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A Simplified Graphical Procedure for Constructing a 10˚ or 20˚ Angle

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Abstract

This paper presents a simplified graphical procedure for constructing, *using an unmarked straightedge and a compass only*, a 10˚ to 20˚ angle, which is in other words, trisecting a 30˚ or 60˚ angle. The procedure, when applied to the 30˚ and 60˚ angles that have been "proven" to be not trisectable, produced a construction having the **identical angular relationship** with Archimedes' Construction, in which the required trisection angles were found to be 10.00000˚ and 20.00000˚ respectively (*i.e.* exactly one-third of the given angle or ∠E'MA = 1/3∠E'CG). Based on this **identical angular relationship** as well as the numerical results obtained, one can only conclude that the geometric requirements for arriving at an exact trisection of the 30˚ or 60˚ angle, and therefore the construction of a 10˚ or 20˚ angle, have been met, notwithstanding the theoretical proofs of Wantzel, Dudley, and others. *Thus, the solution to the age-old trisection problem, with respect to these two angles, has been accomplished.*

Keywords

Archimedes' Construction, College Geometry, Angle Trisection, Trisection of an Angle, Famous Problems in Mathematics. Geometer's Sketch Pad, Mechanisms, Mechanism Analysis, Kinematics, Trisector

1. Introduction

Geometrically, the task of constructing an angle of any specific measurement can be described as finding a way to divide an angle greater than the specified measurement so that when divided into *exactly equal* parts, it will produce the desired construction. In this paper, for example, the task of constructing a 10˚ or 20˚ angle implies selecting a 30˚ or 60˚ angle and trisecting it to obtain the desired construction.

The *trisection of an acute angle problem* (except that of 45˚), *using an unmarked straightedge and a compass only*, has been one of the most intriguing geometric challenges for mathematicians for centuries [1] [2], during which time, it has been classified as one of the three unsolvable problems of Geometry: the other two being the *squaring of a circle* and the *doubling of a cube*. Simply stated and also "proven", *the trisection of an arbitrary acute angle* (*except* 45˚) *cannot be achieved using an unmarked straightedge and compass only* [3] [4]. Or, as stated by Underwood Dudley, author of A Budget of Trisections, "*There is no procedure, using only an unmarked straightedge and a compass to construct one-third of an arbitrary angle*". Yet, there have been countless attempts by a number of mathematicians to either disprove this assertion or devise a construction that is as close as possible to the exact solution. Some of the more notable attempts, in both cases, to be found in the literature, besides A Budget of Trisections [4], include:

- The Trisectors by Underwood Dudley [5],
- web articles on "*The Trisection of an Angle*" by Jim Loy [6], and on "*Angle Trisection*" in the Wikipedia [7],
- Randyrradana Weblog on "*Constructing a* 20 *Degree Angle Using Ruler and Compass*," by Gayathri Khrishna [8],
- Web article "*Trisecting The Angle*" in The Britannica," [9] and
- two articles by this author namely, "*Mechanism Analysis of a Trisector*" [10] and "*A Procedure For Trisecting an Acute Angle*" [11].

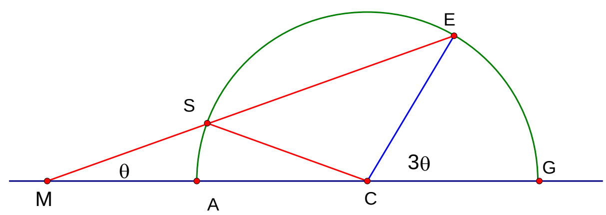
The object of this paper is not to debate the established proofs alluded to, but simply to present a comprehensive graphical procedure, *using only an unmarked straightedge and compass*, that will demonstrate one approach on how a 10˚ or 20˚ angle can be constructed, or, in other words, how a 30˚ or 60˚ angle can be trisected.

The procedure is based on the article "*Mechanism Analysis of a Trisector*" [11], in which a working model of a trisector (see **Figure A1**) was analyzed, using principles of Kinematics [12], instead of conventional mathematics and plane geometry to study the trisection problem. The basis for employing this approach, was the fact that *while it was thought that the angle trisection could not be achieved using an unmarked straightedge and compass, yet a mechanism can be built to perform the task perfectly* [13]. Hence, performing a motion analysis on an actual trisector (see **Figure A1**) seemed a logical rationale for seeking to obtain a fresh insight into understanding the trisection problem.

To be clear, Kinematics [12] is the study of motion and the purpose of the trisector model was simply to study and gain an understanding of its motion. Therefore, it is not a violation of the unmarked straightedge and compass rule. For further details on the motion analysis, see reference article [10].

