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Boundedness for Multilinear Operators of Pseudo-Differential Operators

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Abstract

In this paper, we establish a sharp function estimate for the multilinear integral operators associated to the pseudo-differential operators. As the application, we obtain the $L^p(1<p<\infty)$ norm inequalities for the multilinear operators.

Keywords

Multilinear Operator, Pseudo-Differential Operator, Sharp Function Estimate, BMO

1. Introduction and Results

Let b be a locally integrable function on R^n and T be an integral operator. For a suitable function f, the commutator generated by b and T is defined by $[b,T]f=bT(f)-T(bf)$. The investigation of the commutator begins with Coifman-Rochberg-Weiss pioneering study and classical result (see [1]). The major reason for considering the problem of commutators is that the boundedness of commutator can produce some characterizations of function spaces (see [1] [2]). Now, with the development of the Calderón-Zygmund singular integral operators, their commutators and multilinear operators have been well studied (see [1] [3]-[7]). In [8], Hu and Yang proved a variant sharp function estimate for the multilinear singular integral operators. In [9] [10] [11] [12], C. Pérez, G. Pradolini and R. Trujillo-Gonzalez obtained a sharp weighted estimate for the singular integral operators and their commutators. The boundedness of the pseudo-differential operators was studied by many authors (see [13]-[21]). In [15], the boundedness of the commutators associated to the pseudo-differential operators is obtained. The main purpose of this paper is to study the multilinear pseudo-differential operators as follows.

We say a symbol $\sigma(x,\xi)$ is in the class $S^m_{\rho,\delta}$ or $\sigma\in S^m_{\rho,\delta}$, if for $x,\xi\in R^n$,

$$\left|\frac{\partial^\alpha}{\partial x^\alpha}\frac{\partial^\beta}{\partial\xi^\beta}\sigma(x,\xi)\right|\le C_{\alpha,\beta}\left(1+|\xi|\right)^{m-\rho|\beta|+\delta|\alpha|}.$$

A pseudo-differential operator with symbol $\sigma(x,\xi)\in S^m_{\rho,\delta}$ is defined by

$$T(f)(x)=\int_{R^n}\mathrm{e}^{2\pi ix\cdot\xi}\sigma(x,\xi)\hat{f}(\xi)\mathrm{d}\xi,$$

where *f* is a Schwartz function and $\hat{f}$ denotes the Fourier transform of *f*. We know there exists a kernel $K(x,y)$ such that

$$T(f)(x)=\int_{R^n}K(x,x-y)f(y)\mathrm{d}y,$$

where, formally,

$$K(x,y)=\int_{R^n}\mathrm{e}^{2\pi i(x-y)\cdot\xi}\sigma(x,\xi)\mathrm{d}\xi.$$

In [14], the boundedness of the pseudo-differential operators with symbol $\sigma\in S^{-\beta}_{1-\theta,\delta}\left(\beta<n\theta/2, 0\le\delta<1-\theta\right)$ is obtained. In [14], the boundedness of the pseudo-differential operators with symbol of order 0 and $-\infty$ is obtained. In [17], the sharp function estimate of the pseudo-differential operators with symbol $\sigma\in S^{-n\theta/2}_{1-\theta,\delta}\left(0<\theta<1, 0\le\delta<1-\theta\right)$ is obtained. In [15], the boundedness of the pseudo-differential operators and their commutators with symbol $\sigma\in S^{-n\theta/2}_{1-\theta,\delta}\left(0<\theta<1, 0\le\delta<1-\theta\right)$ is obtained. Our results are motivated by these papers.

Suppose T is a pseudo-differential operator with symbol $\sigma(x,\xi)\in S^m_{\rho,\delta}$. Let m_j be the positive integers ($j=1,\cdots,l$), $m_1+\cdots+m_l=L$ and b_j be the functions on R^n ($j=1,\cdots,l$). Set, for $1\le j\le m$,

$$R_{m_j+1}(b_j;x,y)=b_j(x)-\sum_{|\alpha|\le m_j}\frac{1}{\alpha!}D^\alpha b_j(y)(x-y)^\alpha.$$

The multilinear operator associated to *T* is defined by

$$T_b(f)(x)=\int_{R^n}\frac{\prod_{j=1}^{l}R_{m_j+1}(b_j;x,y)}{|x-y|^L}K(x,x-y)f(y)\mathrm{d}y.$$

Note that when $L=0$, T_b is just the multilinear commutator of T and b_j (see [11]). While when $L>0$, T_b is non-trivial generalizations of the commutator. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors. Hu and Yang (see [8]) proved a variant sharp estimate for the multilinear singular integral operators. In [11], Pérez and Trujillo-Gonzalez prove a sharp estimate for the multilinear commutator when $b_j\in Osc_{expL^{r_j}}\left(R^n\right)$. The main purpose of this paper is to prove a sharp function inequality for the multilinear operators associated to the pseudo-differential operators with symbol $\sigma\in S^{-n\theta/2}_{1-\theta,\delta}\left(0<\theta<1, 0\le\delta<1-\theta\right)$ when $D^\alpha b_j\in BMO\left(R^n\right)$ for all α with $|\alpha|=m_j$. As the application, we obtain the $L^p(p>1)$ norm inequality for the multilinear operators.

