2}\right)!}(2n-4k+1)\mathrm{P}_{n-2k}(x), & n\neq 0.\end{cases} \quad (8.7)$$

These relations can be proved by complete induction from their recurrence and differentiation relations.

9. Expansions of Functions into Series of Spherical Bessel Functions

Expansions of Exponential functions of the special kind (3.24) and (8.2) are favorable as starting point for the derivation of expansions of more general functions using the Fourier decomposition of these functions. This was made in Section 4 for the expansion in Bessel functions of integer argument. In this Section we develop the analogous case of expansions in Spherical Bessel functions using (8.2) as starting point. The Fourier transform of a function and its inversion are defined in the same way as in (4.1).

In analogy to (4.2) using (8.2) we find the expansion of a function in Spherical Bessel functions

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}y\,\tilde{f}(u)\mathrm{e}^{\mathrm{i}xy}=\frac{1}{2\pi}\sum_{n=0}^{+\infty}\mathrm{i}^n(2n+1)\left(\int_{-\infty}^{+\infty}\mathrm{d}y\,\tilde{f}(y)\mathrm{P}_n(\alpha y)\right)\mathrm{j}_n\left(\frac{x}{\alpha}\right), \quad (9.1)$$

or split in Real and Imaginary part

$$\begin{aligned}f(x)=\frac{1}{2\pi}\Bigg\{&\sum_{m=0}^{+\infty}(-1)^m(4m+1)\left(\int_{-\infty}^{+\infty}\mathrm{d}y\,\tilde{f}(y)\mathrm{P}_{2m}(\alpha y)\right)\mathrm{j}_{2m}\left(\frac{x}{\alpha}\right)\\ &+\mathrm{i}\sum_{m=0}^{\infty}(-1)^m(4m+3)\left(\int_{-\infty}^{+\infty}\mathrm{d}y\,\tilde{f}(u)\mathrm{P}_{2m+1}(\alpha y)\right)\mathrm{j}_{2m+1}\left(\frac{x}{\alpha}\right)\Bigg\}.\end{aligned} \quad (9.2)$$

A difference to (4.3) is that for simplicity an over-complete set Chebyshev polynomials of first kind $\mathrm{T}_n(x)$ is used there whereas here it is favorable to use the complete but not over-complete set of Legendre polynomials $\mathrm{P}_n(x)$ with non-negative indices.

Thus a function $f(x)$ can be expanded in Spherical Bessel functions in the following way

$$f(x)=\sum_{n=0}^{\infty}c_n(\alpha)j_n\left(\frac{z}{\alpha}\right), \quad (9.3)$$

where the coefficients of the expansion are determined by

$$c_n(\alpha)=\frac{\mathrm{i}^n}{2\pi}(2n+1)\int_{-\infty}^{+\infty}\mathrm{d}y\,\tilde{f}(y)\mathrm{P}_n(\alpha y). \tag{9.4}$$

The next relations which we write down in the following can be made in full analogy to corresponding relations for Bessel functions and Chebyshev polynomials in Section 5. In this way, in analogy to (5.3) the completeness relation for the pair of Spherical Bessel functions and Legendre polynomials is

$$\sum_{n=0}^{+\infty}(2n+1)\,\mathrm{j}_n\left(\frac{x}{\alpha}\right)\mathrm{i}^n\mathrm{P}_n\left(\mathrm{i}\alpha\frac{\partial}{\partial y}\right)\delta(y)=\delta(x-y). \tag{9.5}$$

from which follows by multiplication with $\mathrm{j}\left(\frac{y}{\alpha}\right)$ and integration over the real axis y a relation from which we conclude the orthogonality

$$\int_{-\infty}^{+\infty}\mathrm{d}y\,\mathrm{j}_m\left(\frac{y}{\alpha}\right)\mathrm{i}^n\mathrm{P}_n\left(\mathrm{i}\alpha\frac{\partial}{\partial y}\right)\delta(y)=\frac{\delta_{m,n}}{m+n+1}. \tag{9.6}$$

According to the rules of the theory of generalized functions this may be rewritten in analogy to (5.7)

$$\left\{\mathrm{i}^n\mathrm{P}_n\left(-\mathrm{i}\alpha\frac{\partial}{\partial y}\right)\mathrm{j}_m\left(\frac{y}{\alpha}\right)\right\}_{y=0}=\int_{-\infty}^{+\infty}\mathrm{d}y\,\delta(y)\,\mathrm{i}^n\mathrm{P}_n\left(-\mathrm{i}\alpha\frac{\partial}{\partial y}\right)\mathrm{j}_m\left(\frac{y}{\alpha}\right)=\frac{\delta_{mn}}{m+n+1}. \tag{9.7}$$

This relation can be used to make expansions into series of generalized functions $\mathrm{i}^n\mathrm{P}_n\left(\mathrm{i}\alpha\frac{\partial}{\partial x}\right)\delta(x)$ which are a modification of the expansion into moment series discussed in Section 2. The coefficients of these expansions are generalized moments of the expanded function $f(x)$.

10. Examples of Expansions into Spherical Bessel Functions

A simple method to find the expansion of the monomials in Spherical Bessel functions is again (see Section 6) to expand both sides of (8.2) into a Taylor series with respect to powers x^n. Inserting the explicit form of the Legendre polynomials (8.3) one finds in this way

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}xz}&=\sum_{n=0}^{\infty}\frac{(\mathrm{i}x)^n}{n!}z^n\\
&=\sum_{m=0}^{\infty}\mathrm{i}^m(2m+1)\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k\left(m-\frac{1}{2}-k\right)!}{k!(m-2k)!\left(-\frac{1}{2}\right)!}(2\alpha x)^{m-2k}\,\mathrm{j}_m\left(\frac{z}{\alpha}\right)\\
&=\sum_{n=0}^{\infty}\frac{(\mathrm{i}2\alpha x)^n}{n!}\sum_{k=0}^{\infty}\frac{\left(n-\frac{1}{2}+k\right)!}{\left(-\frac{1}{2}\right)!k!}(2n+1+4k)\,\mathrm{j}_{n+2k}\left(\frac{z}{\alpha}\right).
\end{aligned} \tag{10.1}$$

If we collect now on both sides the sum terms proportional to x^n we find the following expansion of z^n in Spherical Bessel functions

$$z^n = 2^n \sum_{k=0}^{\infty} \frac{\left(n - \frac{1}{2} + k\right)!}{\left(-\frac{1}{2}\right)! k!} (2n + 1 + 4k) \, \mathrm{j}_{n+2k}(z), \quad (n = 0,1,2,\cdots). \tag{10.2}$$

In particular, from this formula follows for $n = 0$

$$z^0 = \sum_{k=0}^{\infty} \frac{\left(k - \frac{1}{2}\right)!}{k! \left(-\frac{1}{2}\right)!} (1 + 4k) \, \mathrm{j}_{2k}(z) = 1, \tag{10.3}$$

independently of the chosen value z on the middle side.