2. Theory

The procedure being presented is based on the well-known Archimedes' Construction [2] represented in the diagram below, that illustrates the geometric requirements that must be met in order to arrive at an exact trisection, and the general theorem relating to arcs and angles.



Let ∠ECG (or 3∠*θ*) be the required angle to be trisected. With center at C and radius CE describe a semicircle. Given that a line from point E can be drawn to cut the semicircle at S and intersect the extended side GC at some point M such that the distance SM is equal to the radius SC, then from the general theorem relating to arcs and angles,

∠EMG = 1/2 (∠ECG − ∠SCM)

2∠EMG + ∠SCM = ∠ECG

Since ΔCSM is an isosceles Δ

∠SCM = ∠EMG = ∠*θ*

Therefore, 3∠EMG = ∠ECG or 3∠*θ*= ∠ECG or ∠EMA = 1/3∠ECG.

To Summarize:

ONCE, in the construction, the segments SM, SC, E'C, and CG are all equal, and ∠SMA = ∠SCA,

THEN, the EXACT trisection of the given angle ∠E'CG is achieved

or ∠E'MA = 1/3∠E'CG.

3. Procedure

STEP 1 See Figure 1

To illustrate the procedure, we consider the 30˚ and 60˚ angle to be trisected (*i.e.* divided into exactly three equal parts using only an unmarked straightedge and a compass). The construction for this angle is given in following Figure 1, Figure 2(a), Figure 2(b), Figure 3(a), and Figure 3(b).

1) Using CG as the base, erect a perpendicular CC' at C.

2) With center at C and any convenient radius, describe a semicircle from point G cutting perpendicular CC' at E, and terminating at A on GC (extended).

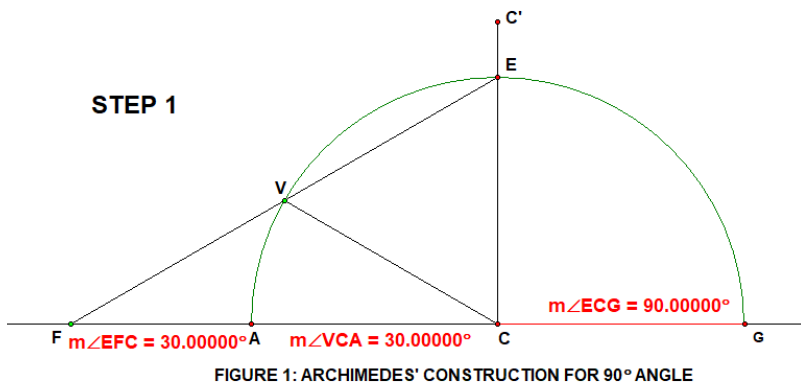


Figure 1. Archimedes' Construction for 90˚ trisection.

3) Using CE as the base, form an equilateral triangle CEV, where V is the vertex.

4) Extend segment EV to meet GC (extended) at a point F.

STEP 2 See Figure 2(a) or Figure 3(a) and Figure 2(b) or Figure 3(b)

1) Using point G as center and CE as radius, describe an arc cutting the semicircle in STEP 1 at point E' to form the given angle ∠E'CG = 30˚ or 60˚

2) Construct segment E'F, cutting the semicircle at V'.

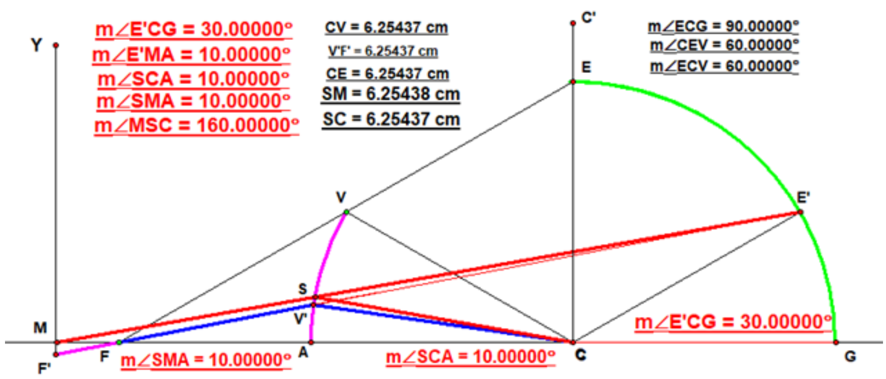
3) Join V' to C with segment V'C and extend segment V'F to V'F' such that V'F' = V'C.

4) At T, erect a perpendicular ray, TY, cutting the baseline FG at a point M.