First, let us introduce some notations. Throughout this paper, $Q=Q(x,d)$ will denote a cube of R^n with sides parallel to the axes, whose center is *x* and side length is d. For a locally integrable function b, the sharp function of b is defined by

$$b^{\#}(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q\left|b(y)-b_Q\right|\mathrm{d}y,$$

where, and in what follows, $b_Q=|Q|^{-1}\int_Q b(x)\mathrm{d}x$. It is well-known that (see [22] [23] [24])

$$b^{\#}(x)\approx\sup_{Q\ni x}\inf_{c\in C}\frac{1}{|Q|}\int_Q\left|b(y)-c\right|\mathrm{d}y$$

and

$$\left\|b-b_{2^kQ}\right\|_{BMO}\le Ck\|b\|_{BMO}\ \text{ for } k\ge 1.$$

We say that b belongs to $BMO(R^n)$ if $b^{\#}$ belongs to $L^{\infty}(R^n)$ and $\|b\|_{BMO}=\|b^{\#}\|_{L^{\infty}}$. Let M be the Hardy-Littlewood maximal operator defined by

$$M(f)(x)=\sup_{x\in Q}\frac{1}{|Q|}\int_Q|f(y)|\mathrm{d}y,$$

we write that $M_p(f)=\left(M\left(f^p\right)\right)^{1/p}$ for $0<p<\infty$.

We shall prove the following theorems.

Theorem 1. Suppose T is the pseudo-differential operator with symbol $\sigma\in S^{-n\theta/2}_{1-\theta,\delta}\left(0<\theta<1,0\le\delta<1-\theta\right)$. Let $D^{\alpha}b_j\in BMO(R^n)$ for all α with $|\alpha|=m_j$ and $j=1,\cdots,l$. Then there exists a constant $C>0$ such that for every $f\in C_0^{\infty}(R^n)$, $2<r<\infty$ and $\tilde{x}\in R^n$,

$$\left(T_b(f)\right)^{\#}(\tilde{x})\le C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}\left\|D^{\alpha_j}b_j\right\|_{BMO}\right)M_r(f)(\tilde{x}).$$

Theorem 2. Suppose T is the pseudo-differential operator with symbol $\sigma\in S^{-n\theta/2}_{1-\theta,\delta}\left(0<\theta<1,0\le\delta<1-\theta\right)$. Let $D^{\alpha}b_j\in BMO(R^n)$ for all α with $|\alpha|=m_j$ and $j=1,\cdots,l$.

1) If $w\in A_{\infty}$ and $2<p<\infty$, then

$$\left\|T_b(f)\right\|_{L^p(w)}\le C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}\left\|D^{\alpha_j}b_j\right\|_{BMO}\right)\left\|M_r(f)\right\|_{L^p(w)}.$$

2) If $w\in A_1$ and $2<p<\infty$, then

$$\left\|T_b(f)\right\|_{L^p(w)}\le C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}\left\|D^{\alpha_j}b_j\right\|_{BMO}\right)\|f\|_{L^p(w)}.$$

2. Proofs of Theorems

To prove the theorems, we need the following lemmas.

Lemma 1. (see [4]) Let b be a function on R^n and $D^{\alpha}b\in L^q(R^n)$ for all α with $|\alpha|=L$ and some $q>n$. Then

$$\left|R_L(b;x,y)\right|\le C|x-y|^L\sum_{|\alpha|=L}\left(\frac{1}{\left|\tilde{Q}(x,y)\right|}\int_{\tilde{Q}(x,y)}\left|D^{\alpha}b(z)\right|^q\mathrm{d}z\right)^{1/q},$$

where $\tilde{Q}$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

Lemma 2. (see [13]) Let T be the pseudo-differential operator with symbol $\sigma \in S^{-n\theta/2}_{1-\theta,\delta}(0<\theta<1, 0\le\delta<1-\theta)$. Then, for every $f\in L^p(R^n), 1<p<\infty$,

$$\|T(f)\|_{L^p} \le C\|f\|_{L^p}.$$

Lemma 3. (see [13]) Let $\sigma \in S^{-n\theta/2}_{1-\theta,\delta}(0<\theta<1, 0\le\delta<1-\theta)$ and K be the kernel of the pseudo-differential operator T with symbol $\sigma \in S^{-n\theta/2}_{1-\theta,\delta}$. Then, for $|x_0-x|\le d<1$ and $k\ge 1$,

$$\left(\int_{(2^k d)^{1-\theta}\le|y-x_0|<(2^{k+1}d)^{1-\theta}} |K(x,x-y)-K(x_0,x_0-y)|^2 \mathrm{d}y\right)^{1/2} \le C\frac{|x_0-x|^{(1-\theta)(m-n/2)}}{(2^k d)^{m(1-\theta)}},$$

provided m is an integer such that $n/2<m<n/2+1/(1-\theta)$.

Lemma 4. (see [13]) Let $\sigma\in S^0_{\rho,\delta}(0<\rho<1)$ and

$$K(x,w)=\int_{R^n} \mathrm{e}^{2\pi i w\cdot\xi}\sigma(x,\xi)\mathrm{d}\xi.$$

Then, for $|w|\ge 1/4$ and any integer $N\ge 1$,

$$|K(x,w)|\le C_N|w|^{-2N}.$$

Proof of Theorem 1. It suffices to prove for $f\in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|}\int_Q |T_b(f)(x)-C_0|\mathrm{d}x \le C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)M_r(f)(x).$$

Without loss of generality, we may assume $l=2$. Fix a cube $Q=Q(x_0,d)$ and $\tilde{x}\in Q$. We consider the following two cases:

