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Canards Flying on Bifurcation

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Abstract

There exists a property "structural stability" for "4-dimensional canards" which is a singular-limit solution in a slow-fast system with a bifurcation parameter. It means that the system includes the possibility to have some critical values on the bifurcation parameter. Corresponding to these values, the pseudo-singular point, which is a singular point in the time-scaled-reduced system should be changed to another one. Then, the canards may fly to another pseudo-singular point, if possible. Can the canards fly? The structural stability gives the possibility for the canards flying. The precise reasons why happen are described in this paper.

Keywords

Canard Solution, Slow-Fast System, Nonstandard Analysis, Hilbert's 16th Problem, Brownian Motion, Stochastic Differential Equation

1. Introduction

In the beginning of bifurcation problem, R. Thom developed catastrophe theory, which stands on a statical model, that is, it consists of multi-variable real functions with parameters ([1] [2] [3]). Shortly, in some equivalent classes, Hessian matrix on non-degenerate critical points is classified like as "fold", "cusp", "swallow's tail", … It is originally based on the behavior of differential equations keeping structural stability, and then the potential function is used as the statical model. Notice that it is of multi-variables essentially.

Although there are many books written on "bifurcation problem", "Catastrophe Theory and its Applications" by T. Poston and I. Stewart [4] is recommended for many readers, because it is written from a geometrical point of view without rigorous proof.

In the slow-fast system with parameters, we took up the pseudo-singular point having structural stability. It means the catastrophe is caused on a dynamic model directly. In fact, when analyzing a concrete example, the constrained surface ($\varepsilon = 0$) in the slow-fast system reduces the Thom's function classified. Throughout this paper, we shall describe the structure precisely. In our previous paper [5], it becomes clear that if the system has "symmetry", there exist two cases. One is the pseudo-singular point is structurally stable and the other case is unstable. In the unstable case, we showed computer simulations since it depends on a parameter. In the stable case, however, there was nothing but giving short comments since it is independent. Note that the canard turns to another one by the parameter.

When the parameter value changes from negative sign to positive one, even if the pseudo singular point satisfies the conditions of canard, it is confirmed that the unstable pseudo-singular point is vanished. In other words, the canards are vanished. Then, the canards on the stable pseudo-singular point just appear. We call it "canards flying". Since "canard" is a singular-limit solution (ε tends to zero) in the slow-fast system, the behavior of the orbit is very complicated when ε takes nearly equal to zero. It gives us a new structure as "dynamical catastrophe".

2. Slow-Fast System with Bifurcation Parameter

Consider the following system:

$$\begin{cases} \varepsilon \dfrac{\mathrm{d}x}{\mathrm{d}t} = h(x, y, \varepsilon) \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = g(x, y, b) \end{cases} \tag{1}$$

where ε is infinitesimal, b is any constant and

$$x = (x_1, x_2) \in R^2,\ y = (y_1, y_2) \in R^2,$$

$$h = (h_1, h_2): R^4 \to R^2,\ g = (g_1, g_2): R^4 \to R^2.$$

Assume that $g(x, y, b) = g(x, by)$, for the simplicity, and the origin is a singular point.

Furthermore we assume that the System (1) sastisfies the following conditions (A1)-(A6):

(A1) h is of class $\mathbf{C}^1$ and g is of class $\mathbf{C}^2$.

(A2) The slow manifold $S = \{(x, y) \in \mathbf{R}^4 \,|\, h(x, y, 0) = 0\}$ is a two-dimensional differential manifold and intersects the set

$$T = \left\{(x, y) \in \mathbf{R}^4 \,\middle|\, \det\left[\frac{\partial h}{\partial x}(x, y, 0)\right] = 0\right\} \tag{2}$$

transversely, where

$$\frac{\partial h}{\partial x} = \begin{bmatrix} \dfrac{\partial h_1}{\partial x_1} & \dfrac{\partial h_1}{\partial x_2} \\ \dfrac{\partial h_2}{\partial x_1} & \dfrac{\partial h_2}{\partial x_2} \end{bmatrix} \tag{3}$$

Then, the pli set

$$PL = \{(x, y) \in S \cap T\} \tag{4}$$

is a one-dimensional differentiable manifold.

