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A Class of Potentials for Hyperbolic Transcendental Entire Maps

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Abstract

We identify a class of transcendental entire maps of finite order, of disjoint-type, satisfying the rapid derivative growth condition. Within this class, we show that there exist hyperbolic transcendental entire maps that generate a large class of potentials which intersect the so-called *tame* potentials and form a distinct class of potentials. The methods and techniques derived from the thermodynamic formalism are applied to these potentials for transcendental entire maps acting on some subset of the Julia set which is conjugated to the shift map over a code space with a countable alphabet endowed with the euclidean induced metric on the complex plane.

Keywords

Thermodynamic Formalism, Entire Transcendental Functions, Tame Potentials

1. Introduction

The study of the thermodynamic formalism of transcendental entire maps has received considerable attention. Notably, the ergodic theory of the exponential family $E_\lambda(z) = \lambda \exp(z)$ has been thoroughly examined for a wide range of parameters, as referenced in [1]-[10] and references therein.

When dealing with transcendental entire functions, several challenges arise in the exploration of the thermodynamic formalism. For instance, the Julia set is never compact, a contrast to the situation with polynomials and rational maps. This discrepancy leads to convergence issues, and standard arguments such as the Schauder-Tychonoff Fixed Point Theorem cannot be applied.

Due to the wide-range nature of the transcendental maps, it becomes essential to narrow the scope and consider suitable sub-classes when exploring thermodynamic formalism. In this context, Mayer and Urbański made significant contributions to their paper [10] by providing a complete understanding of the thermodynamic formalism for a large class of hyperbolic meromorphic functions of finite order satisfying a rapid growth condition, associated with a class of tame potentials. Furthermore, recently the authors in [11] provided an overview of the thermodynamic formalism for transcendental meromorphic and entire functions and their geometric applications. In addition, in [12], they developed an optimal approach to thermodynamic formalism for a wide range of transcendental entire functions whose set of singularities is bounded. Another significant advancement in the field is the development of thermodynamic formalism for random transcendental dynamics. This approach was successfully detailed in [13].

In the present work, we highlight a class of transcendental entire maps, which includes the exponential family, as defined in Section 2. We show that within this class exist hyperbolic transcendental entire maps that generate a large class of potentials for which the thermodynamic formalism can be effectively applied. The key novelty of our study lies in identifying a class of potentials that deviates from those earlier studied in [10]. We find that the techniques and methods from their work can be adapted with minor modifications to our situation, taking advantage of the properties of the symbolic representation of these maps acting on invariant subsets of Julia sets. These code spaces maintain a natural topology that is inherited from the Euclidean topology.

The paper is organized as follows: In Section 2 we define a class of hyperbolic transcendental entire maps and a class of potentials to state Theorem 1. After gathering several dynamic properties in Theorem 2 and properties of potentials in Proposition 1, the proof of Corollary 1 holds.

2. Hyperbolic Transcendental Entire Maps, Potentials and Results

Given a transcendental entire function $f:\mathbb{C}\to\mathbb{C}$, the Fatou set $F(f)$ is the subset of $\mathbb{C}$ where the iterates f^n of f form a normal family, and its complement is namely called the Julia set, which is denoted by $J(f)$.

Denote by $\text{Sing}\left(f^{-1}\right)$ the set of finite singularities of the inverse function f^{-1}, which is the set of critical values (images of critical points) and asymptotic values of f together with their finite limit points. The *post-singular* set $PS(f)$ of f is defined as,

$$PS(f):=\bigcup_{n=0}^{\infty} f^n\left(\text{Sing}\left(f^{-1}\right)\right),$$

and $\rho_f:=\limsup_{z\to\infty}\dfrac{\log\log|f(z)|}{\log|z|}$ is namely called *the order of* f.

Let $\mathscr{T}$ denote the class of transcendental entire functions f satisfying the following properties

1) It is of *finite order*;

2) Satisfies the *rapid derivative growth condition*: There are $\alpha_2 > 0$ $\alpha_1 > \alpha_2$ and $\kappa > 0$ such that for every $z \in J(f) \setminus f^{-1}(\infty)$ we have

$$\left|f'(z)\right| \geq \kappa^{-1} |z|^{\alpha_1} \left|f(z)\right|^{\alpha_2};$$

3) It is of *disjoint type*, that is, the set $\operatorname{Sing}(f^{-1})$ is contained in a compact subset of the immediate basin $B = B(z_0)$ of an attracting fixed point $z_0 \in \mathbb{C}$. This is a strong form of hyperbolicity, which was explicitly studied in [14] for instance.