Case 1. $d\le 1$. In this case, let Q^* be the cube concentric with Q of side length $d^{1-\theta}$. Let $\tilde{Q}=5\sqrt{n}Q^*$ and $\tilde{b}_j(x)=b_j(x)-\sum_{|\alpha|=m_j}\frac{1}{\alpha!}(D^\alpha b_j)_{\tilde{Q}}x^\alpha$, then $R_{m_j}(b_j;x,y)=R_{m_j}(\tilde{b}_j;x,y)$ and $D^\alpha\tilde{b}_j=D^\alpha b_j-(D^\alpha b_j)_{\tilde{Q}}$ for $|\alpha|=m_j$. We write, for $f=f\chi_{\tilde{Q}}+f\chi_{R^n\setminus\tilde{Q}}=f_1+f_2$,

$$T_b(f)(x)=\int_{R^n}\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j;x,y)}{|x-y|^L}K(x,x-y)f(y)\mathrm{d}y$$

$$=\int_{R^n}\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,y)}{|x-y|^L}K(x,x-y)f_1(y)\mathrm{d}y$$

$$-\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{R^n}\frac{R_{m_2}(\tilde{b}_2;x,y)(x-y)^{\alpha_1}D^{\alpha_1}\tilde{b}_1(y)}{|x-y|^L}K(x,x-y)f_1(y)\mathrm{d}y$$

$$-\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{R^n}\frac{R_{m_1}(\tilde{b}_1;x,y)(x-y)^{\alpha_2}D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^L}K(x,x-y)f_1(y)\mathrm{d}y$$

$$+\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{b}_1(y)D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^L}K(x,x-y)f_1(y)\mathrm{d}y$$

$$+\int_{R^n}\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j;x,y)}{|x-y|^L}K(x,x-y)f_2(y)\mathrm{d}y,$$

then

$$\frac{1}{|Q|}\int_Q\left|T_b(f)(x)-T_{\tilde b}(f_2)(x_0)\right|dx$$

$$\le\frac{1}{|Q|}\int_Q\left|\int_{R^n}\frac{\prod_{j=1}^2 R_{m_j}(\tilde b_j;x,y)}{|x-y|^L}K(x,x-y)f_1(y)dy\right|dx$$

$$+\frac{C}{|Q|}\int_Q\left|\sum_{|\alpha_1|=m_1}\int_{R^n}\frac{R_{m_2}(\tilde b_2;x,y)(x-y)^{\alpha_1}}{|x-y|^L}D^{\alpha_1}\tilde b_1(y)K(x,x-y)f_1(y)dy\right|dx$$

$$+\frac{C}{|Q|}\int_Q\left|\sum_{|\alpha_2|=m_2}\int_{R^n}\frac{R_{m_1}(\tilde b_1;x,y)(x-y)^{\alpha_2}}{|x-y|^L}D^{\alpha_2}\tilde b_2(y)K(x,x-y)f_1(y)dy\right|dx$$

$$+\frac{C}{|Q|}\int_Q\left|\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde b_1(y)D^{\alpha_2}\tilde b_2(y)}{|x-y|^L}K(x,x-y)f_1(y)dy\right|dx$$

$$+\frac{1}{|Q|}\int_Q\left|T_{\tilde b}(f_2)(x)-T_{\tilde b}(f_2)(x_0)\right|dx$$

$$:=I_1+I_2+I_3+I_4+I_5.$$

Now, let us estimate I_1, I_2, I_3, I_4 and I_5, respectively. First, for $x\in Q$ and $y\in\tilde Q$, by Lemma 1, we get

$$R_L(\tilde b_j;x,y)\le C|x-y|^L\sum_{|\alpha_j|=L}\|D^{\alpha_j}b_j\|_{BMO}.$$

Now, let $\sigma(x,\xi)=\sigma(x,\xi)|\xi|^{n\theta/2}|\xi|^{-n\theta/2}=q(x,\xi)|\xi|^{-n\theta/2}$, we have $q(x,\xi)\in S^0_{1-\theta,\delta}$, set S be the pseudo-differential operator with symbol $q(x,\xi)$, by the Hardy-Littlewood-Soboleve fractional integration theorem and the L^2-boundedness of S (see [13]), we obtain, for $1/p=1/2-\theta/2$,

$$I_1\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)\frac{1}{|Q|}\int_Q|T(f_1)(x)|dx$$

$$\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)\left(\frac{1}{|Q|}\int_Q|T(f_1)(x)|^p dx\right)^{1/p}$$

$$\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)|Q|^{-1/p}\left(\int_{R^n}|S(f_1)(x)|^2dx\right)^{1/2}$$

$$\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)|Q|^{-1/p}\left(\int_{R^n}|f_1(x)|^2dx\right)^{1/2}$$

$$\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)\frac{|\tilde Q|^{1/2}}{|Q|^{1/p}}\left(\frac{1}{|\tilde Q|}\int_{\tilde Q}|f(x)|^2dx\right)^{1/2}$$

$$\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)\left(\frac{1}{|\tilde Q|}\int_{\tilde Q}|f(x)|^r dx\right)^{1/r}$$

$$\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}b_j\|_{BMO}\right)M_r(f)(\tilde x).$$

For I_2, by Lemma 1 and Hölder's inequality, we get, for $1/r + 1/r' = 1/2$,

$$I_2 \le C \sum_{|\alpha_2|=m_2} \left\| D^{\alpha_2} b_2 \right\|_{BMO} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|Q|} \int_Q \left| T\left(D^{\alpha_1} \tilde{b}_1 f_1 \right)(x) \right|^p \mathrm{d}x \right)^{1/p}$$

$$\le C \sum_{|\alpha_2|=m_2} \left\| D^{\alpha_2} b_2 \right\|_{BMO} \sum_{|\alpha_1|=m_1} |Q|^{-1/p} \left(\int_{R^n} \left| S\left(D^{\alpha_1} \tilde{b}_1 f_1 \right)(x) \right|^2 \mathrm{d}x \right)^{1/2}$$

$$\le C \sum_{|\alpha_2|=m_2} \left\| D^{\alpha_2} b_2 \right\|_{BMO} \sum_{|\alpha_1|=m_1} |Q|^{-1/p} \left(\int_{R^n} \left| D^{\alpha_1} \tilde{b}_1 (x) f_1 (x) \right|^2 \mathrm{d}x \right)^{1/2}$$

$$\le C \sum_{|\alpha_2|=m_2} \left\| D^{\alpha_2} b_2 \right\|_{BMO} \frac{|\tilde{Q}|^{1/2}}{|Q|^{1/p}} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} \left| D^{\alpha_1} b_1 (x) - \left(D^{\alpha} b_j \right)_{\tilde{Q}} \right|^{r'} \mathrm{d}x \right)^{1/r'}$$

$$\times \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} \left| f(x) \right|^r \mathrm{d}x \right)^{1/r}$$

$$\le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \left\| D^{\alpha} b_j \right\|_{BMO} \right) M_r(f)(\tilde{x}).$$