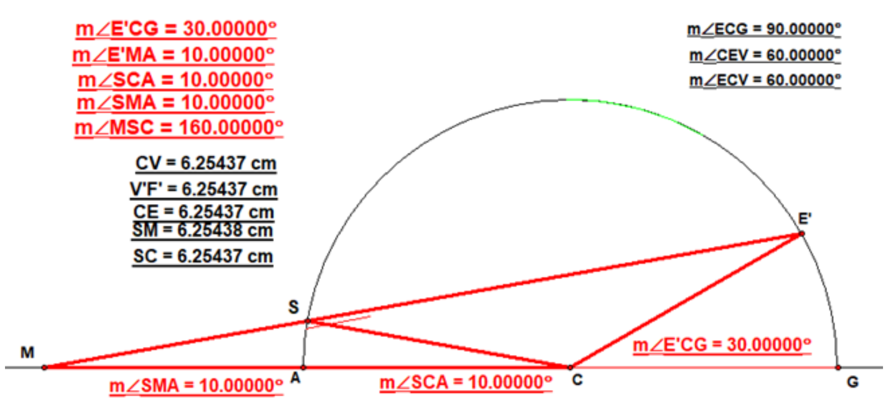
5) Join E' to M with segment E'M, cutting AV at a point S, to form the required trisection angle ∠E'MA = 1/3∠E'CG, in compliance with Archimedes' Construction [2].

6) Join S to C with a segment SC to complete the construction, which makes segment SM equal to segment SC. See Figure 2(b) or Figure 3(b) and note the Identical Angular Relationship with Archimedes' Construction [2] in Section 2 on Theory.

STEP 2

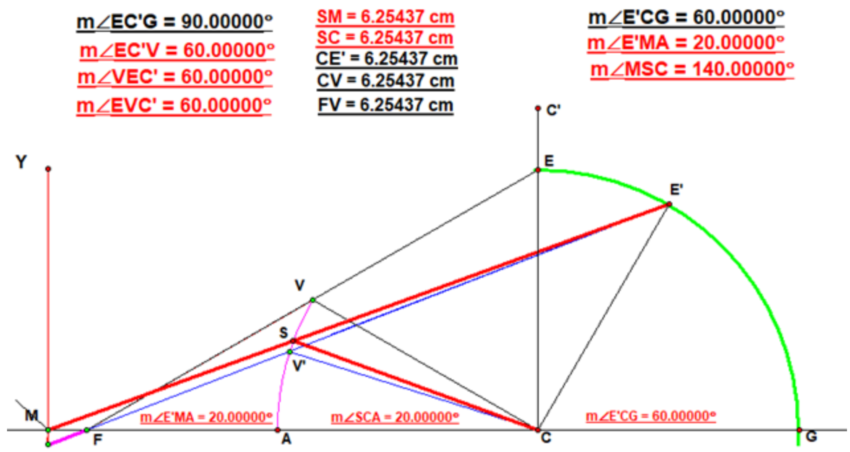


(a)

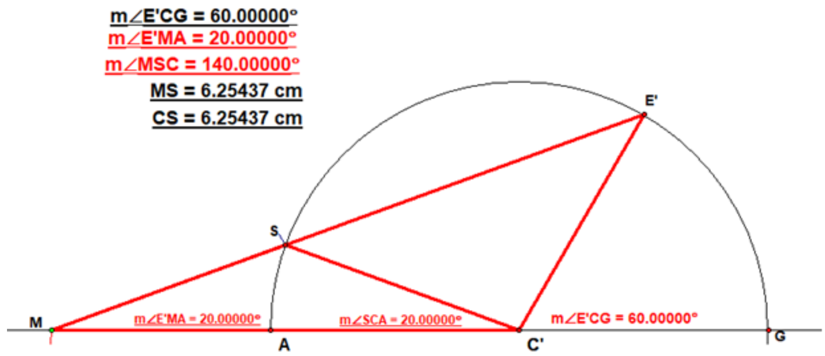


(b)

Figure 2. (a) 30 × 10 Composite Construction Showing Trisection of 30˚ Angle where ∠E'MA = 1/3∠E'CG = 10.00000˚; (b) 30 × 10 Resultant Angles Showing Trisection of 30˚ Angle where ∠E'MA = 1/3∠E'CG = 10.00000˚.



(a)



(b)

Figure 3. (A) 60 × 20 Composite Construction Showing Trisection of 60˚ Angle where ∠E'MA = 1/3∠E'CG = 20.00000˚; FIGURE 2B: 60 × 20 Resultant Angles Showing Trisection of 60˚ Angle where ∠E'MA = 1/3∠E'CG = 20.00000˚.