If we insert the expression for $\mathrm{P}_n(x)$ in (8.7) then by reordering of the sum terms and comparison with (3.24) follows a relation of the Bessel functions expressed by the Spherical Bessel functions and similar its inversion

$$\mathrm{J}_n(z) = \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k \left(k - \frac{1}{2}\right)! \left(n + k - \frac{1}{2}\right)!}{k!(n+k)!} (2n + 4k + 1) \, \mathrm{j}_{n+2k}(z),$$

$$\mathrm{j}_n(z) = \begin{cases} \mathrm{J}_0(z) + 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)(2k+1)} \mathrm{J}_{2k}(z), & n = 0, \\ -\frac{1}{4} \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k \left(k - \frac{3}{2}\right)! (n+k-1)!}{k! \left(n + k + \frac{1}{2}\right)!} (n + 2k) \, \mathrm{J}_{n+2k}(z), & n \neq 1. \end{cases} \tag{10.4}$$

The expansion for $\mathrm{j}_0(z) = \frac{\sin(z)}{z}$ in Bessel functions in (10.4) was necessary to calculate separately and we used the Taylor series of $\frac{\sin(z)}{z}$ inserting there the relation for z^n in (6.2). In similar way from the Taylor series of $\cos(z)$ and $\sin(z)$ one obtains

$$\cos(z) = \mathrm{J}_0(z) + 2 \sum_{n=1}^{\infty} (-1)^n \mathrm{J}_{2n}(z),$$

$$\sin(z) = 2 \sum_{n=0}^{\infty} (-1)^n \mathrm{J}_{2n+1}(z). \tag{10.5}$$

These relations are the special case $x = 1$ in the formulae (3.20).

11. Expansions into Hermite Polynomials and Hermite Functions

For comparison with the preceding expansions we give in this Section widely without proofs a well-known example for the expansion of functions into a series of Hermite functions.

The Hermite polynomials $\mathrm{H}_n(x), (n = 0,1,2,\cdots)$ can be introduced by the Rodrigues or by an alternative definition as follows

$$\begin{aligned} \mathrm{H}_n(x) &\equiv (-1)^n \exp(x^2) \frac{\partial^n}{\partial x^n} \exp(-x^2) \\ &= \exp\left(-\frac{1}{4}\frac{\partial^2}{\partial x^2}\right)(2x)^n \underbrace{\exp\left(\frac{1}{4}\frac{\partial^2}{\partial x^2}\right)1}_{=1} = \left(2x - \frac{\partial}{\partial x}\right)^n 1, \end{aligned} \tag{11.1}$$

with the explicit representation

$$\mathrm{H}_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k n!}{k!(n-2k)!}(2x)^{n-2k}. \tag{11.2}$$

Besides the Hermite polynomials $\mathrm{H}_n(x)$ we define the Hermite functions $\mathrm{h}_n(x)$ by

$$\mathrm{h}_n(x) \equiv \frac{1}{\pi^{\frac{1}{4}}\sqrt{2^n n!}} \exp\left(-\frac{x^2}{2}\right) \mathrm{H}_n(x). \tag{11.3}$$

They satisfy the orthonormality relation

$$\int_{-\infty}^{+\infty} \mathrm{d}x \mathrm{h}_m(x) \mathrm{h}_n(x) = \delta_{mn}, \tag{11.4}$$

and the completeness relation

$$\sum_{n=0}^{\infty} \mathrm{h}_n(x) \mathrm{h}_n(y) = \delta(x-y). \tag{11.5}$$

In the cases up to now discussed, we have had two different sets of functions which are mutually bi-orthogonal. Here both sets of involved functions are the same.

By introduction of a free real parameter α by the substitution $x \to \frac{x}{\alpha}$ the orthonormality relation (11.4) can be generalized to

$$\frac{1}{|\alpha|} \int_{-\infty}^{+\infty} \mathrm{d}x \mathrm{h}_m\left(\frac{x}{\alpha}\right) \mathrm{h}_n\left(\frac{x}{\alpha}\right) = \delta_{mn}, \tag{11.6}$$

and the completeness relation (11.5) to

$$\frac{1}{|\alpha|} \sum_{n=0}^{\infty} \mathrm{h}_n\left(\frac{x}{\alpha}\right) \mathrm{h}_n\left(\frac{y}{\alpha}\right) = \frac{1}{|\alpha|} \delta\left(\frac{x-y}{\alpha}\right) = \delta(x-y). \tag{11.7}$$

For the expansion of a continuously differentiable function $f(x)$ into Hermite functions $\mathrm{h}\left(\frac{x}{\alpha}\right)$ one obtains then

$$f(x) = \sum_{n=0}^{\infty} c_n(\alpha) \mathrm{h}_n\left(\frac{x}{\alpha}\right), \quad c_n(\alpha) = \frac{1}{|\alpha|} \int_{-\infty}^{+\infty} \mathrm{d}x \mathrm{h}_n\left(\frac{x}{\alpha}\right) f(x). \tag{11.8}$$

This can be obtained in well-known way from the orthonormality relation (11.6). However, one may it also derive from the completeness relation (11.7) according to

$$\begin{aligned} f(x) &= \int_{-\infty}^{+\infty} \mathrm{d}y \delta(x-y) f(y) = \int_{-\infty}^{+\infty} \mathrm{d}y \frac{1}{|\alpha|} \sum_{n=0}^{\infty} \mathrm{h}_n\left(\frac{x}{\alpha}\right) \mathrm{h}_n\left(\frac{y}{\alpha}\right) f(y) \\ &= \sum_{n=0}^{\infty} \underbrace{\frac{1}{|\alpha|} \int_{-\infty}^{+\infty} \mathrm{d}y \mathrm{h}_n\left(\frac{y}{\alpha}\right) f(y)}_{=c_n(\alpha)} \mathrm{h}_n\left(\frac{x}{\alpha}\right). \end{aligned} \tag{11.9}$$

where we have changed the ordering of integration and summation.