For I_3, similar to the proof of I_2, we get

$$I_3 \le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \left\| D^{\alpha} b_j \right\|_{BMO} \right) M_r(f)(\tilde{x}).$$

Similarly, for I_4, taking $r_1, r_2 > 1$ such that $1/r + 1/r_1 + 1/r_2 = 1/2$, we obtain

$$I_4 \le C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \left(\frac{1}{|Q|} \int_Q \left| T\left(D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} \tilde{b}_2 f_1 \right)(x) \right|^p \mathrm{d}x \right)^{1/p}$$

$$\le C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} |Q|^{-1/p} \left(\int_{R^n} \left| S\left(D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} \tilde{b}_2 f_1 \right)(x) \right|^2 \mathrm{d}x \right)^{1/2}$$

$$\le C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} |Q|^{-1/p} \left(\int_{R^n} \left| D^{\alpha_1} \tilde{b}_1 (x) D^{\alpha_2} \tilde{b}_2 (x) f_1 (x) \right|^2 \mathrm{d}x \right)^{1/2}$$

$$\le C \frac{|\tilde{Q}|^{1/2}}{|Q|^{1/p}} \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \prod_{j=1}^{2} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} \left| D^{\alpha_j} \tilde{b}_j (x) \right|^{r_j} \mathrm{d}x \right)^{1/r_j} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} \left| f(x) \right|^r \mathrm{d}x \right)^{1/r}$$

$$\le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \left\| D^{\alpha} b_j \right\|_{BMO} \right) M_r(f)(\tilde{x}).$$

For I_5, we write

$$T_{\tilde{b}}(f_2)(x) - T_{\tilde{b}}(f_2)(x_0)$$

$$= \int_{R^n} \left(\frac{K(x, x-y)}{|x-y|^L} - \frac{K(x_0, x_0-y)}{|x_0-y|^L} \right) \prod_{j=1}^{2} R_{m_j}\left(\tilde{b}_j; x, y \right) f_2(y) \mathrm{d}y$$

$$+ \int_{R^n} \left(R_{m_1}\left(\tilde{b}_1; x, y \right) - R_{m_1}\left(\tilde{b}_1; x_0, y \right) \right) \frac{R_{m_2}\left(\tilde{b}_2; x, y \right)}{|x_0-y|^L} K(x_0, x_0-y) f_2(y) \mathrm{d}y$$

$$+ \int_{R^n} \left(R_{m_2}\left(\tilde{b}_2; x, y \right) - R_{m_2}\left(\tilde{b}_2; x_0, y \right) \right) \frac{R_{m_1}\left(\tilde{b}_1; x_0, y \right)}{|x_0-y|^L} K(x_0, x_0-y) f_2(y) \mathrm{d}y$$

$$
\begin{aligned}
&-\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{R^n}\left[\frac{R_{m_2}\left(\tilde{b}_2;x,y\right)(x-y)^{\alpha_1}}{|x-y|^L}K(x,x-y)\right.\\
&\left.-\frac{R_{m_2}\left(\tilde{b}_2;x_0,y\right)(x_0-y)^{\alpha_1}}{|x_0-y|^L}K(x_0,x_0-y)\right]D^{\alpha_1}\tilde{b}_1(y)f_2(y)\mathrm{d}y\\
&-\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{R^n}\left[\frac{R_{m_1}\left(\tilde{b}_1;x,y\right)(x-y)^{\alpha_2}}{|x-y|^L}K(x,x-y)\right.\\
&\left.-\frac{R_{m_1}\left(\tilde{b}_1;x_0,y\right)(x_0-y)^{\alpha_2}}{|x_0-y|^L}K(x_0,x_0-y)\right]D^{\alpha_2}\tilde{b}_2(y)f_2(y)\mathrm{d}y\\
&+\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\int_{R^n}\left[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^L}K(x,x-y)\right.\\
&\left.-\frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^L}K(x_0,x_0-y)\right]D^{\alpha_1}\tilde{b}_1(y)D^{\alpha_2}\tilde{b}_2(y)f_2(y)\mathrm{d}y\\
&=I_5^{(1)}+I_5^{(2)}+I_5^{(3)}+I_5^{(4)}+I_5^{(5)}+I_5^{(6)}.
\end{aligned}
$$

By Lemma 1 and the following inequality (see [24])

$$
\left|b_{Q_1}-b_{Q_2}\right|\le C\log\left(|Q_2|/|Q_1|\right)\|b\|_{BMO}\quad\text{for } Q_1\subset Q_2,
$$

we know that, for $x\in Q$ and $y\in Q\left(x_0,\left(2^{k+1}d\right)^{1-\theta}\right)\setminus Q\left(x_0,\left(2^k d\right)^{1-\theta}\right)$,

$$
\begin{aligned}
\left|R_L\left(\tilde{b};x,y\right)\right|&\le C|x-y|^L\sum_{|\alpha|=L}\left(\left\|D^\alpha b\right\|_{BMO}+\left|\left(D^\alpha b\right)_{\tilde{Q}(x,y)}-\left(D^\alpha b\right)_{\tilde{Q}}\right|\right)\\
&\le Ck|x-y|^L\sum_{|\alpha|=L}\left\|D^\alpha b\right\|_{BMO}.
\end{aligned}
$$