NOTE: While the Geometer's Sketch Pad [14] was employed in developing the construction, the use of this software is not a violation of the unmarked straightedge rule, since its sole purpose was strictly for 1) laying out the lines and arcs for precision, and not measurements, except for determining final results at the very end, and 2) for color coding or organizing the data, which is appropriate for an effective presentation. Otherwise, the construction can easily be hand-drawn.

4. Proof

Referring to Figure 2(b) and Figure 3(b) above, and applying the general theorem relating to arcs and angles (See Section 2 on THEORY of this paper), we get

∠E'MG = 1/2 (∠E'CG − ∠SCM) or ∠E'MA = 1/2 (∠E'CG − ∠SCM)

2∠E'MA = ∠E'CG − ∠SCM

2∠E'MA + ∠SCM = ∠E'CG

Since ∠SCM = ∠E'MA

Then 3∠E'MA = ∠E'CG

Therefore,

for the 30˚ trisection **∠E'MA = 1/3∠E'CG = 1/3(30˚) = 10.00000˚ (QED)**

for the 60˚ trisection **∠E'MA = 1/3∠E'CG = 1/3(60˚) = 20.00000˚ (QED)**

Note that these numerical results obtained by The Geometer's Sketch Pad [14] represent the highest level of accuracy and precision (e.g. five decimal places) attainable by this software.

5. Benefit

The major benefit of having the ability to construct the 30˚ or 60˚ angle is that it has made it possible to construct more angles than ever before, using only an unmarked straightedge and compass only. They include angles of not only multiples of 10 ˚ or 20˚ (*i.e.* 10˚, 20˚, 30˚…), but also multiples of 5˚ and 2.5˚ (*i.e.* 2.5˚, 5˚, 7.5˚…).

6. Summary

A simplified graphical procedure for constructing a 10˚ or 20˚ angle which, in other words, is for trisecting a 30˚ or 60˚ angle, *using an unmarked straightedge and a compass only*, has been presented. The procedure, when applied to these two angles which have been "proven" to be non-trisectable, in each case, produced a construction having an **identical angular relationship** with Archimedes' Construction [2], where the trisection angle was found to be exactly one-third of the respective given angles (*i.e.* ∠E'MA) = 1/3∠E'CG = 10.00000˚ and 20.00000˚, as shown in **Figure 2(b)** or **Figure 3(b)** as well as Section 4 on PROOF in this paper. Based on this **identical angular relationship** between the two constructions and also the numerical results obtained by The Geometer's Sketch Pad [14], one can only conclude that the geometric requirements for arriving at an *exact* trisection of the 30˚ or 60˚ angle, and therefore the construction of a 10˚ or 20˚ angle, have been met, notwithstanding the theoretical proofs of Wantzel, Dudley, and others [1] [2] [3] [4].

To be specific, the construction presented has achieved the desired objectives of constructing a 10˚ *or* 20˚ *angle which is, in other words, dividing a* 30˚ *or* 60˚ *angle into three exactly equal parts using an unmarked straightedge and a compass only. Thus, the solution to the age-old trisection problem with respect to these two angles, has been accomplished.*

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix

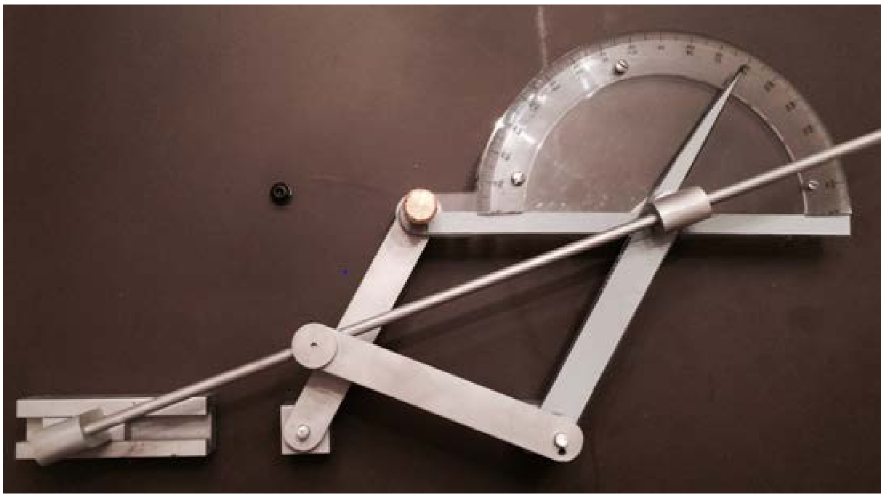

Figure A1. Trisector model.