Note that each $f \in \mathscr{T}$ belongs to the *Eremenko-Lyubich* class

$$\mathcal{B} := \left\{ f : \mathbb{C} \to \mathbb{C} : \operatorname{Sing}(f^{-1}) \text{ is bounded} \right\}.$$

It was proved in [15] that for $f \in \mathcal{B}$ all the Fatou components of f are simply connected. Hence the immediate basin B is simply connected. Moreover each $f \in \mathscr{T}$ is hyperbolic in the sense that the closure of $\overline{PS(f)}$ is disjoint from the Julia set and $\overline{PS(f)}$ is compact. We have that f has no wandering and Baker domains, so B is the only Fatou component of f, see [15] [16] [17].

Examples in the class $\mathscr{T}$ include the family $\lambda \exp(z)$ for $\lambda \in (0, 1/\mathrm{e})$, the family of maps $\lambda \sin(z)$ for $\lambda \in (0,1)$, and $\lambda g(z)$, where $\lambda \in \mathbb{C} \setminus \{0\}$ and g is an arbitrary map of finite order such that $\operatorname{Sing}(g^{-1})$ is bounded and $|\lambda|$ is enough small, other examples are the expanding entire maps $\sum_{j=0}^{p+q} a_j \mathrm{e}^{(j-p)z}$, $p, q > 0$, $a_j \in \mathbb{C}$, studied early in [9].

2.1. Potentials

Fix $f \in \mathscr{T}$. Since the immediate attraction basin $B = B(z_0)$ of an attracting fixed point z_0 is simply connected, there exists a bounded simply connected domain $D \subset \mathbb{C}$, such that its closure $\bar{D} \subset B$ and boundary ∂D is an analytic Jordan curve. Moreover, $\operatorname{Sing}(f^{-1}) \subset D$ and $\overline{f(D)} \subset D$, for more details see ([8], Lemma 3.1). Following [8], the pre-images of $\mathbb{C} \setminus \bar{D}$ by f consists of countably many unbounded connected components called *tracts* of f. We denote the collection of all these tracts by $\mathscr{R}$.

Since the closure of each tract is simply connected, there exists an open simple arc $\alpha : (0, \infty) \to \mathbb{C} \setminus \bar{D}$, which is disjoint from the union of the closures of all tracts and such that $\alpha(t)$ tends to a point of ∂D as t tends to 0^+, and $\alpha(t)$ tends to ∞ as t tends to $+\infty$. We use this curve to define the fundamental domains on each tract as follows: since for every $T \in \mathscr{R}$ the map $f|_T$ is a cover of $\mathbb{C} \setminus \bar{D}$, we have $T \setminus f^{-1}(\alpha)$ is the union of infinitely many disjoint simply connected domains S such that the function

$$f|_S : S \to \mathbb{C} \setminus (\bar{D} \cup \alpha)$$

is bijective. Given $T \in \mathscr{R}$, we denote by $\mathcal{S}_T$ the collection of connected components of $T \setminus f^{-1}(\alpha)$. The elements of

$$\mathcal{S} := \bigcup_{T \in \mathscr{R}} \mathcal{S}_T \tag{1}$$

are called fundamental domains.

For each $S \in \mathcal{S}$, we have that the restriction $f|_S$ is univalent, so we denote its inverse branch by $g_S := \left(f|_S\right)^{-1} : \mathbb{C} \setminus \left(\bar{D} \cup \alpha\right) \to S$. For $n \geq 1$ and each $j \in \{0,1,\cdots,n\}$ denote by S_j an element of $\mathcal{S}$ and put $g_{S_0 S_1 \cdots S_n} = g_{S_0} \circ \cdots \circ g_{S_n}$. Then,

$$g_{S_0 \cdots S_n}\left(\mathbb{C} \setminus \left(\bar{D} \cup \alpha\right)\right) = \left\{z \in \mathbb{C} : f^j(z) \in S_j, \text{for every } j = 0,\cdots,n\right\}. \tag{2}$$

For each sequence $\underline{S} = \left(S_0 S_1 \cdots\right) \in \mathcal{S}^{\mathbb{N}}$, let $K_{\underline{S}} := \bigcap_{n=0}^{\infty} g_{S_0 S_1 \cdots S_n}\left(\mathbb{C} \setminus \left(\bar{D} \cup \alpha\right)\right)$. Then, the Julia set of f is given by the disjoint union of $K_{\underline{S}}$, that is

$$J(f) = \bigsqcup_{\underline{S} \in \mathcal{S}^{\mathbb{N}}} K_{\underline{S}}.$$