As example we consider an exponential function $\mathrm{e}^{\mathrm{i}xz}$ and make an expansion in Hermite functions $h_n\left(\frac{z}{\alpha}\right)$. We do not calculate it here in detail and give only the result for this expansion which is

$$\mathrm{e}^{\mathrm{i}xz} = \exp\left(\mathrm{i}\alpha x \frac{z}{\alpha}\right) = \sqrt{2\pi}\sum_{n=0}^{\infty} \mathrm{i}^n \mathrm{h}_n(\alpha x)\mathrm{h}_n\left(\frac{z}{\alpha}\right), \tag{11.10}$$

where α is a free parameter. It possesses the same principal structure as the expansions (3.24) and (8.2) of the same exponential function. However, the right-hand side is divergent in usual sense but from this does not categorically follow that it is not convergent in a generalized sense. It should be convergent in the sense of weak convergence of generalized functions or linear functionals and then could be used, for example, in Fourier integrals over sufficiently well-behaved functions that we do not discuss here in detail.

There is still another possibility to modify the expansion into Hermite functions $\mathrm{h}_n(x)$ by a free real parameter β by writing the orthonormality (11.4) using (11.3)

$$\frac{1}{\sqrt{\pi 2^{m+n} m!n!}}\int_{-\infty}^{+\infty} \mathrm{d}x \exp\left(-\left(1-\beta^2\right)x^2\right)\mathrm{H}_m(x)\exp\left(-\beta^2 x^2\right)\mathrm{H}_n(x) = \delta_{mn}. \tag{11.11}$$

Then one finds for expansions of the form

$$f(x) = \sum_{n=0}^{\infty} c_n(\beta)\exp\left(-\beta^2 x^2\right)\mathrm{H}_n(x), \tag{11.12}$$

with the coefficients $c_n(\beta)$ determined by

$$c_n(\beta) = \frac{1}{\sqrt{\pi}\,2^n n!}\int_{-\infty}^{+\infty} \mathrm{d}x \exp\left(-\left(1-\beta^2\right)x^2\right)\mathrm{H}_m(x) f(x). \tag{11.13}$$

For $\beta = 0$ this is the expansion of a function directly into a series of Hermite polynomials.

12. Conclusions

We have discussed the expansion of functions in series of Bessel functions and of Spherical Bessel functions of kind which is usually called Neumann series and tried to illuminate the analogies to other known expansions, in particular to Taylor series, momentum series and series in Hermite functions. An important role played what we call a dual partner of the considered sequence of functions which in case of Hermite functions are again the same functions (self-duality). The most of our expansions possess a free eligible parameter α. Orthogonality and completeness relations could be formulated from the dual partners of functions. We tried to illustrate this by the expansion of special sets of functions, in particular, of the monomials $z^n, (n = 0,1,2,\cdots)$ in series of Bessel and Spherical Bessel functions. The kind of convergence of the expansions is usually not discussed but it seems that in most cases there is a sufficient supply of appropriate functions for which the series are convergent in different sense (at least, in the sense of weak convergence of Generalized functions). We think that there exist also other sets of dual partners for expansions of functions and conjecture that the modified Bessel function to the four-dimensional case $N = 4$ of the Helmholtz equation (see **Appendix B**) could be related to the Chebyshev polynomials of second kind $\mathrm{U}_n(x)$ but did not investigate this.

In the four **Appendices** we make some useful collection of formulae (modulated Trigonometric and Hyperbolic functions, *N*-dimensional Radial part of Helmholtz equation) and of the normally-ordered form of the operators which generate the Spherical Bessel functions from an initial member of their sequences. In last **Appendix** we illustrate the differences between Chebyshev polynomials of first and second kind and of Legendre polynomials with their connection to Gegenbauer polynomials and with respect to their over-completeness if we take into account their non-negative and their negative indices.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendices

Appendix A. Expansions of Modulated Trigonometric and Hyperbolic Functions in Bessel Functions

In this **Appendix** we collect for lexicographic purposes formulae which can be derived from the basic definition (3.1) by transformations and substitutions for modulated Trigonometric and Hyperbolic functions.

The following first group are exponential functions with Trigonometric functions in the argument

$$\begin{aligned}
\exp\left(\mathrm{i}z\sin(\theta)\right) &= \sum_{n=-\infty}^{+\infty} \mathrm{J}_n(z)\mathrm{e}^{\mathrm{i}n\theta} = \sum_{n=-\infty}^{+\infty} \mathrm{i}^n\,\mathrm{I}_n(-\mathrm{i}z)\mathrm{e}^{\mathrm{i}n\theta},\\
\exp\left(\mathrm{i}z\cos(\theta)\right) &= \sum_{n=-\infty}^{+\infty} \mathrm{i}^n\,\mathrm{J}_n(z)\mathrm{e}^{\mathrm{i}n\theta} = \sum_{n=-\infty}^{+\infty} \mathrm{I}_n(\mathrm{i}z)\mathrm{e}^{\mathrm{i}n\theta},\\
\exp\left(z\sin(\theta)\right) &= \sum_{n=-\infty}^{+\infty} \mathrm{J}_n(-\mathrm{i}z)\mathrm{e}^{\mathrm{i}n\theta} = \sum_{n=-\infty}^{+\infty} \mathrm{i}^n\,\mathrm{I}_n(-z)\mathrm{e}^{\mathrm{i}n\theta},\\
\exp\left(z\cos(\theta)\right) &= \sum_{n=-\infty}^{+\infty} \mathrm{i}^n\mathrm{J}_n(-\mathrm{i}z)\mathrm{e}^{\mathrm{i}n\theta} = \sum_{n=-\infty}^{+\infty} \mathrm{I}_n(z)\mathrm{e}^{\mathrm{i}n\theta}.
\end{aligned} \tag{A.1}$$