Note that $|x-y|\sim|x_0-y|$ for $x\in\tilde{Q}$ and $y\in R^n\setminus\tilde{Q}$, we obtain

$$
\begin{aligned}
\left|I_5^{(1)}\right|&\le\sum_{k=0}^{\infty}k^2\int_{\left(2^k d\right)^{1-\theta}\le|y-x_0|<\left(2^{k+1}d\right)^{1-\theta}}\left|K(x,x-y)-K(x_0,x_0-y)\right|\\
&\quad\times\frac{1}{|x-y|^m}\prod_{j=1}^{2}\left|R_{m_j}\left(\tilde{b}_j;x,y\right)\right|\left|f(y)\right|\mathrm{d}y\\
&\quad+\sum_{k=0}^{\infty}k^2\int_{\left(2^k d\right)^{1-\theta}\le|y-x_0|<\left(2^{k+1}d\right)^{1-\theta}}\left|\frac{1}{|x-y|^L}-\frac{1}{|x_0-y|^L}\right|\\
&\quad\times\left|K(x_0,x_0-y)\right|\prod_{j=1}^{2}\left|R_{m_j}\left(\tilde{b}_j;x,y\right)\right|\left|f(y)\right|\mathrm{d}y\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\left\|D^\alpha b_j\right\|_{BMO}\right)\sum_{k=0}^{\infty}k^2\left(\int_{|y-x_0|<\left(2^{k+1}d\right)^{1-\theta}}\left|f(y)\right|^2\mathrm{d}y\right)^{1/2}\\
&\quad\times\left(\int_{\left(2^k d\right)^{1-\theta}\le|y-x_0|<\left(2^{k+1}d\right)^{1-\theta}}\left|K(x,x-y)-K(x_0,x_0-y)\right|^2\mathrm{d}y\right)^{1/2}\\
&\quad+C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\left\|D^\alpha b_j\right\|_{BMO}\right)\sum_{k=0}^{\infty}k^2\left(\int_{|y-x_0|<\left(2^{k+1}d\right)^{1-\theta}}\left|f(y)\right|^2\mathrm{d}y\right)^{1/2}\\
&\quad\times\left(\int_{\left(2^k d\right)^{1-\theta}\le|y-x_0|<\left(2^{k+1}d\right)^{1-\theta}}\frac{|x_0-x|^2}{|x_0-y|^2}\left|K(x_0,x_0-y)\right|^2\mathrm{d}y\right)^{1/2},
\end{aligned}
$$

for the second term above, similar to the proof of Lemma 2.1 in [13], we have

$$\left(\int_{(2^k d)^{1-\theta}\le|y-x_0|<(2^{k+1}d)^{1-\theta}} \frac{|x_0-x|^2}{|x_0-y|^2}\left|K(x_0,x_0-y)\right|^2 \mathrm{d}y\right)^{1/2} \le C\frac{|x_0-x|^{(1-\theta)(m-n/2)}}{(2^k d)^{m(1-\theta)}},$$

thus, by Lemma 3 and recall that $n/2<m$,

$$\begin{aligned}\left|I_5^{(1)}\right| &\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\left\|D^\alpha b_j\right\|_{BMO}\right)\sum_{k=0}^{\infty}k^2\frac{d^{(1-\theta)(m-n/2)}}{(2^k d)^{m(1-\theta)}}\left(\int_{|y-x_0|<(2^{k+1}d)^{1-\theta}}\left|f(y)\right|^2\mathrm{d}y\right)^{1/2}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\left\|D^\alpha b_j\right\|_{BMO}\right)\sum_{k=1}^{\infty}k^2 2^{k(1-\theta)(n/2-m)}\left(\frac{1}{\left|Q\left(x_0,(2^k d)^{1-\theta}\right)\right|}\int_{Q\left(x_0,(2^k d)^{1-\theta}\right)}\left|f(y)\right|^r\mathrm{d}y\right)^{1/r}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\left\|D^\alpha b_j\right\|_{BMO}\right)\sum_{k=1}^{\infty}k^2 2^{k(1-\theta)(n/2-m)}M_r(f)(\tilde{x})\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\left\|D^\alpha b_j\right\|_{BMO}\right)M_r(f)(\tilde{x}).\end{aligned}$$

For $I_5^{(2)}$, by the formula (see [4]):

$$R_L(\tilde{b};x,y)-R_L(\tilde{b};x_0,y)=\sum_{|\beta|<L}\frac{1}{\beta!}R_{L-|\beta|}\left(D^\beta\tilde{b};x,x_0\right)(x-y)^\beta$$

and Lemma 1, we have

$$\left|R_L(\tilde{b};x,y)-R_L(\tilde{b};x_0,y)\right|\le C\sum_{|\beta|<L}\sum_{|\alpha|=L}|x-x_0|^{L-|\beta|}|x-y|^{|\beta|}\left\|D^\alpha b\right\|_{BMO},$$

thus

$$\begin{aligned}\left|I_5^{(2)}\right| &\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\left\|D^\alpha b_j\right\|_{BMO}\right)\sum_{k=0}^{\infty}\int_{(2^k d)^{1-\theta}\le|y-x_0|<(2^{k+1}d)^{1-\theta}}k\frac{|x-x_0|}{|x_0-y|}\left|K(x_0,x_0-y)\right|\left|f(y)\right|\mathrm{d}y\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\left\|D^\alpha b_j\right\|_{BMO}\right)\sum_{k=1}^{\infty}k 2^{k(1-\theta)(n/2-m)}\left(\frac{1}{\left|Q\left(x_0,(2^k d)^{1-\theta}\right)\right|}\int_{Q\left(x_0,(2^k d)^{1-\theta}\right)}\left|f(y)\right|^r\mathrm{d}y\right)^{1/r}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\left\|D^\alpha b_j\right\|_{BMO}\right)M_r(f)(\tilde{x}).\end{aligned}$$