(A3) Either the value of g_1 or that of g_2 is nonzero at any point of *PL*.

Note that the pli set *PL* devides the slow manifolds *S**PL* into three parts depending on the signs of the two eigenvalues of $\frac{\partial h}{\partial x}(x, y, 0)$.

First consider the following reduced system which is obtained from (1) with $\varepsilon = 0$:

$$\begin{cases} 0 = h(x, y, 0) \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = g(x, y, b) \end{cases} \tag{5}$$

By differentiating $h(x, y, 0)$ with respect to *t*, we have

$$\frac{\partial h}{\partial x}(x, y, 0)\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial h}{\partial y}(x, y, 0) g(x, y, b) = 0 \tag{6}$$

Then (4) becomes the following:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -\left[\dfrac{\partial h}{\partial x}(x, y, 0)\right]^{-1} \dfrac{\partial h}{\partial y}(x, y, 0) g(x, y, b) \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = g(x, y, b) \end{cases} \tag{7}$$

where $(x, y) \in S \setminus PL$. To avoid degeneracy in (6), we consider the time-scaled-reduced system:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \left\{-\det\left[\dfrac{\partial h}{\partial x}(x, y, 0)\right]^{-1}\right\}\left[\dfrac{\partial h}{\partial x}(x, y, 0)\right]^{-1} \dfrac{\partial h}{\partial y}(x, y, 0) g(x, y, b) \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = \left\{\det\left[\dfrac{\partial h}{\partial x}(x, y, 0)\right]^{-1}\right\} g(x, y, b) \end{cases} \tag{8}$$

The phase portrait of the System (8) is the same as that of (7) except the region where $\det\left[\frac{\partial h}{\partial x}(x, y, 0)\right] = 0$, but only the orientation of the orbit is different.

Definition 1. *A singular point of* (8), *which is on PL, is called a pseudo singular point of* (1). *The set of pseudo singular points is denoted by PS.*

(A4) $\mathrm{rank}\left[\frac{\partial h}{\partial x}(x, y, 0)\right] = 2$, $\mathrm{rank}\left[\frac{\partial h}{\partial y}(x, y, 0)\right] = 2$ for any $(x, y) \in S \setminus PL$.

From (A4), the implicit function theorem guarantees the existence of a unique function $y = \varphi(x)$ such that $h(x, \varphi(x), 0) = 0$. By using $y = \varphi(x)$, we obtain the following system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \left\{-\det\left[\frac{\partial h}{\partial x}(x, \varphi(x), 0)\right]^{-1}\right\}\left[\frac{\partial h}{\partial x}(x, \varphi(x), 0)\right]^{-1} \frac{\partial h}{\partial y}(x, \varphi(x), 0) g(x, \varphi(x), b). \tag{9}$$

(A5) All singular points of (8) are non-degenerate, that is, the linearization of (8) at a singular point has two nonzero eigenvalues.

Now, let us introduce a definition of "symmetry". It is a key word through this paper.

Definition 2. *If* $h_1(x_1,x_2,y_1,y_2,\varepsilon)=h_2(x_2,x_1,y_2,y_1,\varepsilon)$, *and* $g_1(x_1,x_2,y_1,y_2,b)=g_2(x_2,x_1,y_2,y_1,b)$, *then the system is "symmetric" for the subspace* $I=\{(x_1,x_2,y_1,y_2)\mid x_1=x_2,y_1=y_2\}$.

(A6) I intersects PL transversely.

Definition 3. *Let* λ_1,λ_2 *be two eigenvalues of the linearization of* (8) *at a pseudo singular point. The pseudo singular point with real eigenvalues is called a pseudo singular saddle point if* $\lambda_1<0<\lambda_2$ *and a pseudo singular node point if* $\lambda_1<\lambda_2<0$ *or* $\lambda_1>\lambda_2>0$.

The following Theorems 1 and 2 are established in [5] and [6], respectively.