Since f has finite order and of disjoint-type, following [18], the Julia set $J(f)$ is a Cantor bouquet, that is a union of uncountably many pairwise disjoint curves tending to infinity (hair) and each curve is attached to the unique point accessible from the immediate basin *B*, called the endpoint of the hair. More precisely, either $K_{\underline{S}}$ is empty or there is a homeomorphism $h_{\underline{S}} : [0,+\infty) \to K_{\underline{S}}$ such that $\lim_{t \to +\infty} h_{\underline{S}}(t) = \infty$, and such that for every $t > 0$ we have $\lim_{n \to +\infty} f^n\left(h_{\underline{S}}(t)\right) = \infty$. In the latter case $z_{\underline{S}} := h_{\underline{S}}(0)$ is the only point of $K_{\underline{S}}$ accessible[1] from the immediate basin *B*. See also [7], which generalizes previous results for the exponential map having an attracting fixed point of [19].

Following as in [10], let us consider ρ_f be the order of f and $\alpha_1, \alpha_2 > 0$ be the corresponding constants of the rapid derivative growth condition of f. Fix $\tau \in (0, \alpha_2)$ and let $\gamma : \mathbb{C} \setminus \{0\} \to \mathbb{R} \cup \{\infty\}$ defined by $\gamma(z) = \frac{1}{|z|^{\tau}}$. Let θ be the Riemaniann metric on $\mathbb{C} \setminus \{0\}$ defined by

$$\mathrm{d}\theta(z) = \gamma(z)|\mathrm{d}z|,$$

and we derive f with respect to θ instead of the Euclidean metric. So, for each $z \in \mathbb{C} \setminus \{0\}$ we have

$$\left|f'(z)\right|_{\theta} = \left|f'(z)\right| \frac{\gamma \circ f(z)}{\gamma(z)} = \left|f'(z)\right| \frac{|z|^{\tau}}{\left|f(z)\right|^{\tau}}. \tag{3}$$

We consider $\mathscr{C}$, the set of functions ψ from $\bigcup_{S \in \mathcal{S}} S$ to $\mathbb{R}^+$ that are bounded from above and are constant on each element of $\mathcal{S}$. That is, we write

$$\mathscr{C} := \left\{\psi : \bigcup_{S \in \mathcal{S}} S \to \mathbb{R}^+ : \psi \text{ is bounded from above and constant over each } S \in \mathcal{S}\right\}. \tag{4}$$

We define the following class of potentials for f:

$$\mathscr{P}_f = \left\{\varphi_{\psi,t}(z) = \log\psi(z) - t\log\left|f'(z)\right|_{\theta}, \psi \in \mathscr{C}, t > \frac{\rho_f}{\alpha_1 + \tau}\right\}.$$

[1]If U is simply connected domain in the Riemann sphere $\bar{\mathbb{C}}$, we say that a point $z \in \partial U$ is *accessible* from U if there exists a curve $\upsilon : [0,\infty) \to U$ such that $\lim_{t \to \infty} \upsilon(t) = z$.

Observe that this class contains potentials $-t\log|f'|_\theta$, which from (3) are cohomologous to $-t\log|f'|$.

For each $f\in\mathscr{F}$ we denote by $\mathscr{T}_f$ the class of tame potentials defined in [10], that is

$$\mathscr{T}_f := \left\{\varphi = h - t\log|f'|_\theta ; h \text{ is bounded weakly Hölder function}, t > \frac{\rho'}{\alpha_1+\alpha_2}\right\}$$

Note that the class $\mathscr{F}$ does not include most of the functions considered in [10]. However, the class of potentials $\mathscr{P}_f$ determined from $f\in\mathscr{F}$ intersects the class of tame potentials, in which the difference is non-empty. So our first result is the following

Theorem 1. *There exists* $f\in\mathscr{F}$ *such that* $\mathscr{P}_f\cap\mathscr{T}_f\neq\varnothing$ *and* $\mathscr{P}_f\setminus\mathscr{T}_f\neq\varnothing$.

2.2. Symbolic Representation

Let $\Sigma_{\mathbb{Z}} := \left\{\underline{s} = (s_0 s_1\cdots) : s_j\in\mathbb{Z}, \text{for all } j\geq 0\right\}$ be the full shift space, and the shift metric is defined as follows, for some $\theta\in(0,1)$,

$$d(\underline{s},\underline{t}) = \theta^{\inf\{k:s_k\neq t_k\}\cup\{\infty\}}. \tag{5}$$

For every $n\geq 1$, we denote a finite word $s_0\cdots s_{n-1}$ in $\mathbb{Z}^n$ simply by s^*, so we follows the following notation for cylinders

$$\left[s^*\right] = \left\{\underline{w}\in\Sigma_{\mathbb{Z}} : w_i = s_i, 0\leq i\leq n-1\right\}$$

and for $s\in\mathbb{Z}$, we simply denote $[s] = \left\{\underline{w}\in\Sigma_{\mathbb{Z}} : w_0 = s\right\}$. Let $\sigma:\Sigma_{\mathbb{Z}}\to\Sigma_{\mathbb{Z}}$ be the left-sided shift map, given by $\sigma(s_0 s_1\cdots) = (s_1 s_2\cdots)$.