Similarly,

$$\left|I_5^{(3)}\right|\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\left\|D^\alpha b_j\right\|_{BMO}\right)M_r(f)(\tilde{x}).$$

For $I_5^{(4)}$, recall that $\left|b_{2^kQ}-b_{2Q}\right|\le Ck\|b\|_{BMO}$, similar to the proofs of $I_5^{(1)}$ and $I_5^{(2)}$, we get, for $1/r+1/r'=1/2$

$$\left|I_5^{(4)}\right| \le C\sum_{|\alpha_1|=m_1}\int_{R^n}\left|\frac{(x-y)^{\alpha_1}}{|x-y|^L}-\frac{(x_0-y)^{\alpha_1}}{|x_0-y|^L}\right|\left|R_{m_2}\left(\tilde{b}_2;x,y\right)\right|\left|K(x,x-y)\right|\left|D^{\alpha_1}\tilde{b}_1(y)\right|\left|f_2(y)\right|\mathrm{d}y$$

$$+C\sum_{|\alpha_1|=m_1}\int_{R^n}\left|R_{m_2}\left(\tilde{b}_2;x,y\right)-R_{m_2}\left(\tilde{b}_2;x_0,y\right)\right|\frac{\left|(x_0-y)^{\alpha_1}\right|}{|x_0-y|^L}\left|K(x,x-y)\right|\left|D^{\alpha_1}\tilde{b}_1(y)\right|\left|f_2(y)\right|\mathrm{d}y$$

$$+C\sum_{|\alpha_1|=m_1}\int_{R^n}\left|K(x,x-y)-K(x_0,x_0-y)\right|\frac{\left|(x_0-y)^{\alpha_1}\right|}{|x_0-y|^L}\left|R_{m_2}\left(\tilde{b}_2;x_0,y\right)\right|\left|D^{\alpha_1}\tilde{b}_1(y)\right|\left|f_2(y)\right|\mathrm{d}y$$

$$\le C\sum_{|\alpha|=m_2}\left\|D^\alpha b_2\right\|_{BMO}\sum_{|\alpha_1|=m_1}\sum_{k=1}^{\infty}k2^{k(1-\theta)(n/2-m)}$$

$$\times\left(\frac{1}{\left|Q\left(x_0,\left(2^k d\right)^{1-\theta}\right)\right|}\int_{Q\left(x_0,\left(2^k d\right)^{1-\theta}\right)}\left|f(y)D^{\alpha_1}\tilde{b}_1(y)\right|^2\mathrm{d}y\right)^{1/2}$$

$$\le C\sum_{|\alpha|=m_2}\left\|D^\alpha b_2\right\|_{BMO}\sum_{k=1}^{\infty}k2^{k(1-\theta)(n/2-m)}\left(\frac{1}{\left|Q\left(x_0,\left(2^k d\right)^{1-\theta}\right)\right|}\int_{Q\left(x_0,\left(2^k d\right)^{1-\theta}\right)}\left|f(y)\right|^r\mathrm{d}y\right)^{1/r}$$

$$\times\sum_{|\alpha_1|=m_1}\left(\frac{1}{\left|Q\left(x_0,\left(2^k d\right)^{1-\theta}\right)\right|}\int_{Q\left(x_0,\left(2^k d\right)^{1-\theta}\right)}\left|D^{\alpha_1}b_1(y)-\left(D^{\alpha_1}b_1\right)_{\tilde{Q}}\right|^{r'}\mathrm{d}y\right)^{1/r'}$$

$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\left\|D^\alpha b_j\right\|_{BMO}\right)\sum_{k=1}^{\infty}k^2 2^{k(1-\theta)(n/2-m)}M_r(f)(\tilde{x})$$

$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\left\|D^\alpha b_j\right\|_{BMO}\right)M_r(f)(\tilde{x}).$$

Similarly,

$$\left|I_5^{(5)}\right|\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\left\|D^\alpha b_j\right\|_{BMO}\right)M_r(f)(\tilde{x}).$$

For $I_5^{(6)}$, similar to the proof of $I_5^{(1)}$, we get, for $1/r+1/r_1+1/r_2=1/2$,

$$\left|I_5^{(6)}\right|\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\int_{R^n}\left|\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^L}-\frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^L}\right|$$

$$\times\left|K(x,x-y)\right|\left|D^{\alpha_1}\tilde{b}_1(y)\right|\left|D^{\alpha_2}\tilde{b}_2(y)\right|\left|f_2(y)\right|\mathrm{d}y$$

$$+C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\int_{R^n}\left|K(x,x-y)-K(x_0,x_0-y)\right|\frac{\left|(x_0-y)^{\alpha_1+\alpha_2}\right|}{|x_0-y|^L}$$

$$\times\left|D^{\alpha_1}\tilde{b}_1(y)\right|\left|D^{\alpha_2}\tilde{b}_2(y)\right|\left|f_2(y)\right|\mathrm{d}y$$

$$\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\sum_{k=1}^{\infty}2^{k(1-\theta)(n/2-m)}$$

$$\times\left(\frac{1}{\left|Q\left(x_0,\left(2^k d\right)^{1-\theta}\right)\right|}\int_{Q\left(x_0,\left(2^k d\right)^{1-\theta}\right)}\left|f(y)D^{\alpha_1}\tilde{b}_1(y)D^{\alpha_2}\tilde{b}_2(y)\right|^2\mathrm{d}y\right)^{1/2}$$

$$\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\sum_{k=1}^{\infty}2^{k(1-\theta)(n/2-m)}\left(\frac{1}{\left|Q\left(x_0,\left(2^k d\right)^{1-\theta}\right)\right|}\int_{Q\left(x_0,\left(2^k d\right)^{1-\theta}\right)}|f(y)|^r\,\mathrm{d}y\right)^{1/r}$$

$$\times\prod_{j=1}^{2}\left(\frac{1}{\left|Q\left(x_0,\left(2^k d\right)^{1-\theta}\right)\right|}\int_{Q\left(x_0,\left(2^k d\right)^{1-\theta}\right)}\left|D^{\alpha_j}b_j(y)-\left(D^{\alpha_j}b_j\right)_{\tilde{Q}}\right|^{r_j}\mathrm{d}y\right)^{1/r_j}$$

