Advances in Pure Mathematics, 2023, 13, 473-481
https://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

3D Fractals, Axiom of Algebra $(\Delta_n)^n = +1$

Emmanuel Cadier Anaxhaoza

Shiva's Technologies Institute, Paris, France
Email: moonambassy@gmail.com

How to cite this paper: Anaxhaoza, E.C. (2023) 3D Fractals, Axiom of Algebra $(\Delta_n)^n = +1$. *Advances in Pure Mathematics*, **13**, 473-481.
https://doi.org/10.4236/apm.2023.137031

Received: June 18, 2023
Accepted: July 22, 2023
Published: July 25, 2023

Abstract

After having laid down the Axiom of Algebra, bringing the creation of the square root of −1 by Euler to the entire circle and thus authorizing a simple notation of the nth roots of unity, the author uses it to organize homogeneous divisions of the limited development of the exponential function, that is opening the way to the use of a whole bunch of new primary functions in Differential Calculus. He then shows how new supercomplex products in dimension 3 make it possible to calculate fractals whose connexity depends on the product considered. We recall the geometry of convex polygons and regular polygons.

Keywords

Psychedelic, Axiom of Algebra (AA), Generalization of the Sign, Quantum Physics, Self-Derivative, Exponential, Cosinus, Sinus, Stable Groups for Derivation Operation, Differential Calculation Theory, Supercomplex Products, Regular Polygons, 3D Fractals, Mathematical Imagery, Geometry of Regular Polygons

1. Introduction

"Round Trip". The transcendental character of the circle finds an echo in both philosophy and mathematics. The circle, symbol of transcendence and unity, is also revealed to be transcendental in its mathematical representation thanks to the Lindemann-Weierstrass theorem. The definition of the points of the unit circle as so many rotations by Euler's expressions $e^{i\theta}$ highlights this intimate relationship. The new notation introduced here brings back the definition of any angle to its part of the unit circle.

2. Generalization of the Sign

With the newly introduced symbols, the multiplication by minus one is equivalent at going alongside the diameter, through the center, until the other side.

2.1. The Replacement of the Notation "*i*" by the Δ_4 Symbol and Its New Relevance in Differential Calculus

Delta4 is also the partition of the circle into 4, or the quarter of a circle. The notation i introduced by Leonhard Euler [1] arises from the equation $i^2 = -1$.

$(\Delta_4)^2 = -1$ in the same way as *i*. But the new notation introduces the notion of partitioning the circle into *n* parts or "th" of a circle; and thus we obtain a more general law: let

$$(\Delta_n)^n = +1, n \in \mathbb{N}$$

This corresponds to making *n* rotations of an nth partition of the circle in the direct or anti-clockwise direction around the zero origin, or center of the circle, and thus return to the starting point of the circle with radius 1, or point +1 of the real line. We also had $i^4 = +1$ but that did not bring out the notion that *i* is a quarter of a circle. We will better understand the new notation with the diagram (Figure 1).

We use to denote delta "*n*" a triangle which circumscribes the integer *n*. The points of the circle of radius 1, nth roots of the unit are thus the set of the $(\Delta_n)^k$, $n \in \mathbb{N}$, $k \in \mathbb{Z}$, $|k| \le n$. We can join a $(\Delta_n)^k$ by going through the diameter and adding the minus sign in front of it. If, starting from point +1 we perform the rotation in the opposite direction, then we take the inverse of Δ_n.

So $\frac{-1}{\Delta_6} = \Delta_6^{\ 2} = \Delta_3$ as $\Delta_6^{\ 3} = -1$;

More generally $\Delta_n^{\ \frac{n}{2}} = -1$, for $n \ge 2$.

Thus $\frac{1}{\Delta_8^{\ 3}} = \Delta_8^{\ 5} = -\Delta_8$ and $\frac{1}{\Delta_6^{\ 2}} = \Delta_6^{\ 4} = -\Delta_6$, in the same way we have: $-i = \frac{1}{i}$, or $\frac{1}{\Delta_4} = -\Delta_4$. $\Delta_2 = \frac{1}{\Delta_2}$ is like writing $-1 = -1$.

