Scientific Research Publishing

Advances in Pure Mathematics, 2023, 13, 463-472
https://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

Proof of Riemann Conjecture Based on Contradiction between Xi-Function and Its Product Expression

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How to cite this paper: Chen, C.M. (2023) Proof of Riemann Conjecture Based on Contradiction between Xi-Function and Its Product Expression. *Advances in Pure Mathematics*, **13**, 463-472.
https://doi.org/10.4236/apm.2023.137030

Received: May 16, 2023
Accepted: July 18, 2023
Published: July 21, 2023

Abstract

Riemann proved three results: analytically continue $\zeta(s)$ over the whole complex plane $s=\sigma+it$ with a pole $s=1$; (Theorem A) functional equation $\xi(t)=G(s_0)\zeta(s_0)$, $s_0=1/2+it$ and (Theorem B) product expression $\xi_1(t)$ by all roots of $\xi(t)$. He stated **Riemann conjecture (RC): All roots of $\xi(t)$ are real.** We find a mistake of Riemann: he used the same notation $\xi(t)$ in two theorems. Theorem B must contain complex roots; it conflicts with RC. Thus theorem B can only be used by contradiction. Our research can be completed on $s_0=1/2+it$. Using all real roots r_k and (true) complex roots $z_j=t_j+i\alpha_j$ of $\xi(z)$, define product expressions $w(t)$, $w(0)=\xi(0)$ and $Q(t)>0$, $Q(0)=1$ respectively, so $\xi_1(t)=w(t)Q(t)$. Define infinite point-set $L(\omega)=\{t:t\ge 10 \text{ and } |\zeta(s_0)|=\omega\}$ for small $\omega>0$. If $\xi(t)$ has complex roots, then $\omega=\omega Q(t)$ on $L(\omega)$. Finally in a large interval of the first module $|z_1|\gg 1$, we can find many points $t\in L(\omega)$ to make $Q(t)<1/2$. This contraction proves RC. In addition, Riemann hypothesis (RH) for ζ also holds, but it cannot be proved by ζ.

Keywords

Riemann Conjecture, Xi-Function, Functional Equation, Product Expression, Multiplicative Group, Contradiction

1. Two Different Research Routes

D. Hilbert (1900) proposed 23 problems in The Second International Congress of Mathematicians and for the first time stated [1]

"…, *it still remains to prove the correctness of an exceedingly important statement of Riemann, viz., that the zero points of the function* $\zeta(s)$ *defined by the series*

$$\zeta(s)=1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\cdots$$

all have the real part $1/2$, *except the well-known negative integral real zeros…*"

Since then it has been accepted as a classical formulation:

Riemann Hypothesis (RH). *The nontrivial zeros of* $\zeta(s)$ *all have the real part* $1/2$.

(But Riemann didn't say that!) Due to Hilbert's high prestige, the mathematicians all have focused on ζ. S. Smale [2] (1998) proposed 18 problems in "*Mathematical problems for the next century*". RH was listed as the first. In 2000, Clay Mathematics Institute opened seven Millennium Problems, including RH, see official reviews E. Bombieri [3] (2000) and P. Sarnak [4] (2005). In the 20th century, extremely large scale computations for ζ confirm that RH holds (up to $t=10^9\sim10^{13}$) [5] [6] [7], which have enhanced our belief. But, many scholars have different opinions about RH to be true or false, see [3] [4]. This is difficult position today.

Hilbert said [8], "*If I were to awaken after having slept for a thousand years, my first question would be: Has the Riemann hypothesis proven*?" A century has passed, J. Conrey [9] (2003) pointed out that: "*In my belief, RH is a genuinely arithmetic problem, likely don't succumb to the method of analysis*". It seems that using "the hard analysis" has come to the end!

For this, we turn to the entire function $\xi(t)=G(s_0)\zeta(s_0)$, $s_0=1/2+it$. But its product expression must contain complex roots, which contradicts RC. We find that the product expression is the most suitable tool for studying "no complex roots". Finally, RC can be proved by contradiction.