$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\left\|D^{\alpha}b_j\right\|_{BMO}\right)M_r(f)(\tilde{x}).$$

Thus

$$|I_5|\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\left\|D^{\alpha}b_j\right\|_{BMO}\right)M_r(f)(\tilde{x}).$$

Case 2. $d>1$. In this case, let $\tilde{Q}=5\sqrt{n}Q$ and $\tilde{b}_j(x)=b_j(x)-\sum_{|\alpha|=m_j}\frac{1}{\alpha!}\left(D^{\alpha}b_j\right)_{\tilde{Q}}x^{\alpha}$, then $R_{m_j}(b_j;x,y)=R_{m_j}(\tilde{b}_j;x,y)$ and $D^{\alpha}\tilde{b}_j=D^{\alpha}b_j-\left(D^{\alpha}b_j\right)_{\tilde{Q}}$ for $|\alpha|=m_j$. Write, for $f=f\chi_{\tilde{Q}}+f\chi_{R^n\setminus\tilde{Q}}=f_1+f_2$,

$$\frac{1}{|Q|}\int_Q|T_b(f)(x)|\,\mathrm{d}x$$

$$\le\frac{1}{|Q|}\int_Q\left|\int_{R^n}\frac{\prod_{j=1}^{2}R_{m_j}(\tilde{b}_j;x,y)}{|x-y|^L}K(x,x-y)f_1(y)\,\mathrm{d}y\right|\mathrm{d}x$$

$$+\frac{C}{|Q|}\int_Q\left|\sum_{|\alpha_1|=m_1}\int_{R^n}\frac{R_{m_2}(\tilde{b}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^L}D^{\alpha_1}\tilde{b}_1(y)K(x,x-y)f_1(y)\,\mathrm{d}y\right|\mathrm{d}x$$

$$+\frac{C}{|Q|}\int_Q\left|\sum_{|\alpha_2|=m_2}\int_{R^n}\frac{R_{m_1}(\tilde{b}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^L}D^{\alpha_2}\tilde{b}_2(y)K(x,x-y)f_1(y)\,\mathrm{d}y\right|\mathrm{d}x$$

$$+\frac{C}{|Q|}\int_Q\left|\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{b}_1(y)D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^L}K(x,x-y)f_1(y)\,\mathrm{d}y\right|\mathrm{d}x$$

$$+\frac{1}{|Q|}\int_Q\left|T_{\tilde{b}}(f_2)(x)\right|\mathrm{d}x$$

$$:=J_1+J_2+J_3+J_4+J_5.$$

Similar to the proof of I_1, I_2, I_3 and I_4, we get, by the $L^p(1<p<\infty)$-boundedness of T (see Lemma 2),

$$J_1\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\left\|D^{\alpha_j}b_j\right\|_{BMO}\right)\left(\frac{1}{|Q|}\int_{R^n}|T(f_1)(x)|^r\,\mathrm{d}x\right)^{1/r}$$

$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\left\|D^{\alpha_j}b_j\right\|_{BMO}\right)|Q|^{-1/r}\left(\int_{R^n}|f_1(x)|^r\,\mathrm{d}x\right)^{1/r}$$

$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\left\|D^{\alpha_j}b_j\right\|_{BMO}\right)\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|f(x)|^r\,\mathrm{d}x\right)^{1/r}$$

$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\left\|D^{\alpha_j}b_j\right\|_{BMO}\right)M_r(f)(\tilde{x});$$

$$J_2 \le C\sum_{|\alpha_2|=m_2}\left\|D^{\alpha_2}b_2\right\|_{BMO}\sum_{|\alpha_1|=m_1}\left(\frac{1}{|Q|}\int_{R^n}\left|T\left(D^{\alpha_1}\tilde{b}_1 f_1\right)(x)\right|^2\mathrm{d}x\right)^{1/2}$$

$$\le C\sum_{|\alpha_2|=m_2}\left\|D^{\alpha_2}b_2\right\|_{BMO}\sum_{|\alpha_1|=m_1}|Q|^{-1/2}\left(\int_{R^n}\left|D^{\alpha_1}\tilde{b}_1(x)f_1(x)\right|^2\mathrm{d}x\right)^{1/2}$$

$$\le C\sum_{|\alpha_2|=m_2}\left\|D^{\alpha_2}b_2\right\|_{BMO}\sum_{|\alpha_1|=m_1}\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}\left|D^{\alpha_1}b_1(x)-\left(D^{\alpha}b_1\right)_{\tilde{Q}}\right|^{r'}\mathrm{d}x\right)^{1/r'}$$

$$\times\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|f(x)|^r\mathrm{d}x\right)^{1/r}$$

$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\left\|D^{\alpha}b_j\right\|_{BMO}\right)M_r(f)(\tilde{x});$$

$$J_3 \le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\left\|D^{\alpha}b_j\right\|_{BMO}\right)M_r(f)(\tilde{x});$$

$$J_4 \le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}|Q|^{-1/2}\left(\int_{R^n}\left|T\left(D^{\alpha_1}\tilde{b}_1 D^{\alpha_2}\tilde{b}_2 f_1\right)(x)\right|^2\mathrm{d}x\right)^{1/2}$$

$$\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}|Q|^{-1/2}\left(\int_{R^n}\left|D^{\alpha_1}\tilde{b}_1(x)D^{\alpha_2}\tilde{b}_2(x)f_1(x)\right|^2\mathrm{d}x\right)^{1/2}$$

$$\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\prod_{j=1}^{2}\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}\left|D^{\alpha_j}\tilde{b}_j(x)\right|^{r_j}\mathrm{d}x\right)^{1/r_j}\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|f(x)|^r\mathrm{d}x\right)^{1/r}$$