Observe that by definition the set $\mathcal{S}$ given in (1) is countably infinite, so we identify $\mathcal{S}$ with $\mathbb{Z}$. Put

$$X := \left\{\underline{S}\in\mathcal{S}^{\mathbb{N}} : K_{\underline{S}}\neq\varnothing\right\}\subseteq\Sigma_{\mathbb{Z}}. \tag{6}$$

Let

$$Z = \bigcup_{\underline{S}\in X} K_{\underline{S}}.$$

From (2), we have for each $\underline{S}\in\mathcal{S}^{\mathbb{N}}$, $f\left(K_{\underline{S}}\right) = K_{\sigma(\underline{S})}$, then the function f on the Julia set $J(f)$ is semi-conjugate to σ on *X*, however f on the set

$$\mathcal{EP} := \left\{z_{\underline{S}} = h_{\underline{S}}(0) : \underline{S}\in X\right\},$$

is conjugate to $\sigma|_X$. Hence the set *X* is completely σ-invariant.

The set $\mathcal{EP}$ defined above, is the set of *endpoints* of hairs $K_{\underline{S}}$ and it satisfies the following properties, it is the set of accessible points from the immediate attraction basin *B*. It is totally disconnected, however $\mathcal{EP}\cup\{\infty\}$ is connected, see [14]. Moreover, following [20], the Hausdorff dimension of this set is equal to two, generalizing previous results of Karpińska [3] for the exponential map $f_\lambda(z) = \lambda e^z$ with parameters $\lambda\in(0,1/e)$. This exponential map is probably the best-known example in the family $\mathscr{F}$, its Julia set is a Cantor bouquet and the set of endpoints is modeled by the symbolic space of all allowable sequences, see

[19] and [21].

In the following, we state some properties concerning the dynamics $\left(X,\sigma|_X\right)$, endowed with a metric inherited from the euclidean metric on $J(f)$. It does not necessarily generate the topology induced by the cylinder sets.

Let $H : X\times[0,+\infty)\to J(f)$, and $H|_{X\times\{0\}} : X\times\{0\}\to \mathcal{EP}$ defined by $H(\underline{s},0)=h_{\underline{s}}(0)$, we have that H induces a metric ρ on X,

$$\rho(\underline{s},\underline{w}) := \left|h_{\underline{s}}(0)-h_{\underline{w}}(0)\right|.$$

The shift map $\sigma : X\to X$ is continuous with respect to ρ.

Given $\underline{s}\in\Sigma_{\mathbb{Z}}$ and $w^*\in\mathbb{Z}^n$ let us write $w^*\underline{s}=\left(w_0\cdots w_{n-1}s_0s_1\cdots\right)$. For a set $A\subset\Sigma_{\mathbb{Z}}$ write $w^*A=\left\{w^*\underline{s}:\underline{s}\in A\right\}$. For $\underline{s}\in X$ and $\delta>0$ we define the following sets with respect to the metric ρ.

$$B(\underline{s},\delta) := \left\{\underline{w}\in X:\rho(\underline{s},\underline{w})<\delta\right\}=\left\{\underline{w}\in X:\left|h_{\underline{s}}(0)-h_{\underline{w}}(0)\right|<\delta\right\},$$

$$\bar{B}(\underline{0},\delta) := \left\{\underline{s}\in X:\rho(\underline{s},\underline{0})\le\delta\right\}.$$

$$\mathbb{B}_0(\underline{s},\delta) := \left\{\underline{w}\in X:\rho(\underline{s},\underline{w})<\delta\ \&\ b_0=a_0\right\}.$$

For every $n\ge1$ and $\underline{s}\in X$ define

$$\mathbb{B}_n(\underline{s},\delta) := \left\{\underline{w}\in X:\sigma^j(\underline{w})\in\mathbb{B}_0\left(\sigma^j(\underline{s}),\delta\right),\text{for all } j=0,1,\cdots,n\right\}.$$

The set X endowed with the metric ρ is non-compact, however, it can be approximated by an increasing sequence of compact and invariant subsets. Indeed, for all $N\ge1$, define

$$\Sigma_N := \left\{\underline{s}=(s_0s_1\cdots)\in X:\text{for } j\ge0, s_j\in\{-N,\cdots,N\}\right\},$$

so, the following holds

Theorem 2.