In addition, RH also holds by $\zeta(s_0)=\xi(t)/G(s_0)$, but it cannot directly be proved by ζ. Because ζ is not an entire function, there is only one way, *i.e.* study the series summation $\zeta\neq0$, which has surpassed ability of the existing analysis.

We clarify three notations used in this paper:

1) Euler $\zeta(s)$-function is analytic in the whole complex plane with a pole $s=1$.

2) Riemann took $s_0=1/2+it$ to define $\xi(t)=G(s_0)\zeta(s_0)$ (not $\xi(s)$ used in literatures).

3) We construct $\xi_1(t)$ by all roots of $\xi(t)$ (Riemann had a mistake to use the same $\xi(t)$).

2. Follow Riemann's Thought

In Riemann's paper, only two pages focused on RC [10], pp. 300-302.

Euler (1737) proved product formula of primes

$$\zeta(\sigma)=\sum_{n=1}^{\infty}\frac{1}{n^{\sigma}}=\prod_{p\in primes}\left(1-\frac{1}{p^{\sigma}}\right)^{-1},\ \sigma>1. \tag{1}$$

Taking $s=\sigma+it,\sigma>1$ and $y=n^2\pi x$ in gamma integral

$$\Gamma\left(\frac{s}{2}\right)=\int_0^{\infty}y^{s/2-1}\mathrm{e}^{-y}\mathrm{d}y=n^s\pi^{s/2}\int_0^{\infty}x^{s/2-1}\mathrm{e}^{-n^2\pi x}\mathrm{d}x,$$

and summing over *n*, Riemann had

$$\zeta(s)=\sum_{n=1}^{\infty}n^{-s}=\pi^{s/2}\Gamma^{-1}\left(\frac{s}{2}\right)\int_0^{\infty}x^{s/2-1}\psi(x)\mathrm{d}x,\ \psi(x)=\sum_{n=1}^{\infty}\mathrm{e}^{-n^2\pi x},$$

where Jacobi function $\psi(x)$ satisfies $2\psi(x)+1=x^{-1/2}\left(2\psi\left(\frac{1}{x}\right)+1\right)$. By $z=1/x$, there is

$$\int_0^1 z^{s/2-1}\psi(z)\mathrm{d}z=\frac{1}{s(s-1)}+\int_1^{\infty}x^{-s/2-1/2}\psi(x)\mathrm{d}x.$$

The singularity $x=0$ has been eliminated. Riemann got an integral representation

$$\zeta(s)=\pi^{s/2}\Gamma^{-1}\left(\frac{s}{2}\right)\left\{\frac{1}{s(s-1)}+\int_1^{\infty}\left(x^{s/2-1}+x^{-(s+1)/2}\right)\psi(x)\mathrm{d}x\right\}, \tag{2}$$

which has been analytically continued over the whole complex plane except for a pole $s=1$ Whereas $\Gamma^{-1}(s/2)$ has zeros $s=-2,-4,\cdots$, called trivial zeros of $\zeta(s)$, no interest for us.

Multiplying (2) by $G(s)$, Riemann directly took $s=1/2+it$ and got

$$\xi(t)=G(s)\zeta(s),\ G(s)=\frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right),\ |G(s)|\approx Ct^{7/4}\mathrm{e}^{-t\pi/4}. \tag{3}$$

(Many scholars have accepted another notation $\xi(s)=G(s)\zeta(s)$, [10] p. 17, but Riemann's notation $\xi(t)$ is more concise in research, see (4), (5) and (6)). Inserting ζ into (3) and applying integration by parts twice, one has [10] p. 17,