The following second group are exponential functions with Hyperbolic functions in the argument

$$\begin{aligned}
\exp\left(z\,\mathrm{sh}(t)\right) &= \sum_{n=-\infty}^{+\infty} \mathrm{J}_n(z)\mathrm{e}^{nt} = \sum_{n=-\infty}^{+\infty} \mathrm{i}^n\,\mathrm{I}_n(-\mathrm{i}z)\mathrm{e}^{nt},\\
\exp\left(\mathrm{i}z\,\mathrm{ch}(t)\right) &= \sum_{n=-\infty}^{+\infty} \mathrm{i}^n\,\mathrm{J}_n(z)\mathrm{e}^{nt} = \sum_{n=-\infty}^{+\infty} \mathrm{I}_n(\mathrm{i}z)\mathrm{e}^{nt},\\
\exp\left(\mathrm{i}z\,\mathrm{sh}(t)\right) &= \sum_{n=-\infty}^{+\infty} \mathrm{J}_n(\mathrm{i}z)\mathrm{e}^{nt} = \sum_{n=-\infty}^{+\infty} \mathrm{i}^n\,\mathrm{I}_n(z)\mathrm{e}^{nt},\\
\exp\left(z\,\mathrm{ch}(u)\right) &= \sum_{n=-\infty}^{+\infty} \mathrm{i}^n\,\mathrm{J}_n(-\mathrm{i}z)\mathrm{e}^{nt} = \sum_{n=-\infty}^{+\infty} \mathrm{I}_n(z)\mathrm{e}^{nt}.
\end{aligned} \tag{A.2}$$

We now consider Trigonometric and Hyperbolic functions modulated by trigonometric or Hyperbolic functions.

First group: Trigonometric functions modulated by Trigonometric functions[5]

$$\begin{aligned}
\cos\left(z\cos(\theta)\right) &= \sum_{m=-\infty}^{+\infty} (-1)^m\,\mathrm{J}_{2m}(z)\cos(2m\theta),\\
\sin\left(z\cos(\theta)\right) &= \sum_{m=-\infty}^{+\infty} (-1)^m\,\mathrm{J}_{2m+1}(z)\cos\left((2m+1)\theta\right),\\
\cos\left(z\sin(\theta)\right) &= \sum_{m=-\infty}^{+\infty} \mathrm{J}_{2m}(z)\cos(2m\theta),\\
\sin\left(z\sin(\theta)\right) &= \sum_{m=-\infty}^{+\infty} \mathrm{J}_{2m+1}(z)\sin\left((2m+1)\theta\right).
\end{aligned} \tag{A.3}$$

Second group: Trigonometric functions modulated by Hyperbolic functions

[5]This group is mostly given in reference books, e.g., [6], (9.1.42-9.1.45). The next groups can be obtained by simple transformations but for convenience we give them too.

$$\cos\left(z\operatorname{ch}(t)\right)=\sum_{m=-\infty}^{+\infty}(-1)^m \mathrm{J}_{2m}(z)\operatorname{ch}(2mt),$$

$$\sin\left(z\operatorname{ch}(t)\right)=\sum_{m=-\infty}^{+\infty}(-1)^m \mathrm{J}_{2m+1}(z)\operatorname{ch}\left((2m+1)t\right)$$

$$\cos\left(z\operatorname{sh}(t)\right)=\sum_{m=-\infty}^{+\infty}(-1)^m \mathrm{I}_{2m}(z)\operatorname{ch}(2mt),$$

$$\sin\left(z\operatorname{sh}(t)\right)=\sum_{m=-\infty}^{+\infty}(-1)^m \mathrm{I}_{2m+1}(z)\operatorname{sh}\left((2m+1)t\right). \tag{A.4}$$

Third group: Hyperbolic functions modulated by Trigonometric functions

$$\operatorname{ch}\left(z\cos(\theta)\right)=\sum_{m=-\infty}^{+\infty}\mathrm{I}_{2m}(z)\cos(2m\theta),$$

$$\operatorname{sh}\left(z\cos(\theta)\right)=\sum_{m=-\infty}^{+\infty}\mathrm{I}_{2m+1}(z)\cos\left((2m+1)\theta\right),$$

$$\operatorname{ch}\left(z\sin(\theta)\right)=\sum_{m=-\infty}^{+\infty}(-1)^m \mathrm{I}_{2m}(z)\cos(2m\theta),$$

$$\operatorname{sh}\left(z\sin(\theta)\right)=\sum_{m=-\infty}^{+\infty}(-1)^m \mathrm{I}_{2m+1}(z)\sin\left((2m+1)\theta\right), \tag{A.5}$$

Fourth group: Hyperbolic functions modulated by Hyperbolic functions

$$\operatorname{ch}\left(z\operatorname{ch}(t)\right)=\sum_{m=-\infty}^{+\infty}\mathrm{I}_{2m}(z)\operatorname{ch}(2mt),$$

$$\operatorname{sh}\left(z\operatorname{ch}(t)\right)=\sum_{m=-\infty}^{+\infty}\mathrm{I}_{2m+1}(z)\operatorname{ch}\left((2m+1)t\right),$$

$$\operatorname{ch}\left(z\operatorname{sh}(t)\right)=\sum_{m=-\infty}^{+\infty}\mathrm{J}_{2m}(z)\operatorname{ch}(2mt),$$

$$\operatorname{sh}\left(z\operatorname{sh}(t)\right)=\sum_{m=-\infty}^{+\infty}\mathrm{J}_{2m+1}(z)\operatorname{sh}\left((2m+1)t\right). \tag{A.6}$$

The formulae (A.3)-(A.4) can be transformed in various way. One can make substitutions according to (3.3) and can apply the identities (3.17). Furthermore, the Bessel functions with negative indices can be substituted according to (3.2) and (3.4) by Bessel functions with positive indices.