1) *For all* $N\ge1$, $\Sigma_N\subset X$, Σ_N *is compact with respect to* ρ *and invariant by* σ. *Moreover, for each compact subset* Λ *of X with respect to the metric* ρ, *so that* $\sigma(\Lambda)\subset\Lambda$, *we have, there exists* $N_0\ge1$, *such that* $\Lambda\subset\Sigma_{N_0}$.

2) *There exists* δ_0 *such that the following condition holds:*

There exist $C>0$ *and* $\lambda>1$ *such that for every* $n\in\mathbb{N}$ *and* $\underline{s},\underline{t}\in X$ *and* $u^*\in\mathbb{Z}^n$, *if* $\rho(\underline{s},\underline{t})<\delta_0$ *then we have*

$$\rho\left(u^*\underline{s},u^*\underline{t}\right)\le C\lambda^{-n}\rho(\underline{s},\underline{t}).$$

3) *For every* $R>0$ *there exists* $n\ge1$ *such that for every* $\underline{s}\in B(\underline{0},R)$, *we have* $\sigma^n\left(B(\underline{s},\delta_0)\right)\supset B(\underline{0},R)$. *Thus* (X,σ) *is topologically mixing*

4) *The set* $\bigcup_{N\ge1}\Sigma_N$ *is dense in X.*

We recall what a conformal measure means; consider a measurable endomorphism $T:Y\to Y$ on a measurable space $(Y,\mathscr{B})$ and a measurable non-negative function g on Y. A measure m on $(Y,\mathscr{B})$ is called g-conformal for T on g if for all measurable set A which $T(A)$ is measurable and $T|_A$ is invertible we have

$$m\left(T(A)\right)=\int_A g\mathrm{d}m. \tag{7}$$

Observe that (7) implies that $m \circ T$ is absolutely continuous with respect to m on the σ-algebra $\mathscr{B} \cap A$, for every set $A \in \mathscr{B}$ such that $T: A \to T(A)$ is a measurable isomorphism.

Corollary 1. *Let $f \in \mathscr{T}$. Then for every potential $\phi \in \mathscr{P}_f$ we have the following properties.*

1) *The topological pressure* $P(\phi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{z \in f^{-n}(w)} \exp\left(\sum_{j=0}^{n-1} \phi \circ f^j(z)\right)$.

exists and is independent of $w \in J(f)$.

2) *There exists a unique* $\mathrm{e}^{P(\phi)-\phi}$ *-conformal measure ν_ϕ of f.*

3) *There exists a unique probability Gibbs state μ_ϕ. That is, μ_ϕ is f-invariant and equivalent to ν_ϕ. Moreover, both measures are ergodic and supported on the radial Julia set $J_r(f)$, where*

$$J_r(f) = \left\{ z \in J(f) : \lim_{n\to\infty} f^n(z) < \infty \right\}.$$

4) *The density $\rho_\phi = \mathrm{d}\mu_\phi / \mathrm{d}\nu_\phi$ is a nowhere vanishing continuous and bounded function on the Julia set $J(f)$.*

3. Proof of Results

3.1. Proof of Theorem 1

Consider the exponential family $\left\{ f_\lambda(z) = \lambda \mathrm{e}^z, \lambda \in (0, 1/\mathrm{e}) \right\}$. Each f_λ belongs to $\mathscr{T}$, because it has order equal to 1, satisfies the rapid derivative growth condition with $\alpha_1 = 0$ and $\alpha_2 = 1$, and since 0 is the only singular value of f_λ. Thus this map is hyperbolic. Moreover the potentials $-t \log|z| = -t \log|f_\lambda'(z)| + \log \gamma_1 - \log \gamma_1 \circ f_\lambda$, where $\gamma_1 = |z|^{-t}$, are tame potentials and also belong to the class $\mathscr{P}_{f_\lambda}$.

On the other hand, let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$, then

$$f_\lambda\left(\left\{ z : \operatorname{Re} z < \ln\left(\frac{1}{\lambda}\right) \right\}\right) = \mathbb{D} \setminus \{0\},$$

and since $1 < \ln\left(\frac{1}{\lambda}\right)$ we have $\overline{f_\lambda(\mathbb{D})} \subset \mathbb{D}$. Moreover, since the immediate basin B of the attracting fixed point is the only Fatou component of f_λ we have $\overline{\mathbb{D}} \subset B$.