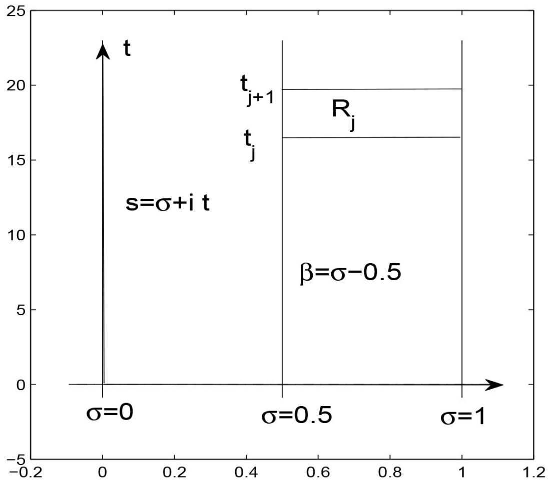
$$\begin{aligned}\xi(t)&=\frac{1}{2}+\frac{s(s-1)}{2}\int_1^{\infty}\left(x^{s/2-1}+x^{-s/2-1/2}\right)\psi(x)\mathrm{d}x,\\&=r_1+\int_1^{\infty}\left(x^{s/2-1}+x^{-s/2-1/2}\right)g(x)\mathrm{d}x,\ g(x)=2x^2\psi''+3x\psi',\end{aligned}$$

where $r_1=\frac{1}{2}+\psi(1)+4\psi'(1)=0$. Riemann got a real function [10] pp. 301-302,

$$\xi(t)=2\int_1^{\infty}\cos\left(\frac{t}{2}\ln x\right)x^{-3/4}g(x)\mathrm{d}x, \tag{4}$$

and said, "*This function is finite for all finite values of t and can be developed as a power series in t^2 which converges very rapidly*". Here Riemann used translation $\beta=\sigma-1/2$ and rotation $s=1/2+iz$, $z=t-i\beta$, see **Figure 1**, and got an even entire function $\xi(z)$ by (4). We state

Theorem A. *The entire function* $\xi(z)$ *satisfies functional equation*



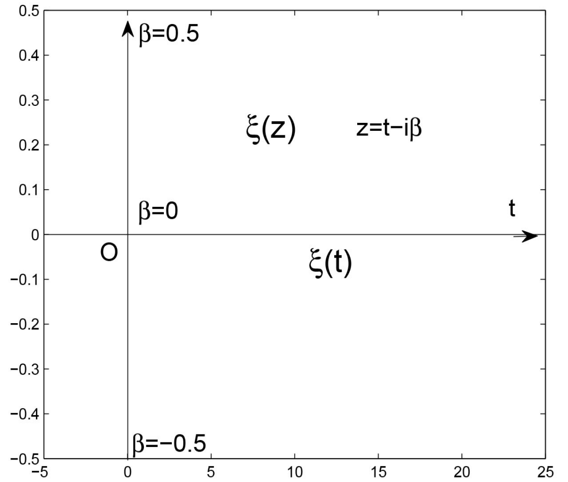


Figure 1. Translation $\beta=\sigma-1/2$ and rotation $z=t-i\beta$.

$$\xi(z)=G(s)\zeta(s),\ s=\sigma+it=1/2+iz,\ z=t-i\beta,\ \beta=\sigma-1/2, \tag{5}$$

which has symmetry $\xi(z)=\xi(-z)$ and conjugate $\xi(\bar{z})=\overline{\xi(z)}$.

Riemann continued:

"…, *the function* $\xi(t)$ *can vanish only when the imaginary part of t lies between* $\frac{1}{2}i$ *and* $-\frac{1}{2}i$. *The number of roots of* $\xi(t)=0$ *whose real parts lie between* 0 *and T is about*

$$=\frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi},\ \{\textit{remark. proved by Mangoldt}, 1905\}$$

…. One finds in fact about this many real roots within these bounds and it is very likely that all of the roots are real. One would of course like to have a rigorous proof of this, but I have put aside the research for such a proof after some fleeting vain attempts, …"

He proposed an important statement in critical strip $\Omega=\{z=t-i\beta:|\beta|\le 1/2, 0\le t<\infty\}$,

Riemann conjecture (RC). *All the roots of function* $\xi(z)$ *are real.*

Riemann finally said,

"*If one denotes by* α *the roots of the equation* $\xi(\alpha)=0$, *then one can express* $\log\xi(t)$ *as*

$$\sum\log\left(1-\frac{t^2}{\alpha^2}\right)+\log\xi(0),$$

because, since the density of roots of size *t* grows only like $\log(t/2\pi)$ as *t* grows, this expression converges and for infinite *t* is only infinite like $t\log t$; Thus it differs from $\log\xi(t)$ by a function of t^2 which is continuous and finite for finite *t* and which, when divided by t^2, is infinitely small for infinite *t*. This difference is therefore a constant, the value of which can be determined by setting $t=0$."