Appendix B. The *N*-Dimensional Laplace and Corresponding Wave-Equation Operator in Spherical Coordinates in Coordinate-Invariant Form

In an N-dimensional Euclidean space we split the N-dimensional position vector $\boldsymbol{r}$ into the product of radius coordinate r with a unit vector $\boldsymbol{n}$ in direction of the vector $\boldsymbol{r}$ and make this for clarity in the first equations in index-less and at once in index representation

$$\boldsymbol{r}=r\boldsymbol{n},\quad r_i=rn_i,\quad r=|\boldsymbol{r}|\equiv\sqrt{\boldsymbol{r}^2}=\sqrt{r_ir_i},$$

$$\boldsymbol{n}\equiv\frac{\boldsymbol{r}}{|\boldsymbol{r}|},\quad n_i=\frac{r_i}{r},\quad \boldsymbol{n}^2=n_in_i=1,\quad \boldsymbol{rn}=r_in_i\equiv r. \tag{B.1}$$

The unit vector $\boldsymbol{n}$ possesses $N-1$ independent coordinates equivalently to the unit sphere in N-dimensional Euclidean space. Then the operator of vectorial differentiation (Nabla) $\nabla \equiv \frac{\partial}{\partial \boldsymbol{r}}$ is

$$\nabla = \frac{\partial |\boldsymbol{r}|}{\partial \boldsymbol{r}} \frac{\partial}{\partial |\boldsymbol{r}|} + \frac{\partial \boldsymbol{n}}{\partial \boldsymbol{r}} \frac{\partial}{\partial \boldsymbol{n}}, \quad \nabla_i = \frac{\partial r}{\partial r_i} \frac{\partial}{\partial r} + \frac{\partial n_j}{\partial r_i} \frac{\partial}{\partial n_j}, \tag{B.2}$$

with the result

$$\nabla = \boldsymbol{n} \frac{\partial}{\partial |\boldsymbol{r}|} + \frac{1}{|\boldsymbol{r}|} (\mathrm{I} - \boldsymbol{n} \cdot \boldsymbol{n}) \frac{\partial}{\partial \boldsymbol{n}}, \quad \nabla_i = n_i \frac{\partial}{\partial r} + \frac{1}{r} (\delta_{ij} - n_i n_j) \frac{\partial}{\partial n_j}. \tag{B.3}$$

For the Laplace operator ∇^2 we find separated in Radius plus Spherical coordinates

$$\begin{aligned} \nabla^2 &= \nabla_i \nabla_i \\ &= \left(n_i \frac{\partial}{\partial r} + \frac{1}{r} (\delta_{ij} - n_i n_j) \frac{\partial}{\partial n_j} \right) \left(n_i \frac{\partial}{\partial r} + \frac{1}{r} (\delta_{ik} - n_i n_k) \frac{\partial}{\partial n_k} \right) \\ &= \left(\frac{\partial^2}{\partial r^2} + \frac{N-1}{r} \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \left((\delta_{jk} - n_j n_k) \frac{\partial^2}{\partial n_j \partial n_k} - (N-1) n_k \frac{\partial}{\partial n_k} \right). \end{aligned} \tag{B.4}$$

The spherical part is here derived and represented in coordinate-invariant form (see, e.g., Madelung [20], p. 243, in special spherical coordinates). The wave-equation operator is $\nabla^2 + \kappa^2$ where for the temporal part a Fourier transformation is made which adds a term $\kappa^2 \equiv \frac{\omega^2}{c^2}$ to the Laplace operator which becomes then a Helmholtz operator. In the form (B.4) of the Laplace operator a separation of the variables is not possible. One has first to multiply it with r^2. Then one obtains for a solution $F(\boldsymbol{r}) = R(r)\Phi(\boldsymbol{n})$ of the wave equation

$$\begin{aligned} 0 &= r^2 \left\{ \nabla^2 + \kappa^2 \right\} F(\boldsymbol{r}) \\ &= \left\{ r^2 \left(\frac{\partial^2}{\partial r^2} + \frac{N-1}{r} \frac{\partial}{\partial r} + \kappa^2 \right) + \left((\delta_{jk} - n_j n_k) \frac{\partial^2}{\partial n_j \partial n_k} - (N-1) n_k \frac{\partial}{\partial n_k} \right) \right\} R(r) \Phi(\boldsymbol{n}) \\ &= \Phi(\boldsymbol{n}) \left\{ r^2 \left(\frac{\partial^2}{\partial r^2} + \frac{N-1}{r} \frac{\partial}{\partial r} + \kappa^2 \right) \right\} R(r) \\ &\quad + R(r) \left\{ (\delta_{jk} - n_j n_k) \frac{\partial^2}{\partial n_j \partial n_k} - (N-1) n_k \frac{\partial}{\partial n_k} \right\} \Phi(\boldsymbol{n}). \end{aligned} \tag{B.5}$$

After division of this equation by $R(r)\Phi(\boldsymbol{n})$ it leads to two separate equations

$$\begin{aligned} 0 &= \left\{ r^2 \left(\frac{\partial^2}{\partial r^2} + \frac{N-1}{r} \frac{\partial}{\partial r} + \kappa^2 \right) - c \right\} R(r), \\ 0 &= \left\{ (\delta_{jk} - n_j n_k) \frac{\partial^2}{\partial n_j \partial n_k} - (N-1) n_k \frac{\partial}{\partial n_k} + c \right\} \Phi(\boldsymbol{n}), \end{aligned} \tag{B.6}$$

where c is a constant. It is favorable to represent this constant in the form $c = \nu(\nu + N - 2)$. The equation for the radial part is then

$$0 = r^2 \left\{ \frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r} + \kappa^2 - \frac{\nu(\nu+N-2)}{r^2} \right\} R(r), \tag{B.7}$$

with one of two possible linearly independent solutions

$$R(r) = \frac{\mathrm{J}_{\nu+\frac{N}{2}-1}(\kappa r)}{r^{\frac{N}{2}-1}}, \quad (r \geq 0). \tag{B.8}$$