Since $f_\lambda^{-1}(\mathbb{C} \setminus \overline{\mathbb{D}}) = \left\{ z : \operatorname{Re} z > \ln\left(\frac{1}{\lambda}\right) \right\}$, the only tract of f_λ is the half plane $T = \left\{ z : \operatorname{Re} z > \ln\left(\frac{1}{\lambda}\right) \right\}$. Let us consider the ray $\alpha : (0, \infty) \to \mathbb{C} \setminus \overline{\mathbb{D}}$ defined by $\alpha(t) = -(1+t)$, then

$$f_\lambda^{-1}(\alpha(0,\infty)) = \bigcup_{k \in \mathbb{Z}} \left\{ x + (2k-1)\pi i : x > \ln\left(\frac{1}{\lambda}\right) \right\},$$

and for each $k \in \mathbb{Z}$, put $S_k := \left\{ z : \operatorname{Re} z > \ln\left(\frac{1}{\lambda}\right), (2k-1)\pi < \operatorname{Im} z < (2k+1)\pi \right\}$.

Then $T \setminus f_\lambda^{-1}(\alpha(t))$ is the disjoint union of the fundamental domains S_k.

Following [6], let $c: J(f_\lambda) \to \mathbb{R}^+$ be a function such that for each $k \in \mathbb{Z}$, this function is constant on $J(f_\lambda) \cap (S_{-k} \cup S_k)$ and we denote by c_k its value on this set. Furthermore we assume that the sequence $(c_k)_{k\in\mathbb{Z}}$ of positive numbers satisfies

$$\lim_{k\to\infty} \frac{\log c_k}{\log k} = -\infty. \tag{8}$$

Define $\varphi(z) := \log\left(c(z)|z|^{-t}\right)$, where $t > 0$, $c(z) = c_k$ if $z \in S_{-k} \cup S_k$ and the sequence $(c_k)_{k\in\mathbb{Z}}$ satisfies (8). Observe that any potential as above $\varphi(z) = \log\left(c(z)|z|^{-t}\right)$ satisfies $\lim_{k\to+\infty} c_k \to 0$, so, φ is not a tame potential, however, this belongs to the class $\mathscr{P}_{f_\lambda}$ since the function c is bounded on each S_k and $|f_\lambda'(z)|_\theta = |z|$.

3.2. Proof of Theorem 2

1) The classical Denjoy-Carleman-Ahlfors Theorem [22] implies that transcendental entire functions of finite order have only a finite number of tracts. We will assume for simplicity that for $f \in \mathscr{F}$ there is only one tract T, and there is no complication in the generalization to a finite number of tracts.

Let $N \geq 1$ and denote by $\mathcal{S}_N$ the union of $2N+1$ fundamental domains in T, that is

$$\mathcal{S}_N = \bigcup_{k=-N}^{N} S_k$$

and define

$$K_N := \left\{z \in \mathcal{S}_N : \text{for every } j \geq 0, f^j(z) \in \mathcal{S}_N\right\}.$$

Let $\left(\underline{s}^m\right)_{m\geq 1}$ be a sequence in Σ_N. Taking a subsequence if it is necessary, one can assume that for some R large enough, the subsequence $\left(\underline{s}^m\right)_{m\geq 1}$ is contained in $\Sigma_N \cap B(\underline{0}, R)$. So for every $m \geq 1$ there is there is an endpoint $z_{\underline{s}^m} = h_{\underline{s}^m}(0) \in K_N \cap B(z_{\underline{0}}, R)$. Since $K_N \cap B(z_{\underline{0}}, R)$ is bounded and K_N is closed in $J(f)$ we have there is a subsequence converging to some point $z \in K_N$. Let $\underline{s}$ be the itinerary associated to z, then $\underline{s} \in \Sigma_+^N$ and $\rho\left(\underline{s}^{m_j}, \underline{s}\right) = \left|z_{\underline{s}^{m_j}} - z\right| \to 0$, $j \to \infty$.

On the other hand, let Λ be a compact subset of X with respect to the metric ρ with $\sigma(\Lambda) \subset \Lambda$. Let $\underline{s} \in \Lambda$ and let $z \in H(\Lambda)$ with itinerary $\underline{s}$, then since the compact subset $H(\Lambda)$ intersects only a finite numbers of tracts (see ([8], Lemma 3.2)), there exists $N_0 \geq 1$ such that for every $j \geq 0$ we have $|s_j| \leq N_0$. Therefore $\Lambda \subset \Sigma_{N_0}$.

2) Follows from the derivative grown condition and the uniformly expanding property of f, see ([10], Proposition 4.4).