We have seen that Riemann wanted to prove the following

Theorem B. *Assuming that* $\{z_j\}$ *are all roots of* $\xi(z)$, *there is a product expression*

$$\xi_1(z) = \xi(0)\prod_{j=1}^{\infty}\left(1-\frac{z^2}{z_j^2}\right),\ z = t - i\beta,\ \xi(0) = 0.497120778\cdots. \tag{6}$$

It should point out that Riemann's proof is not rigorous, but the conclusion is correct. For this, J. Hadamard [11] (1893) studied the product expression for general entire function, and (6) holds for any even entire function $f(z)$ of order 1, whereas $\xi_1(z)$ only is a special case. **Why** $\xi_1(z)$ **must contain all roots?** Because, let $W(z) = \prod_{j=1}^{\infty}\left(1 - z^2/z_j^2\right)$, then $F(z) = \xi_1(z)/W(z)$ must be an even entire function of order 1 without zeros (this is very important!) and $\ln F(z) = A + Bz, B = 0$. A simplified proof is given in [10] pp. 39-47.

3. A Mistake of Riemann and Our New Thinking

(Theorem A) functional equation $\xi(z) = G(s)\zeta(s)$ is a deep expression generated by $\zeta(s)$, which can calculate all real roots, and no complex roots are found. We have a difficult process to understand Theorem A. We once considered geometrically the peak-valley structure of $\xi(z) = u + iv$. If $u(t,0)$ is single-peak in its root interval, we for the first time have proved RC, [12] [13] [14]. How to prove the single peak? Both attempts are unsuccessful [15] [16]. It seems that the conditions are lacking. We have to consider Theorem B, so a mistake of Riemann is discovered.

(Theorem B) product expression $\xi_1(z)$ must contain all roots of $\xi(t)$, which puts us in a dilemma: if there are no complex roots, then RC is assumed. If there are complex roots, it conflicts with RC. We realize that the former cannot be allowed, while the latter still has the possibility of further research, that is, to give a disproof. Theorems A and B are two different concepts, but Riemann had used the same notation $\xi(t)$, this is a hidden mistake, which is the reason why he failed. (Unfortunately, nobody found this mistake and continued to use it as the same function). Therefore Theorem B only can be used in contradiction.

E. Bombieri [3] pointed out that "*we do not have algebraic and geometric models to guide our thinking, and entirely new ideas may be needed to study these intriguing objects*". This is a valuable advice. Our research shows that geometric analysis based on Theorem A is unsuccessful, but RC can be proved by algebraic structure based on theorems A and B.

Finally we recall that if $\xi(z)$ has a finite number of complex roots, RC is proved by contradiction [17]. This paper suggests a general framework, which still leads to contradiction, even if $\xi(z)$ has infinitely many complex roots.

4. What Contradiction Do the Complex Roots Bring?

The following research can be completed on the symmetry line $s_0 = 1/2 + it$ (*i.e.* $z = t$).