For some first low cases of the dimension N these are the equations

$$\begin{aligned}
N=1: \quad & 0 = \left\{ \frac{\partial^2}{\partial r^2} + \kappa^2 - \frac{\nu(\nu-1)}{r^2} \right\} r^{\frac{1}{2}} \mathrm{J}_{\nu-\frac{1}{2}}(\kappa r), \\
N=2: \quad & 0 = \left\{ \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \kappa^2 - \frac{\nu^2}{r^2} \right\} \mathrm{J}_{\nu}(\kappa r), \\
N=3: \quad & 0 = \left\{ \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \kappa^2 - \frac{\nu(\nu+1)}{r^2} \right\} \frac{\mathrm{J}_{\nu+\frac{1}{2}}(\kappa r)}{r^{\frac{1}{2}}}, \\
N=4: \quad & 0 = \left\{ \frac{\partial^2}{\partial r^2} + \frac{3}{r}\frac{\partial}{\partial r} + \kappa^2 - \frac{\nu(\nu+2)}{r^2} \right\} \frac{\mathrm{J}_{\nu+1}(\kappa r)}{r}.
\end{aligned} \tag{B.9}$$

We do not discuss a second linearly independent solution of these equations. We also cannot discuss the equations for the spherical parts of the Helmholtz equations.

Appendix C. Explicit Representations of Spherical Bessel Functions $\mathrm{j}_n(z)$ for Low Numbers n

According to (7.8) the Spherical Bessel functions can be successively obtained from the relations

$$\begin{aligned}
\mathrm{j}_n(z) &= (-1)^n z^n \left(\frac{1}{z}\frac{\partial}{\partial z} \right)^n \frac{\sin(z)}{z} = (-1)^n z^{n-1} \left(\frac{\partial}{\partial z}\frac{1}{z} \right)^n \sin(z), \\
\mathrm{j}_{-n-1}(z) &= z^n \left(\frac{1}{z}\frac{\partial}{\partial z} \right)^n \frac{\cos(z)}{z} = z^{n-1} \left(\frac{\partial}{\partial z}\frac{1}{z} \right)^n \cos(z),
\end{aligned} \tag{C.1}$$

Using these raising and lowering relations we find explicitly for positive indices n

$$\begin{aligned}
\mathrm{j}_0(z) &= \frac{\sin(z)}{z}, \\
\mathrm{j}_1(z) &= -\frac{z\cos(z) - \sin(z)}{z^2}, \\
\mathrm{j}_2(z) &= -\frac{(z^2-3)\sin(z) + 3z\cos(z)}{z^3}, \\
\mathrm{j}_3(z) &= \frac{z(z^2-15)\cos(z) - 3(2z^2-5)\sin(z)}{z^4}, \\
\mathrm{j}_4(z) &= \frac{(z^4-45z^2+105)\sin(z) + 5z(2z^2-21)\cos(z)}{z^5},
\end{aligned}$$

$$\mathrm{j}_5(z) = -\frac{z\left(z^4 - 105z^2 + 945\right)\cos(z) - 15\left(z^4 - 28z^2 + 63\right)\sin(z)}{z^6}, \qquad \text{(C.2)}$$

and for negative indices n|[6]

$$\mathrm{j}_{-1}(z) = \frac{\cos(z)}{z},$$
$$\mathrm{j}_{-2}(z) = -\frac{z\sin(z) + \cos(z)}{z^2},$$
$$\mathrm{j}_{-3}(z) = -\frac{\left(z^2 - 3\right)\cos(z) - 3z\sin(z)}{z^3},$$
$$\mathrm{j}_{-4}(z) = \frac{z\left(z^2 - 15\right)\sin(z) + 3\left(2z^2 - 5\right)\cos(z)}{z^4},$$
$$\mathrm{j}_{-5}(z) = \frac{\left(z^4 - 45z^2 + 105\right)\cos(z) - 5z\left(2z^2 - 21\right)\sin(z)}{z^5},$$
$$\mathrm{j}_{-6}(z) = -\frac{z\left(z^4 - 105z^2 + 945\right)\sin(z) + 15\left(z^4 - 28z^2 + 63\right)\cos(z)}{z^6}. \qquad \text{(C.3)}$$

The Spherical Bessel functions $\mathrm{j}_n(z)$ with non-negative indices n are regular at $z = 0$ and that with negative indices n are singular at $z = 0$.

Operators of the considered form composed from multiplication and differentiation operators are called normally ordered (or disentangled) if all multiplication operators stand in front of the differential operators and in opposite case anti-normally ordered. The normally ordered form of the operators $\left(\frac{1}{z}\frac{\partial}{\partial z}\right)^n$ and $\left(\frac{\partial}{\partial z}\frac{1}{z}\right)^n$ is

$$\left(\frac{1}{z}\frac{\partial}{\partial z}\right)^n = \sum_{k=0}^{n} \frac{(-1)^k (n-1+k)!}{2^k k!(n-1-k)!} \frac{1}{z^{n+k}} \frac{\partial^{n-k}}{\partial z^{n-k}} = \frac{1}{z}\left(\frac{\partial}{\partial z}\frac{1}{z}\right)^{n-1} \frac{\partial}{\partial z},$$
$$\left(\frac{\partial}{\partial z}\frac{1}{z}\right)^n = \sum_{k=0}^{n} \frac{(-1)^k (n+k)!}{2^k k!(n-k)!} \frac{1}{z^{n+k}} \frac{\partial^{n-k}}{\partial z^{n-k}} = z\left(\frac{1}{z}\frac{\partial}{\partial z}\right)^n \frac{1}{z}. \qquad \text{(C.4)}$$

This can be proved by complete induction but one has only to prove one of the two disentanglement relations (C.4), for example, the second because then the first follows automatically from the right-hand relation (C.4). For some completeness let us give also the anti-normally ordered form of these operators

$$\left(\frac{1}{z}\frac{\partial}{\partial z}\right)^n = \sum_{k=0}^{n} \frac{(n+k)!}{2^k k!(n-k)!} \frac{\partial^{n-k}}{\partial z^{n-k}} \frac{1}{z^{n+k}},$$
$$\left(\frac{\partial}{\partial z}\frac{1}{z}\right)^n = \sum_{k=0}^{n} \frac{(n-1+k)!}{2^k k!(n-1-k)!} \frac{\partial^{n-k}}{\partial z^{n-k}} \frac{1}{z^{n+k}}. \qquad \text{(C.5)}$$

[6]Instead of $\mathrm{j}_{-n}(z)$ as a second series of linearly independent solutions of the differential equation for Spherical Bessel functions (7.6) (see (B.9), case $N = 3$) and in analogy to a second linearly independent series of Bessel functions $\mathrm{Y}_n(z)$ to $\mathrm{J}_n(z)$ are often introduced the functions $y_n(z) \equiv (-1)^{n+1}\mathrm{j}_{-n-1}(z)$ in their standard notation.