3) This a standard fact described in ([10], Lemma 4.2). However, we include a short proof. That is, for each $R > 0$ there exists $n \geq 1$ such that for every $w \in B(z_{\underline{0}}, R)$, $f^n(B(w, \delta_0)) \supset B(z_{\underline{0}}, R)$, then the property follows. Let $\underline{s}, \underline{t} \in X$

and $U=B(\underline{s},\delta_1)$, $V=B(\underline{t},\delta_2)$, for $\delta_1,\delta_2>0$. By expanding property in Part 2, follows there is $n_1>0$ such $\sigma^{n_1}(U)\supseteq B(\sigma^{n_1}\underline{s},\delta_0)$. Let R be large enough such that $\sigma^{n_1}\underline{s},\underline{t}\in B(\underline{0},R)$ and there is N_1 such that

$$\sigma^{N_1}\left(B\left(\sigma^{n_1}\underline{s},\delta_0\right)\right)\supset B(\underline{0},R).$$

So, $\sigma^{N_1+n_1}(U)\supset\sigma^{N_1}B(\sigma^{n_1}\underline{s},\delta_0)\supset B(\underline{0},R)$. Taking $m>N_1+n_1$, we have for every $k\geq m$, $\sigma^k(U)\cap V\neq\varnothing$.

4) This property can be inferred from the general property of the density of periodic sources in the Julia set, as referred in ([23], Theorem 4). However, we include here a short proof: Let $\underline{s}=(s_0s_1\cdots)\in X$ and $\varepsilon>0$. Then, there exists $n_1>0$ such that

$$B\left(\sigma^{n_1}\underline{s},\delta_0\right)\subseteq\sigma^{n_1}B(\underline{s},\varepsilon).$$

It is enough to take $n_1>0$ such that $C\lambda^{-n_1}\delta_0<\varepsilon$. So, we have

$$s_0s_1\cdots s_{n_1-1}B\left(\sigma^{n_1}\underline{s},\delta_0\right)\subseteq B\left(\underline{s},C\lambda^{-n_1}\delta_0\right)\subseteq B(\underline{s},\varepsilon).$$

Hence, $B(\sigma^{n_1}\underline{s},\delta_0)\subseteq\sigma^{n_1}B(\underline{s},\varepsilon)$. Since $X=\bigcup_{R>0}B(\underline{0},R)$, then for some $R>0$ we have $\sigma^{n_1}\underline{s}\in B(\underline{0},R)$. Moreover from Part 3, there is $n_2>0$ such that $B(\underline{0},R)\subset\sigma^{n_2}B(\sigma^{n_1}\underline{s},\delta_0)$. Therefore, for $n=n_1+n_2$ we follow that $B(\underline{0},R)\subset\sigma^nB(\underline{s},\varepsilon)$. Hence, the set $\sigma^nB(\underline{s},\varepsilon)$ contains the sequence $\underline{0}=00\cdots$. Let $w^*=w_0\cdots w_{n-1}\in\mathbb{Z}^n$ such that $w^*B(\underline{0},R)\subset B(\underline{s},\varepsilon)$, then $w_0\cdots w_{n-1}\underline{0}\in B(\underline{s},\varepsilon)$. Then, taking $N:=\max\{|w_0|,\cdots,|w_{n-1}|\}$ we conclude $w_0w_1\cdots w_n\underline{0}\in\Sigma_N$.

3.3. Proof of Corollary 1

Let $\mathrm{CB}(J(f),\mathbb{R})$ be denote the Banach space of bounded continuous functions on $J(f)$. For each potential $\varphi\in\mathscr{P}_f$, the transfer operator associated to φ and denoted by $\mathscr{L}_\varphi$ acts continuously on $\mathrm{CB}(J(f),\mathbb{R})$. So for each $\psi\in\mathrm{CB}(J(f),\mathbb{R})$,

$$\mathscr{L}_\varphi\psi(z)=\sum_{f(w)=z}\psi(w)\exp(\varphi(w)).$$

To prove Corollary 1 one can adapt with minor modifications the approach given in [10] on the thermodynamic formalism for a large class of hyperbolic meromorphic functions $f:\mathbb{C}\to\overline{\mathbb{C}}$ of finite order ρ' satisfying a rapid growth condition and for a class of tame potentials, to the family of transcendental entire maps and potentials under study. Therefore, following as in [10], we have that the following proposition remains valid for potentials ϕ_c with c being only bounded from above.