Denote the complex roots $z_j = t'_j + i\alpha_j$ of $\xi(z)$, where $0 < |\alpha_j| \le 1/2$,

$R_j = |z_j| \ge K \gg 1$ (As no complex roots in $t \le 10^{10}$), by Theorem A, $\xi(z)$ has four conjugate complex roots $\pm(t'_j \pm i\alpha_j)$. By Theorem B, $\xi_1(z)$ must contain fourth degree factor $q_j(z)$, we have on $z = t$

$$q_j(t) = \left(1 - \frac{t^2}{(t'_j + i\alpha_j)^2}\right)\left(1 - \frac{t^2}{(t'_j - i\alpha_j)^2}\right) = \left(1 - \frac{t^2}{R_j^2}\right)^2 + 4t^2\frac{\alpha_j^2}{R_j^4} > 0. \quad (7)$$

Denote all real roots t_k and (true)complex roots z_j of $\xi(z)$, by theorem B, we have

$$\xi_1(t) = w(t)Q(t), w(t) = \xi(0)\prod_{k=1}^{\infty}\left(1 - \frac{t^2}{t_k^2}\right), Q(t) = \prod_{j=1}^{\infty} q_j(t), Q(0) = 1, \quad (8)$$

where $w(t)$ only depends on real roots t_k and $Q(t)$ only depends on complex roots z_j, which are independent of each other. The product expression forms a multiplicative group, satisfying the exchange law and association law. This algebraic model is the base for proving RC.

We recall $\xi(z) = G(s)\zeta(s)$, whether $\xi(z)$ has complex roots or not, the expressions (2) and (4) do not change. This is an important fact.

So our attention focused on the product expression $\xi_1(z)$. There are two cases.

1). Assuming no complex roots, $Q(t) = 1$, $\xi(t) = G(s_0)\zeta(s_0) = w(t)$, then

$$\zeta(s_0) = w(t)/G(s_0). \quad (9)$$

where $w(t)$ and $G(s_0)$ are independent of complex roots.

2). Assuming there are complex roots, $G(s_0)\zeta(s_0) = \xi(t) = \xi_1(t) = w(t)Q(t)$, we have

$$\zeta(s_0) = (w(t)/G(s_0))Q(t). \quad (10)$$

So the contradiction must appear in these t, which make $\zeta(s_0) \ne 0$ and $Q(t) \ne 1$.

5. Properties of the Infinite Point-Set $L(\omega)$

Lemma 1. *For any* $t \ge 10$, *the function* $\zeta(1/2 + it)$ *is unbounded.*

See [10], p. 184. Its roots are very irregular, the first root is $t_1 = 14.1347\cdots$, other roots $t_n \approx 2\pi n/\ln(n/2\pi)$ are increasingly dense, and its average spacing $\Delta_n \approx 2\pi/\ln n$ is getting smaller and smaller. We have

Lemma 2. *For any fixed small number* $\omega > 0$, *define an infinite point-set*

$$L(\omega) = \{t : t \ge 10 \text{ and } |\zeta(1/2 + it)| = \omega\}, \quad (11)$$

Its average spacing $\delta = O(1/\ln t)$ is getting smaller and smaller.

Figure 2 shows the curves $|\zeta(1/2 + it)|$ and $L(1)$ in an interval $[900, 1000]$.