$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\left\|D^{\alpha}b_j\right\|_{BMO}\right)M_r(f)(\tilde{x}).$$

For J_5, we write

$$T_{\tilde{b}}(f_2)(x)=\int_{R^n}\frac{\prod_{j=1}^{2}R_{m_j}\left(\tilde{b}_j;x,y\right)}{|x-y|^L}K(x,x-y)f_2(y)\mathrm{d}y$$

$$-\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{R^n}\frac{R_{m_2}\left(\tilde{b}_2;x,y\right)(x-y)^{\alpha_1}}{|x-y|^L}K(x,x-y)D^{\alpha_1}\tilde{b}_1(y)f_2(y)\mathrm{d}y$$

$$-\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{R^n}\frac{R_{m_1}\left(\tilde{b}_1;x,y\right)(x-y)^{\alpha_2}}{|x-y|^L}K(x,x-y)D^{\alpha_2}\tilde{b}_2(y)f_2(y)\mathrm{d}y$$

$$+\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^L}K(x,x-y)D^{\alpha_1}\tilde{b}_1(y)D^{\alpha_2}\tilde{b}_2(y)f_2(y)\mathrm{d}y,$$

similar to the proof of I_5 and by using lemma 4, we get

$$\left|T_{\tilde{b}}(f_2)(x)\right|\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\left\|D^{\alpha}b_j\right\|_{BMO}\right)\sum_{k=0}^{\infty}k^2\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}|x-y|^{-2n}|f(y)|\mathrm{d}y$$

$$+C\sum_{|\alpha|=m_2}\left\|D^{\alpha}b_2\right\|_{BMO}\sum_{|\alpha_1|=m_1}\sum_{k=0}^{\infty}k\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}|x-y|^{-2n}\left|D^{\alpha_1}\tilde{b}_1(y)\right||f(y)|\mathrm{d}y$$

$$
\begin{aligned}
&+C\sum_{|\alpha|=m_1}\left\|D^{\alpha}b_1\right\|_{BMO}\sum_{|\alpha_2|=m_2}\sum_{k=0}^{\infty}k\int_{2^{k+1}\tilde{Q}\backslash 2^k\tilde{Q}}|x-y|^{-2n}\left|D^{\alpha_2}\tilde{b}_2(y)\right||f(y)|\mathrm{d}y\\
&+C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\backslash 2^k\tilde{Q}}|x-y|^{-2n}\left|D^{\alpha_1}\tilde{b}_1(y)\right|\left|D^{\alpha_2}\tilde{b}_2(y)\right||f(y)|\mathrm{d}y\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\left\|D^{\alpha}b_j\right\|_{BMO}\right)d^{-n}\sum_{k=1}^{\infty}k^2 2^{-kn}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|^r\,\mathrm{d}y\right)^{1/r}\\
&+C\sum_{|\alpha|=m_2}\left\|D^{\alpha}b_2\right\|_{BMO}d^{-n}\sum_{k=1}^{\infty}k2^{-kn}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|^r\,\mathrm{d}y\right)^{1/r}\\
&\times\sum_{|\alpha_1|=m_1}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}\left|D^{\alpha_1}b_1(y)-\left(D^{\alpha_1}b_1\right)_{\tilde{Q}}\right|^{r'}\mathrm{d}y\right)^{1/r'}\\
&+C\sum_{|\alpha|=m_1}\left\|D^{\alpha}b_1\right\|_{BMO}d^{-n}\sum_{k=1}^{\infty}k2^{-kn}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|^r\,\mathrm{d}y\right)^{1/r}\\
&\times\sum_{|\alpha_2|=m_2}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}\left|D^{\alpha_2}b_2(y)-\left(D^{\alpha_1}b_2\right)_{\tilde{Q}}\right|^{r'}\mathrm{d}y\right)^{1/r'}\\
&+C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}d^{-n}\sum_{k=1}^{\infty}2^{-kn}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|^r\,\mathrm{d}y\right)^{1/r}\\
&\times\prod_{j=1}^{2}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}\left|D^{\alpha_j}b_j(y)-\left(D^{\alpha_j}b_j\right)_{\tilde{Q}}\right|^{r_j}\mathrm{d}y\right)^{1/r_j}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\left\|D^{\alpha}b_j\right\|_{BMO}\right)\sum_{k=1}^{\infty}k^2 2^{-kn}M_r(f)(\tilde{x})\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\left\|D^{\alpha}b_j\right\|_{BMO}\right)M_r(f)(\tilde{x}),
\end{aligned}
$$

thus

$$
|J_5|\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\left\|D^{\alpha}b_j\right\|_{BMO}\right)M_r(f)(\tilde{x}).
$$

This completes the proof of Theorem 1.

Proof of Theorem 2. (a) follows from Theorem 1. For **(b)**, Choose $1<r<p$ in Theorem 1, we get

$$
\begin{aligned}
\left\|T_b(f)\right\|_{L^p}&\le\left\|M\left(T_b(f)\right)\right\|_{L^p}\le C\left\|\left(T_b(f)\right)^{\#}\right\|_{L^p}\\
&\le C\prod_{j=1}^{l}\left(\sum_{|\alpha|=m_j}\left\|D^{\alpha}b_j\right\|_{BMO}\right)\left\|M_r(f)\right\|_{L^p}\\
&\le C\prod_{j=1}^{l}\left(\sum_{|\alpha|=m_j}\left\|D^{\alpha}b_j\right\|_{BMO}\right)\|f\|_{L^p}.
\end{aligned}
$$

This finishes the proof.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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