Proposition 1. *Given* $f\in\mathscr{F}$, *we have each potential* $\varphi_{c,t}\in\mathscr{P}_f$ *satisfies the following properties.*

1) $\sup_{\underline{w}\in X}\mathscr{L}_{\varphi_{c,t}\circ H}(\mathbb{1})(\underline{w})<\infty$

2) $\lim_{R\to\infty}\sum_{s\in\mathbb{Z}}\exp\left(\sup_{\underline{w}\in[s]\cap(X\setminus B(\underline{0},R))}\varphi_{c,t}\circ H(\underline{w})\right)=0$

3) $\lim_{z_{\underline{w}}\to\infty} \mathscr{L}_{\varphi_{c,t}\circ H} \mathbb{1}(\underline{w}) = 0$

Proof. Let $\varphi_{c,t} = \log c - t\log|f'|_\theta$ be a potential in $\mathscr{P}_f$ and $\psi \in \mathrm{CB}(J(f),\mathbb{R})$,

$$\begin{aligned}\mathscr{L}_{\varphi_{c,t}\circ H}\psi\circ H(\underline{w}) &= \sum_{\sigma(\underline{s})=\underline{w}} \psi(z_{\underline{s}})c(z_{\underline{s}})\left|f'(z_{\underline{s}})\right|_\theta^{-t} \\ &= \sum_{\sigma(\underline{s})=\underline{w}} \psi(z_{\underline{s}})c(z_{\underline{s}})\left|f'(z_{\underline{s}})\right|^{-t}\left|z_{\underline{s}}\right|^{-\tau t}\left|f(z_{\underline{s}})\right|^{\tau t} \\ &= \left|z_{\underline{w}}\right|^{\tau t}\sum_{\sigma(\underline{s})=\underline{w}} \psi(z_{\underline{s}})c(z_{\underline{s}})\left|f'(z_{\underline{s}})\right|^{-t}\left|z_{\underline{s}}\right|^{-\tau t}.\end{aligned}$$

Since f satisfies the derivative growth condition, we have

$$\begin{aligned}\mathscr{L}_{\varphi_{c,t}\circ H}(\mathbb{1})(\underline{w}) &\le \kappa^t\left|z_{\underline{w}}\right|^{\tau t}\sum_{\sigma(\underline{s})=\underline{w}} c(z_{\underline{s}})\left|z_{\underline{s}}\right|^{-\alpha_1 t}\left|f(z_{\underline{s}})\right|^{-\alpha_2 t}\left|z_{\underline{s}}\right|^{-\tau t} \\ &= \kappa^t\left|z_{\underline{w}}\right|^{\tau t}\sum_{\sigma(\underline{s})=\underline{w}} c(z_{\underline{s}})\left|z_{\underline{s}}\right|^{-\alpha_1 t}\left|z_{\underline{w}}\right|^{-\alpha_2 t}\left|z_{\underline{s}}\right|^{-\tau t} \\ &\le \frac{\kappa^t}{\left|z_{\underline{w}}\right|^{t(\alpha_2-\tau)}}\sum_{\sigma(\underline{s})=\underline{w}} c(z_{\underline{s}})\left|z_{\underline{s}}\right|^{-(\tau+\alpha_1)t} \\ &\le \frac{\kappa^t}{\left|z_{\underline{w}}\right|^{t(\alpha_2-\tau)}}\sup_{\underline{s}\in\mathcal{S}^{\mathbb{N}}} c(z_{\underline{s}})\sum_{\sigma(\underline{s})=\underline{w}}\left|z_{\underline{s}}\right|^{-(\tau+\alpha_1)t}.\end{aligned}$$

Since f is a transcendental entire function of finite order ρ and $t > \rho/(\tau+\alpha_1)$, then the Borel-Picard Theorem (see ([10], Theorem 3.5)) states that the series has the exponent of convergence equal to ρ. So the last sum is finite. Following ([10], Proposition 3.6), there exists $\mathcal{M}_t > 0$ such for all $\underline{w} \in X$ we have

$$\mathscr{L}_{\varphi_{c,t}\circ H}(\mathbb{1})(\underline{w}) \le \frac{\mathcal{M}_t}{\left|z_{\underline{w}}\right|^{t(\alpha_2-\tau)}}\sup_{\underline{s}\in\mathcal{S}^{\mathbb{N}}} c(z_{\underline{s}}). \tag{9}$$

So, the Equation (9) implies $\lim_{R\to\infty}\sum_{s\in\mathbb{Z}}\exp\left(\sup_{\underline{w}\in[s]\cap(X\setminus B(\underline{0},R))}\varphi_{c,t}\circ H(\underline{w})\right) = 0$ and $\lim_{z_{\underline{w}}\to\infty}\mathscr{L}_{\varphi_{c,t}\circ H}\mathbb{1}(\underline{w}) = 0$. □

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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