Lemma 3. *Whether* $\xi(z)$ *has complex roots or not, we always have*

$$|w(t)/G(1/2 + it)| = \omega > 0, \ t \in L(\omega). \quad (12)$$

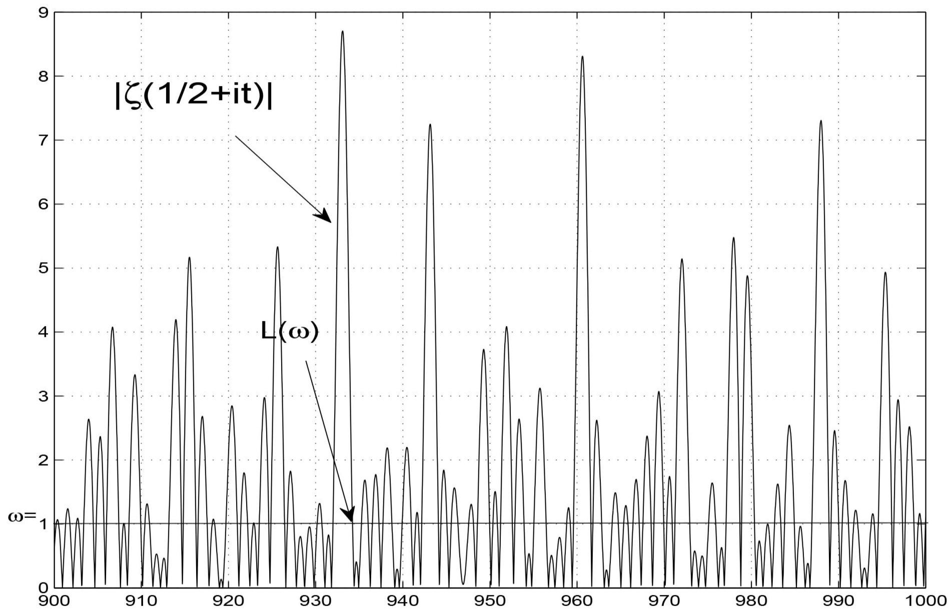


Figure 2. Point-set $L(\omega)$ in $[900,1000]$.

Proof. Firstly assume no complex roots, by $\zeta(s_0)=w(z)/G(s_0)$ and lemma 2, then (12) holds. Once (12) is obtained, we found that $|w(t)|$ and $|G(s_0)|$ are independent of complex roots, so (12) still holds, even if $\xi(z)$ has the complex roots. The lemma is proved.

Remark 1. This reason is similar to the following fact. Under the starting condition $Re(s)>1$, Riemann had analytically continued $\zeta(s)$ to (2) in the whole complex plane, except for a pole $s=1$. So this prerequisite $Re(s)>1$ naturally does not work.

Lemma 4. *If* $\xi(z)$ *has the complex roots, we have an important equality*

$$\omega=\left|\zeta(1/2+it)\right|=\omega Q(t),\ t\in L(\omega). \tag{13}$$

Proof. In this case, we have $\zeta(s_0)=\left(w(t)/G(s_0)\right)Q(t)$. So (13) is derived from Lemma 3.

Below we can find $t\in L(\omega)$ to make $Q(t)<1/2$, which leads the contradiction.

6. Proof of Riemann Conjecture

By contradiction. If these modules $R_j=Kg_j, g_1=1$, where $\{g_j\}$ is non-decreasing sequence and the convergence of $Q(t)$ should be guaranteed. Using a transform $t=Kx$, we have

$$Q(t)=F(x)=\prod_{j=1}^{\infty}p_j(x),\ \ p_j(x)=\left(1-\frac{x^2}{g_j^2}\right)^2+4x^2\frac{\alpha_j^2K^{-2}}{g_j^4}>0, \tag{14}$$

In $I_0=[0,1]$, all $p_j(x)\le 1$ and $F(x)\le 1$. No matter how the complex roots are distributed, we only need to consider the first module $R_1=Kg_1, g_1=1$ and its factor

$$p_1(x) = (1-x^2)^2 + 4x^2\alpha_j^2 K^{-2},\ \ p_1(1) = 4\alpha_j^2 K^{-2} \le K^{-2} \ll 1. \tag{15}$$

Take $\rho = 1/4$ and $x \in I_\rho = [1-p, 1]$, we get

$$Q(t) = F(x) \le p_1(x) \le 1/4 + K^{-2} < 1/2 < 1,\ \ x \in I_\rho. \tag{16}$$

(should discuss $p_1^m(x)$ for multiple root g_1) Finally return to $t = Kx \in [K - K\rho, K]$, its length $K\rho$ is very large, in which there surely are many points $t \in L(\omega)$, and the contradiction is derived from Lemma 4. Therefore RC is proved.

Remark 2. As RC holds, then the positivity $Re\left(\frac{\xi_\beta(z)}{\xi(z)}\right) > 0, \beta > 0$, proved by Hinkkanen [18] (1997) and Lagarias [19] (1999) holds. We have

Theorem 1 [15] [16]. *The strict monotone* $|\xi(t - i\beta)| > |\xi(t - i\beta_0)|$ *holds for* $|\beta| > |\beta_0|$.