2.2. The Exponential, or Self-Derivative, Function

The exponential, or self-derivative, function has for limited development the sum of modulus $\frac{x^n}{n!}$ more 1, so:

$$\exp(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \ -\infty < x < \infty$$

We consider the limited development

$$\exp(\Delta_4 x) = 1 + \frac{\Delta_4 x}{1!} + \frac{(\Delta_4 x)^2}{2!} + \frac{(\Delta_4 x)^3}{3!} + \frac{(\Delta_4 x)^4}{4!} + \cdots, \ -\infty < x < \infty$$

We apply the above calculation rules to the $(\Delta_4)^n$ and we obtain as with *i*, the factoring of Δ_4 as follows:

$$\exp(\Delta_4 x) = \cos(x) + \Delta_4 \sin(x)$$

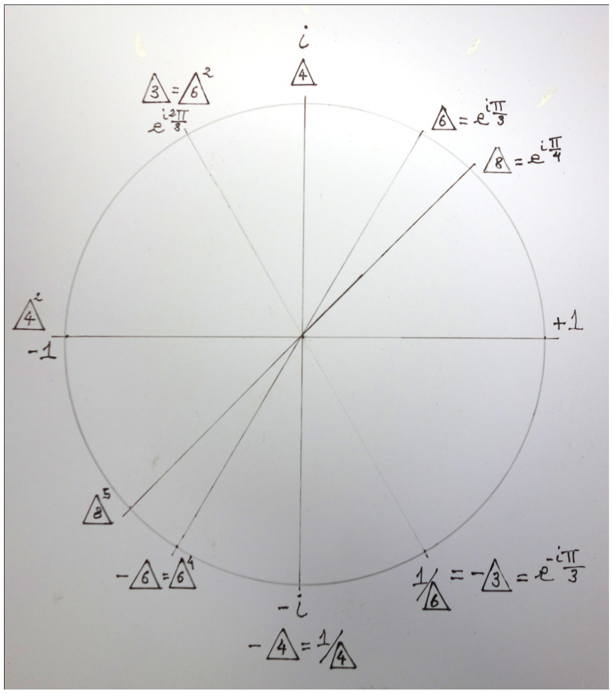

Figure 1. The delta "*n*" power *k*.

With

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots, \ -\infty < x < \infty$$

and

$$\sin(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \ -\infty < x < \infty$$

where cos(*x*) is the real part and sin(*x*) the imaginary part of the complex number $\exp(\Delta_4 x)$.

In the same way we will consider the number $\exp(\Delta_6 x)$

We take $n = 6$ so that $\frac{n}{2}$ is defined and that we can simplify by minus one, so:

$$\exp(\Delta_6 x) = 1 + \frac{\Delta_6 x}{1!} + \frac{(\Delta_6 x)^2}{2!} + \frac{(\Delta_6 x)^3}{3!} + \frac{(\Delta_6 x)^4}{4!} + \cdots, \ -\infty < x < \infty$$

To model the factoring of $\exp(\Delta_4 x)$, we now have two successive factors Δ_6 and $\Delta_6{}^2 = \Delta_3$, as follows:

$$\exp(\Delta_6 x) = ga(x) + \Delta_6 fa(x) + \Delta_6{}^2 zu(x)$$

With

$$ga(x) = F_{\Delta_6^0}(x) = 1 - \frac{x^3}{3!} + \frac{x^6}{6!} - \frac{x^9}{9!} + \cdots, \ -\infty < x < \infty$$

$$fa(x) = F_{\Delta_6}(x) = \frac{x}{1!} - \frac{x^4}{4!} + \frac{x^7}{7!} - \frac{x^{10}}{10!} + \cdots, \ -\infty < x < \infty$$

And

$$zu(x) = F_{\Delta_6^2}(x) = \frac{x^2}{2!} - \frac{x^5}{5!} + \frac{x^8}{8!} - \frac{x^{11}}{11!} + \cdots, \ -\infty < x < \infty$$

ga(*x*), *fa*(*x*) and *zu*(*x*) form a stable group for the derivation operation:

$$\{ga(x)\}' = -zu(x), \ \{fa(x)\}' = ga(x) \ \text{and} \ \{zu(x)\}' = fa(x)$$

The second derivatives are therefore:

$$\{ga(x)\}'' = -fa(x), \ \{fa(x)\}'' = -zu(x) \ \text{and} \ \{zu(x)\}'' = ga(x)$$

And the three-fold derivatives:

$$\{ga(x)\}''' = -ga(x), \ \{fa(x)\}''' = -fa(x) \ \text{and} \ \{zu(x)\}''' = -zu(x)$$

The group is therefore stable for the derivation operation.