Theorem 1. *Let* (x_0,y_0) *be a pseudo singular saddle or node point. If* $\operatorname{trace}\left[\frac{\partial h}{\partial x}(x_0,y_0,0)\right]<0$, *then there exists a solution which first follows the attractive part and the repulsive part after crossing PL near the pseudo singular point.*

Remark 1. *The solution in Theorem 1 is called "canard".*

Theorem 2. *If* $\det\left[\frac{\partial f}{\partial x}(0,0)\right]=0$, *and* $\operatorname{trace}\left[\frac{\partial f}{\partial x}(0,0)\right]<0$, *then canards near the subspace I has a centre manifold.*

Remark 2. *The condition* $\operatorname{trace}\left[\frac{\partial h}{\partial x}(x_0,y_0,0)\right]<0$ *implies that one of eigenvalues of* $\left[\frac{\partial h}{\partial x}(x_0,y_0,0)\right]$ *is equal to zero and the other one is negative. Notice that the system has two kinds of vector fields: one is* 2*-dimensional slow and the other is* 2*-dimensional fast one. The condition provides the state of the fast vector field.*

Remark 3. *The singular solution in Theorem* 1 *is called a canard in* $\mathbf{R}^4$ *with* 2*-dimensional slow manifold. As a result, it causes a delayed jumping. The study of canards requires still more precise topological analysis on the slow vector field.*

Remark 4. *On the subspace I, the following system is established for some b. I is an invariant manifold.*

$$\begin{cases}\varepsilon\dfrac{\mathrm{d}x_1}{\mathrm{d}t}=h_1(x_1,y_1,\varepsilon)\\ \dfrac{\mathrm{d}y_1}{\mathrm{d}t}=g_1(x_1,y_1,b)\end{cases} \quad g_1(x_1,x_2,y_1,y_2,b)= \tag{10}$$

Remark 5. *On the set PL,* $\det\left[\frac{\partial h}{\partial x}\right]=0$ *is satisfied and at* $(x_0,y_0)\in PS$ *the following equation is established:*

$$\left\{\frac{\partial h_1}{\partial x_1}(x_0,\varphi(x_0),0)\frac{\partial h_2}{\partial x_2}(x_0,\varphi(x_0),0)-\frac{\partial h_1}{\partial x_2}(x_0,\varphi(x_0),0)\frac{\partial h_2}{\partial x_1}(x_0,\varphi(x_0),0)\right\}g_1(x_0,\varphi(x_0),b)=0. \tag{11}$$

Note that there exists $y=\phi(x)$ *because of assuming* $\operatorname{rank}\left[\frac{\partial h}{\partial y}\right]=2$.

3. Structural Stability

When and why does the pseudo singular point have structural stability? A geometrical point of view to make it clear is shown in this section.

Lemma 1. *The matrix* $\left[\frac{\partial h}{\partial x}\right]$ *is symmetric.*

Proof. Because the system is symmetric for the set *I*, it is obvious from elementary calculus. □

From (A6), the subspace *I* intersects *PL* transversely. Lemma 1 ensures that I^c also intersects *PL* transversely, where I^c is the orthogonal complement of *I*. Since the matrix $\left[\frac{\partial h}{\partial y}\right]$ is also symmetric, for the sake of simplicity, suppose that $\left[\frac{\partial h}{\partial y}\right]$ is identity without loss of generality.

Using Remark 5, the following lemma is established in [5].

Lemma 2. *Let* $(x_0,y_0)\in PS$ *be on* $I\cap PL$, *then it depends on the parameter b. On the other hand, on* $I^c\cap PL$, *it is independent of b.*

From Lemma 2, we establish the following thorems.

Theorem 3. *There exists a pseudo singular point* $(x^0,y^0)\in PS$, *which is one of a coupled points near the subspace I, when satisfying* $b_0>b>0$. *When* $b_0=0$, *it is just on I.*

Theorem 4. *Let* $(x_0,y_0)\in PS$ *be a saddle or node point. Then, if* $(x_0,y_0)\in I^c\cap PL$, *the pseudo singular point is structurally stable. If* $(x_0,y_0)\in I\cap PL$ *it is structurally unstable.*

The next theorem is the main result.