This is a stronger conclusion than RC. Ancient Greek Aristotle thought, "*Order and symmetry were important elements of beauty*". We say, **the symmetry and order of ξ are mathematical beauty of Riemann conjecture.**

7. Innovations and Contributions of Our Work

We look back why studying RC is so difficult. There were several mistakes. Studying RC can be compared to climbing Mount Qomolangma. The latter has two paths, *i.e.*, along the north slope and the south slope. There are also two paths to studying RC.

The first path. Riemann (1859) had taken three steps: analytically continuation $\zeta(s)$, entire function $\xi(t)$ and product expression $\xi_1(t)$. But he made a mistake in step 3. He said, "*I have put aside the research for such a proof after some fleeting vain attempts*". Edwards mentioned [10] p. 164, "*Siegel states quite positively that the Riemann papers contain no steps toward a proof of the Riemann hypothesis.*" Now it is impossible to know what attempts are done by Riemann. Actually, Riemann had already approached to a proof of RC.

The second path. Hilbert (1900) suggested to study ζ, which only is the first step of Riemann. Which continuation is this? It's also not clear. Riemann proved

$$2\sin\pi s\Gamma(s)\zeta(s) = i\int_{-\infty}^{\infty} \frac{(-x)^{s-1}\,\mathrm{d}x}{\mathrm{e}^x - 1}$$

and $\Gamma(s/2)\pi^{-s/2}\zeta(s)$ remains unchanged when s is replaced by $1-s$ (This integral is applied to derive R-S formula by him [20]). Riemann said, "*This property of the function motivated me to consider the integral* $\Gamma(s/2)$ *instead of the integral* $\Gamma(s)$ *in the general term of* $\sum n^{-s}$, *which leads to a very convenient expression of the function* $\zeta(s)$". Then he had gotten (2) and (4). Perhaps, Riemann thought that RH cannot be proved by ζ. A. Selberg [8] pointed out, "*There have probably been very few attempts at proving the Riemann conjecture, because, simply, no one had ever had any really good idea for how to go about it*". We feel that studying ζ is a misguiding.

In order to prove RC, we have made the following innovations and contributions.

1) We have adopted entirely new method of research, Liuhui methodology (A.D.263): "computation can detect unknown" is a correct and reliable intuition, see [17].

2) Following Riemann's thought, we study $\xi(t)$, but not $\zeta(s)$. The entire research can be completed on the symmetric line $s_0 = 1/2 + it$ (or $z = t$).

3) Find a mistake of Riemann, he used the same notation $\xi(t)$ in theorems A and B. Therefore, Theorem B can only be used by contradiction.

4) We propose a general proof by contradiction, which consists of three steps:

a) Using all real roots and (true) complex roots of ξ to construct $\xi_1(t) = w(t)Q(t)$, this multiplicative group is the most suitable tool for studying "no complex roots".

b) Define the infinite point-set $L(\omega)$ for small $\omega > 0$. There always is $|w(t)/G(s_0)| = \omega$ on $L(\omega)$. If $\xi(t)$ has complex roots, then we have an important equality $\omega = \omega Q(t) > 0$.

c) No matter how the complex roots are distributed, in a large interval $[(3/4)R_1, R_1]$ we can find many points $t \in L(\omega)$ to make $Q(t) < 1/2$. This contradiction proves RC.

5) By $\xi(z) = G(s)\zeta(s)$, RH is also true, but it cannot be directly proved with ζ, which is not an entire function. There is an essential difference between using summation or multiplication.

6) Our work opens up a broad perspective, *i.e.*, we can propose

General RC. *For a broad class of even or odd entire function* $f(z)$, *all roots are real.*

For example, Sarnak [4] discussed $L(s,\chi)$ and Grand-RH (related to Goldbach conjecture), which has been continued to an even entire function. Our method is useful.

Acknowledgements

The author express sincere gratitude to the referee's for their valuable and constructive comments. Special thanks to Prof. Zhengtin Hou and Prof. Xinwen Jiang for their precious opinion in many discussions.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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