In the same way we had for cos(*x*) and sin(*x*):

$\cos(x) = F_{\Delta_4^0}(x)$ and $\sin(x) = F_{\Delta_4}(x)$, Δ_4^0 being an index which numbers the function like a number and we cannot simplify the writing apart Δ_n^1 (Δ_n power 1) which is written Δ_n, a triangle which circumscribes the natural integer of the nth of a circle.

With

$$\{\cos(x)\}' = -\sin(x) \ \text{and} \ \{\sin(x)\}' = \cos(x)$$

Thus $\{\cos(x)\}'' = -\cos(x)$ and $\{\sin(x)\}'' = -\sin(x)$

The group is stable for the derivation operation.

We can say that *ga*(*x*), *fa*(*x*) and *zu*(*x*) are primary solutions of the differential equation of the function f of the variable $x \in \mathbb{R}$ so that $f^{(3)}(x) = -f(x)$.

We can also consider hyperbolic functions, and as we have

$$\exp(x) = \cosh(x) + \sinh(x);$$

We have:

$$\exp(\Delta_3 x) = 1 + \frac{\Delta_3 x}{1!} + \frac{(\Delta_3 x)^2}{2!} + \frac{(\Delta_3 x)^3}{3!} + \frac{(\Delta_3 x)^4}{4!} + \ldots, -\infty < x < \infty$$

We apply the preceding calculation rules to the $(\Delta_3)^n$ and we get the following factoring where the name of the function followed by "*h*" means the hyperbolic of the function, *i.e.* the limited development with only some +.

$$\exp(\Delta_3 x) = gah(x) + \Delta_3 fah(x) + (\Delta_3)^2 zuh(x)$$

$$\exp(\Delta_3 x) = F_{\Delta_6^0} h(x) + \Delta_3 F_{\Delta_6} h(x) + (\Delta_3)^2 F_{\Delta_6^2} h(x)$$

We have also: *gah*(*x*), *fah*(*x*) and *zuh*(*x*) are primary solutions of the differential equation of the function f of the variable $x \in \mathbb{R}$ so that $f^{(3)}(x) = f(x)$.

Continuing the values of *n*, after Δ_6 we have Δ_8, if we first take n even, so that $\Delta_n^{\frac{n}{2}} = -1$

And so

$$\exp(\Delta_8 x)=1+\frac{\Delta_8 x}{1!}+\frac{(\Delta_8 x)^2}{2!}+\frac{(\Delta_8 x)^3}{3!}+\frac{(\Delta_8 x)^4}{4!}+\cdots$$

Either:

$$\exp(\Delta_8 x)=F_{\Delta_8^0}(x)+\Delta_8 F_{\Delta_8}(x)+(\Delta_8)^2 F_{\Delta_8^2}(x)+(\Delta_8)^3 F_{\Delta_8^3}(x)$$

With:

$$F_{\Delta_8^0}(x)=1-\frac{x^4}{4!}+\frac{x^8}{8!}-\frac{x^{12}}{12!}+\cdots,\ -\infty<x<\infty$$

$$F_{\Delta_8}(x)=\frac{x}{1!}-\frac{x^5}{5!}+\frac{x^9}{9!}-\frac{x^{13}}{13!}+\cdots,\ -\infty<x<\infty$$

$$F_{\Delta_8^2}(x)=\frac{x^2}{2!}-\frac{x^6}{6!}+\frac{x^{10}}{10!}-\frac{x^{14}}{14!}+\cdots,\ -\infty<x<\infty$$

And:

$$F_{\Delta_8^3}(x)=\frac{x^3}{3!}-\frac{x^7}{7!}+\frac{x^{11}}{11!}-\frac{x^{15}}{15!}+\cdots,\ -\infty<x<\infty$$

We observe the stability of the group for the derivation operation.

$\sqrt{i}=\Delta_8$ defines the eighth of a circle and generates the octagon.

The powers $\Delta_n{}^k, n\in\mathbb{N}, k\in\mathbb{N}, k\le n$ define the vertex points of regular polygons inscribed in the unit circle.