The sum term to $k = n$ in the first formula of (C.4) and second formula of (C.5) vanishes for $n \neq 0$ and provides only for $n = 0$ a non-vanishing term.

According to (C.1) taking into account (C.4) the Spherical Bessel functions may be represented in terms of Sine and Cosine functions by the following formulae

$$\mathrm{j}_n(z) = (-1)^n z^{n-1}\left(\frac{\partial}{\partial z}\frac{1}{z}\right)^n \sin(z) = \sum_{k=0}^{n}\frac{(-1)^{n-k}(n+k)!}{2^k k!(n-k)!}\frac{1}{z^{k+1}}\frac{\partial^{n-k}}{\partial z^{n-k}}\sin(z),$$

$$\mathrm{j}_{-n-1}(z) = z^{n-1}\left(\frac{\partial}{\partial z}\frac{1}{z}\right)^n \cos(z) = \sum_{k=0}^{n}\frac{(-1)^{k}(n+k)!}{2^k k!(n-k)!}\frac{1}{z^{k+1}}\frac{\partial^{n-k}}{\partial z^{n-k}}\cos(z). \quad \text{(C.6)}$$

The coefficients in front of the Sine and Cosine functions in (C.2) and (C.3) can also be determined from the relations (C.4) but this is a little difficile because the signs of the derivatives of Sine and Cosine are alternating and interfere with the signs in (C.4). For the polynomial coefficients in the numerators of (C.2) on the left-hand side $p_n(z)$ and on the right-hand side $q_n(z)$ we get from above to below

$$p_n(z) = (-1)^{\frac{n(n+1)}{2}}\sum_{l=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^l(n+2l)!}{2^{2l}(2l)!(n-2l)!}z^{n-2l},$$

$$q_n(z) = (-1)^n\sum_{l=0}^{\left[\frac{n-1}{2}\right]}\frac{(-1)^l(n+2l+1)!}{2^{2l+1}(2l+1)!(n-2l-1)!}z^{n-2l-1}. \quad \text{(C.7)}$$

The operators $\frac{1}{z}\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial z}\frac{1}{z}$ do not commute. Their commutator is

$$\left[\frac{1}{z}\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\frac{1}{z}\right] \equiv \frac{1}{z}\frac{\partial^2}{\partial z^2}\frac{1}{z} - \frac{\partial}{\partial z}\frac{1}{z^2}\frac{\partial}{\partial z} = \frac{2}{z^4}, \quad \text{(C.8)}$$

and their algebra is not closed in the sense of a Lie algebra and needs infinitely many additional multiplication operators $\frac{1}{z^{2n}}, (n = 2,3,\cdots)$ for closing.

Appendix D. A Peculiarity of Chebyshev Polynomials of First Kind $\mathrm{T}_n(z)$ within Gegenbauer Polynomials $\mathrm{C}_n^{\nu}(z)$ and Their Special Cases of Legendre Polynomials $\mathrm{P}_n(z)$ and Chebyshev Polynomials of Second Kind $\mathrm{U}_n(z)$

In this **Appendix** we make a comparison of Chebyshev polynomials and Legendre polynomials within the Gegenbauer polynomials and underline the peculiar role of Chebyshev polynomials of first kind $\mathrm{T}_n(z)$ which illuminates why often in formulae with the last the case $n = 0$ has to be considered separately from the other cases $n \neq 0$.

The Gegenbauer polynomials $\mathrm{C}_n^{\nu}(z)$ and their special cases of Chebyshev polynomials of first and second kind $\mathrm{T}_n(z)$ and $\mathrm{U}_n(z)$ as well as Legendre polynomials $\mathrm{P}_n(z)$ can be continued in natural way from non-negative indices n to negative indices n using properties of the factorials. They become in this way overcomplete and the part with negative indices can be expressed by that with positive indices. For the Gegenbauer polynomials this symmetry is

$$\mathrm{C}_{-n}^{\frac{m}{2}}(z)=(-1)^{m-1}\mathrm{C}_{n-m}^{\frac{m}{2}}(z),\quad (m=0,1,2,\cdots),\quad \Rightarrow\quad \mathrm{C}_{-n}^{0}(z)=-\mathrm{C}_{n}^{0}(z)=0. \quad (\mathrm{D}.1)$$

Genuine polynomials from the Gegenbauer polynomials $\mathrm{C}_n^{\nu}(z)$ with upper index $\nu=0$ can be only defined by a limiting procedure. These are the Chebyshev polynomials of first kind $\mathrm{T}_n(z)$ which can be alternatively defined by the limiting procedure

$$\mathrm{T}_n(z)\equiv\begin{cases}1, & n=0,\\ \frac{n}{2}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathrm{C}_n^{\varepsilon}(z), & n\neq 0.\end{cases} \quad (\mathrm{D}.2)$$

The member $\mathrm{T}_0(z)$ cannot even be defined by this limiting procedure and has to be a fixed constant for the set $\mathrm{T}_n(z),(n=0,1,2,\cdots)$ to be a complete set within the continuous functions and this constant is chosen by convention as $c=1$, likely the best choice. The monomial $\mathrm{T}_0(z)$ is also not unique by the general formula (3.11). The set of polynomials $\mathrm{T}_{-n}(z),(n=0,1,2,\cdots)$ form also such a complete set of functions which, however, shares the element $\mathrm{T}_0(z)$ with that of non-negative indices. The other sets of polynomials related to the Gegenbauer polynomials $\mathrm{C}_n^{\nu}(z)$ which are the Legendre polynomials $\mathrm{P}_n(z)$ and the Chebyshev polynomials of second kind $\mathrm{U}_n(z)$ are defined in simple way by the Gegenbauer polynomials. They possess also a complete part of polynomials with negative indices but this part does not have an intersection with elements from the non-negative part. In case of the polynomials $\mathrm{U}_n(z)$ the part with negative indices is separated by a "zero-polynomial" $\mathrm{U}_{-1}(z)=0$ and in the Gegenbauer polynomials with higher upper semi-integer and integer indices the number of the vanishing polynomials grows in unit steps.