For the pentagon, we have:

$$\exp(\Delta_5 x)=1+\frac{\Delta_5 x}{1!}+\frac{(\Delta_5 x)^2}{2!}+\frac{(\Delta_5 x)^3}{3!}+\frac{(\Delta_5 x)^4}{4!}+\cdots$$

Either with the hyperbolics of the functions $F_{\Delta_{10}^k}(x), 0\le k\le\frac{n}{2}-1$, defined for the decagon:

$$\exp(\Delta_5 x)=F_{\Delta_{10}^0}h(x)+\Delta_5 F_{\Delta_{10}}h(x)+\Delta_5{}^2 F_{\Delta_{10}^2}h(x)+\Delta_5{}^3 F_{\Delta_{10}^3}h(x)+\Delta_5{}^4 F_{\Delta_{10}^4}h(x)$$

Thus we define the limited developments for all the groups of functions for any n even and we use the hyperbolics when n is odd.

$$F_{\Delta_{10}^0}h(x)=1+\frac{x^5}{5!}+\frac{x^{10}}{10!}+\cdots,\ -\infty<x<\infty$$

$$F_{\Delta_{10}}h(x)=x+\frac{x^6}{6!}+\frac{x^{11}}{11!}+\cdots,\ -\infty<x<\infty$$

$$F_{\Delta_{10}^2}h(x)=\frac{x^2}{2!}+\frac{x^7}{7!}+\frac{x^{12}}{12!}+\cdots,\ -\infty<x<\infty$$

$$F_{\Delta_{10}^3}h(x)=\frac{x^3}{3!}+\frac{x^8}{8!}+\frac{x^{13}}{13!}+\cdots,\ -\infty<x<\infty$$

$$F_{\Delta_{10}^4}h(x)=\frac{x^4}{4!}+\frac{x^9}{9!}+\frac{x^{14}}{14!}+\cdots,\ -\infty<x<\infty$$

3. Products More Than Complex

With $(a,b,c,d,e,f)\in\mathbb{R}^6$, in dimension 3, we have supercomplex products with the hexagon:

$$\left\{a+\Delta_6 b+\Delta_6{}^2 c\right\}*\left\{d+\Delta_6 e+\Delta_6{}^2 f\right\}$$
$$=\left(ad-bf-ec\right)+\Delta_6\left(ae+bd-cf\right)+\Delta_6{}^2\left(cd+be+af\right)$$

So for the square:

$$\left\{a+\Delta_6 b+\Delta_6{}^2 c\right\}^2=a^2-2bc+\Delta_6\left(2ab-c^2\right)+\Delta_6{}^2\left(2ac+b^2\right)$$

We also have with the triangle Δ_3:

$$\left\{a+\Delta_3 b+\Delta_3{}^2 c\right\}*\left\{d+\Delta_3 e+\Delta_3{}^2 f\right\}$$
$$=\left(ad+bf+ec\right)+\Delta_3\left(ae+bd+cf\right)+\Delta_3{}^2\left(cd+be+af\right)$$

So for the square:

$$\left\{a+\Delta_3 b+\Delta_3{}^2 c\right\}^2=a^2+2bc+\Delta_3\left(2ab+c^2\right)+\Delta_3{}^2\left(2ac+b^2\right)$$

The $\Delta_n{}^k, n\in\mathbb{N}, k\in\mathbb{Z}, |k|\le n$ are complex numbers, nth roots of the unit, defined by the formula:

$$\Delta_n{}^k=\mathrm{e}^{\frac{i2k\pi}{n}}$$

where i is the imaginary number of Euler also called Δ_4.

The first product with Δ_6 and $\Delta_6{}^2$, and the other stable products in dimension 3, having coefficients not all positive, make it possible to define calculations of fractals in three dimensions by taking the z points of the space of coordinates a, b, c and of modulus a norm for the metric space.

With $(a,b,c,d,e,f,g,h)\in\mathbb{R}^8$, in dimension 4, one has among other things "hypercomplex" product of the square:

$$\left\{a+\Delta_4 b+\Delta_4{}^2 c+\Delta_4{}^3 d\right\}*\left\{e+\Delta_4 f+\Delta_4{}^2 g+\Delta_4{}^3 h\right\}$$
$$=ae+bh+cg+df+\Delta_4\left(af+be+ch+dg\right)+\Delta_4{}^2\left(ag+bf+ce+dh\right)$$
$$+\Delta_4{}^3\left(de+cf+bg+ah\right)$$

As $\Delta_4{}^2=-1$ and $\Delta_4{}^3=-\Delta_4$ we can write:

With $(z_1,z_2,z_3,z_4,z_5,z_6)\in\mathbb{C}^6$ we have:

$$\left(z_1-z_2\right)*\left(z_3-z_4\right)=z_5-z_6$$

The product of two complex differences is a complex difference and a complex difference is the product of two complex differences.

We can also, in dimension 4, define hypercomplex products with the powers of Δ_8, as long as we choose 4 powers allowing a stable product. This is always the case if we choose the 4 vertices of the octagon on 4 distinct diameters.

It seems that we can define more general formulas concerning the $\Delta_n{}^k$, which apply even when k is rational, for $\frac{n}{k}$ integer, $n\in\mathbb{N}$:

$$\Delta_n{}^k=\Delta_{\frac{n}{k}} \text{ if } k \text{ is positive, } n\ge 2, k\le n$$

and

$$\Delta_n{}^k=\Delta_{\frac{n}{|k|}}{}^{-1} \text{ if } k \text{ is negative, } n\ge 2, |k|\le n$$

For mathematical imagery, in dimension 3, we can have fun counting the number of different STABLE products for its emission polygon; with component vector { Δ_6^0, Δ_6, Δ_6^2 } as previously calculated; or { Δ_6^{-1}, Δ_6^0, Δ_6 }, or { Δ_6^3, Δ_6^4, Δ_6^5 } which is the same result as { Δ_6^0, Δ_6, Δ_6^2 } but which defines a different product for the components of the vector put back in factor.

$$\begin{aligned}&\left\{\Delta_6^3 a+\Delta_6^4 b+\Delta_6^5 c\right\}*\left\{\Delta_6^3 d+\Delta_6^4 e+\Delta_6^5 f\right\}\\&=ad-bf-ce+\Delta_6\left(ae+bd-cf\right)+\Delta_6^2\left(af+be+cd\right)\\&=\Delta_6^3\left(bf+ce-ad\right)+\Delta_6^4\left(cf-ae-bd\right)+\Delta_6^5\left(-af-be-cd\right)\end{aligned}$$

The products must remain in the predefined components to be qualified as STABLE. It would appear that the product is stable for three vertices of the hexagon located on three different diameters of the circumscribed circle. If we take { Δ_6^0, Δ_6^2, Δ_6^4 } it comes down to taking

{ Δ_3^0, Δ_3, Δ_3^2 } as previously calculated, or so { Δ_6^0, Δ_6^2, $-\Delta_6$ }, the product has only positive coefficients. We will see that if we take the three vertices symmetrical with respect to the imaginary axis, which are also symmetrical by the central symmetry of the center of the circle, let { Δ_6^3, Δ_6, Δ_6^{-1} } we have, for the product, only negative coefficients.

$$\begin{aligned}&\left\{\Delta_6^3 a+\Delta_6 b+\Delta_6^{-1} c\right\}*\left\{\Delta_6^3 d+\Delta_6 e+\Delta_6^{-1} f\right\}\\&=\Delta_6^3\left(-ad-bf-ce\right)+\Delta_6\left(-ea-bd-cf\right)+\Delta_6^{-1}\left(-af-eb-cd\right)\end{aligned}$$

By continuing this reasoning we can return to the product of two complex numbers and with two points of the plane of respective coordinates (*a*, *b*) and (*c*, *d*) we have:

$(a+\Delta_4 b)*(c+\Delta_4 d)=ac-bd+\Delta_4(bc+ad)$, what we already knew. In addition we have:

$$\left\{\Delta_4^2 a+\Delta_4 b\right\}*\left\{\Delta_4^2 c+\Delta_4 d\right\}=\Delta_4^2\left(bd-ac\right)+\Delta_4\left(-ad-bc\right)$$

which seems to define a new product of complex numbers, which will give fractals with the same symmetry as the traditional product of complex numbers. Because it will be understood that the square given by the first product above is symmetrical whether it is $(\pm)ib$, *i.e.* the product is the same for the vector { Δ_4^0, Δ_4 } and { Δ_4^0, Δ_4^3 }, therefore the Mandelbrot fractal is symmetrical along the real line.

$$\left\{a+\Delta_4^3 b\right\}*\left\{c+\Delta_4^3 d\right\}=ac-bd+\Delta_4^3\left(bc+ad\right)$$

And the product for vectors

$\left\{\Delta_4^2 a+\Delta_4 b\right\}$ and $\left\{\Delta_4^2 a+\Delta_4^3 b\right\}$ are the same in the same way.

$$\begin{aligned}\left\{\Delta_4^2 a+\Delta_4^3 b\right\}*\left\{\Delta_4^2 c+\Delta_4^3 d\right\}&=ac-bd+\Delta_4\left(ad+bc\right)\\&=\Delta_4^2\left(bd-ac\right)+\Delta_4^3\left(-ad-bc\right)\end{aligned}$$

The real line is the axis of symmetry of the figure when the building uses the square.