The above mentioned peculiarities of the Chebyshev polynomials of first kind $\mathrm{T}_n(z)$ are the reason that in many formulae related to the polynomials the case $n=0$ and the cases $n\neq 0$ have to be considered separately. We see this in the following **Table D1** & **Table D2**:

Table D1. Chebyshev and Legendre polynomials for positive and negative lower indices.

n	$\mathrm{T}_n(z)=\frac{n}{2}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathrm{C}_n^{\varepsilon}(z)$	$\mathrm{P}_n(z)=\mathrm{C}_n^{\frac{1}{2}}(z)$	$\mathrm{U}_n(z)=\mathrm{C}_n^{1}(z)$
-6	$32z^6-48z^4+18z^2-1,$	$\frac{1}{8}(63z^5-70z^3+15z),$	$-16z^4+12z^2-1$
-5	$16z^5-20z^3+5z,$	$\frac{1}{8}(35z^4-30z^2+3),$	$-4(2z^3-z)$
-4	$8z^4-8z^2+1,$	$\frac{1}{2}(5z^3-3z),$	$-4z^2+1$
-3	$4z^3-3z,$	$\frac{1}{2}(3z^2-1),$	$-2z$
-2	$2z^2-1,$	$z,$	-1

Continued

-1	$z,$	$1,$	0
0	$1,$	$1,$	1
1	$z,$	$z,$	$2z$
2	$2z^2-1,$	$\frac{1}{2}\left(3z^2-1\right),$	$4z^2-1$
3	$4z^3-3z,$	$\frac{1}{2}\left(5z^3-3z\right),$	$4\left(2z^3-z\right)$
4	$8z^4-8z^2+1,$	$\frac{1}{8}\left(35z^4-30z^2+3\right),$	$16z^4-12z^2+1$
5	$16z^5-20z^3+5z,$	$\frac{1}{8}\left(63z^5-70z^3+15z\right),$	$2\left(16z^5-16z^3+3z\right)$
6	$32z^6-48z^4+18z^2-1,$	$\frac{1}{16}\left(231z^6-315z^4+105z^2-5\right),$	$64z^6-80z^4+24z^2-1$

(D.3)

Table D2. Gegenbauer polynomials of higher upper integer and semi-integer indices.

n	$\mathrm{C}_n^{\frac{3}{2}}(z)$	$\mathrm{C}_n^{2}(z)$	$\mathrm{C}_n^{\frac{5}{2}}(z)$
-6	$\frac{5}{2}\left(7z^3-3z\right),$	$-2\left(6z^2-1\right),$	$5z$
-5	$\frac{3}{2}\left(5z^2-1\right),$	$-4z,$	1
-4	$3z,$	$-1,$	0
-3	$1,$	$0,$	0
-2	$0,$	$0,$	0
-1	$0,$	$0,$	0
0	$1,$	$1,$	1
1	$3z,$	$4z,$	$5z$
2	$\frac{3}{2}\left(5z^2-1\right),$	$2\left(6z^2-1\right),$	$\frac{5}{2}\left(7z^2-1\right)$
3	$\frac{5}{2}\left(7z^3-3z\right),$	$4\left(8z^3-3z\right),$	$\frac{35}{2}\left(3z^3-z\right)$
4	$\frac{15}{8}\left(21z^4-14z^2+1\right),$	$80z^4-48z^2+3,$	$\frac{35}{8}\left(33z^4-18z^2+1\right)$
5	$\frac{21}{8}\left(33z^5-30z^3+5z\right),$	$8\left(24z^5-20z^3+3z\right),$	$\frac{21}{8}\left(143z^5-110z^3+15z\right)$
6	$\frac{7}{16}\left(429z^6-495z^4+135z^2-5\right),$	$4\left(112z^6-120z^4+30z^2-1\right),$	$\frac{105}{16}\left(143z^6-143z^4+33z^2-1\right)$

(D.4)

For convenience, the relation of Ultraspherical polynomials $\mathrm{P}_n^{(\alpha,\alpha)}(z)$ as special case $\alpha=\beta$ of Jacobi polynomials $\mathrm{P}_n^{(\alpha,\beta)}(z)$ to Gegenbauer polynomials $\mathrm{C}_n^{\nu}(z)$ is

$$\mathrm{P}_n^{(\alpha,\alpha)}(z) = \frac{(n+\alpha)!(2\alpha)!}{(n+2\alpha)!\alpha!}\mathrm{C}_n^{\alpha+\frac{1}{2}}(z) = \frac{2^{2\alpha}(n+\alpha)!\left(\alpha-\frac{1}{2}\right)!}{(n+2\alpha)!\left(-\frac{1}{2}\right)!}\mathrm{C}_n^{\alpha+\frac{1}{2}}(z), \tag{D.5}$$

and its inversion

$$\mathrm{C}_n^{\nu}(z) = \frac{(n+2\nu-1)!\left(\nu-\frac{1}{2}\right)!}{\left(n+\nu-\frac{1}{2}\right)!(2\nu-1)!}\mathrm{P}_n^{\left(\nu-\frac{1}{2},\nu-\frac{1}{2}\right)}(z) = \frac{(n+2\nu-1)!\left(-\frac{1}{2}\right)!}{2^{2\nu-1}\left(n+\nu-\frac{1}{2}\right)!(\nu-1)!}\mathrm{P}_n^{\left(\nu-\frac{1}{2},\nu-\frac{1}{2}\right)}(z), \tag{D.6}$$

where the doubling formula of the Gamma function is used.