The inverse of n in the delta is also a solution.

The points of the unit circle are full symbols that indicate the sign.

In dimension 3, the connexity of Julia and Mandelbrot fractals [2] depends on the supercomplex product we choose, according to the number of occurrence of minus one within the supercomplex product considered.

Figure 2 is illustrating the calculations.

The circumscribed triangle can be a circumscribed circle if there is no ambiguity with other mathematical symbols.

The 360˚ of the circle corresponds to (delta360)^{360} = +1, the same for grades or gradians (delta400)^{400} = +1.

$$\Delta_{360}{}^{360} = +1$$

$$\Delta_{400}{}^{400} = +1$$

In these last cases the numbers are written base ten is implied, or one can add a point A if one has the place.

4. Geometry of Regular Polygons

Throughout the centuries, the geometry of the regular polygons inscribed in the circle of radius r has been established [3].

The sum of the internal angles between 2 faces in radians, of the regular polygon of n faces, is equal to $(n-2)*\pi$.

The sum of the external angles between 2 faces, of the regular polygon of n faces, is equal to $(n+2)*\pi$.

These sums hold true for irregular convex polygons.

For regular polygons, the perimeter is equal to $2nr\sin\frac{\pi}{n}$.

The area is equal to $nr^2\sin\frac{\pi}{n}\cos\frac{\pi}{n}$, knowing that $\sin 2x = 2\sin x\cos x$, the area is also equal to $\frac{nr^2\sin\frac{2\pi}{n}}{2}$.

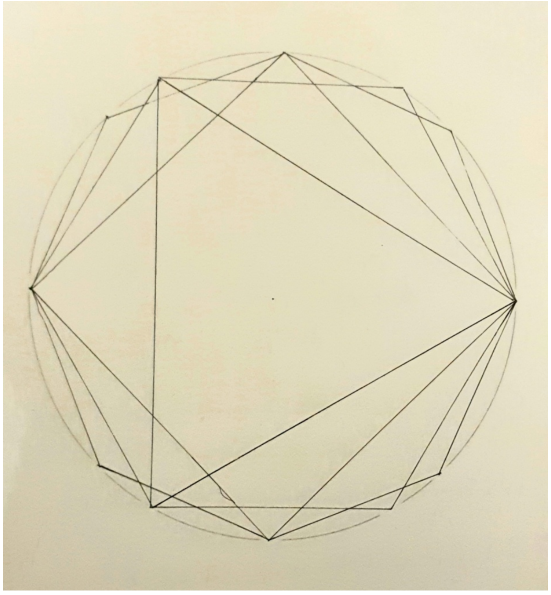

Figure 2. The even regular polygons and the triangle.

Obviously we know that these 2 functions respectively tend to $2\pi * r$ and $\pi * r^2$ when n tends to infinity.

It is noticed that Euler's characteristic theorem [4] adds to Euclid's *Elements* the equation which links vertices V, edges E and faces F of any polyhedron according to $V - E + F = 2$.

5. Conclusions

It is regretted that modern school education too often forgets the geometry of polygons and regular polygons, seeing students remember the particular value of a famous function, namely the sum of the internal angles of a triangle is worth π, but ignoring its general value as the sum of the interior angles of a convex polygon $(n-2)\pi$.

The Theory of Differential Calculus should restructure entirely around stable groups for the operation of derivation.

The mathematical imagery of fractals in 3 dimensions, made possible by supercomplex products, must certainly be very spectacular.

Recall

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$ga(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{3k}}{(3k)!}$$

$$fa(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{3k+1}}{(3k+1)!}$$

$$zu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{3k+2}}{(3k+2)!}$$

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Euler, L. (1748) Introductio in Analysin Infinitorum. Marcum-Michaelem Bousquet, Lausanne.

[2] Mandelbrot, B. (1975) Les Objets fractals: forme, hasard et dimension. Flammarion, Paris.

[3] Euclid (around 300 B.C.) Στοιχεῖα "The Elements".

[4] Euler, L. (1752) Elementa Doctrinæ Solidorum. Saint Petersburg.