Theorem 5. *Let a critical value* $b_0=\inf_{1\le k\le N} b_k$ *be positive, where each* $b_k\,(k=1,2,\cdots,N)$ *is also a critical value and N is the hyperfinte number. If the structurally stable pseudo-singular point for* $b<0$ *and the unstable pseudo-singular point satisfy the canard conditions in Theorem* 1, *then there exists the canard flying.*

Proof. If $b_0>b>0$, then the pseudo-singular point on *I* is unstable by Lemma 2. If $b<0$, the pseudo-singular point on I^c is stable and saddle or node because of the conditions in Theorem 1. Suppose the canard conditions in Theorem 1, then the other unstable point is vanishing, that is, the canard is flying. □

Theorem 5 plays a central role to establish canards flying under Theorems 3 and 4 which are already shown in K-T [5].

4. Concrete Example

4.1. Modefied Coupled FitzHugh-Nagumo Equations

Consider the following typical example of modefied coupled FitzHugh-Nagumo equations. See [7] for more details.

$$\begin{cases} \varepsilon \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = x_2 + y_1 - \dfrac{x_1^3}{3} \\ \varepsilon \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = x_1 + y_2 - \dfrac{x_2^3}{3} \\ \dfrac{\mathrm{d}y_1}{\mathrm{d}t} = -\dfrac{1}{c}\left(x_1 + by_1\right) \\ \dfrac{\mathrm{d}y_2}{\mathrm{d}t} = -\dfrac{1}{c}\left(x_2 + by_2\right) \end{cases} \tag{12}$$

The next equation is the time-scaled-reduced system corresponding to (19).

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\begin{pmatrix} -x_2^2 & -1 \\ -1 & -x_1^2 \end{pmatrix} \begin{pmatrix} -\dfrac{1}{c}\left(x_1 + by_1\right) \\ -\dfrac{1}{c}\left(x_2 + by_2\right) \end{pmatrix} \triangleq f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}. \tag{13}$$

There exsist pseudo singular points $\left(x_0, y_0\right) \in PS$ of the System (9) which are obtained by the following. If $\left(x_0, y_0\right) = \left(x_{01}, x_{02}, y_{01}, y_{02}\right)$ exists on neighborhood of $I \cap PL$, then $x_1 x_2 = 1$ holds.

Remark 6. *In the previous paper* [5], *the definition of the function f in* p.605 *is incorrect. As the sign of* "*f*" *is negative*, Equation (13) *is correct.*

Remark 7. *Notice that in Lemma* 2, *if* $\left(x_0, y_0\right) \in I \cap PL$, *then the critical value* $b = b_0 = \dfrac{3}{2}$ *holds. When* $0 < b < \dfrac{3}{2} = b_0$, *the corresponding pseudo singular points are on neighborhood of I but not on I.*

If $\left(x_0, y_0\right) \in I^c \cap PL$, then $x_1 x_2 = -1$ holds. Therefore,

$$\begin{cases} x_{01} = \pm 1 \\ x_{02} = \mp 1 \\ y_{01} = \dfrac{x_{01}^3}{3} - x_{02} \\ y_{02} = \dfrac{x_{02}^3}{3} - x_{01} \end{cases} \tag{14}$$

Remark 8. *The solution of* (14) *is structurally stable. Then*

$$\left[\frac{\partial h}{\partial x}\right]_{(x_1, x_2) = (1, -1)} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \tag{15}$$

and

$$\left[\frac{\partial f}{\partial x}\right]_{(x_1, x_2) = (1, -1)} = -\frac{1}{c} \begin{pmatrix} 1 & -1 - \dfrac{8}{3}b \\ -1 - \dfrac{8}{3}b & 1 \end{pmatrix}. \tag{16}$$

Then,

$$\operatorname{trace}\left[\frac{\partial h}{\partial x}\right]_{(x_1, x_2) = (1, -1)} = \operatorname{trace}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = -2. \tag{17}$$

Therefore, the assumptions of Theorem 1 *are satisfied and a canard exsits. On the other hand, the characteristic equation is*

$$(1+\lambda)^2-\left(1+\frac{8}{3}b\right)^2=0. \tag{18}$$

The solution of (18) *is*

$$\lambda=-1\pm\left(1+\frac{8}{3}b\right). \tag{19}$$

Therefore, if $b<-\frac{3}{4}$, *then the psudo singular point is saddle. On the other hand, if* $-\frac{3}{4}<b<0$, *then it is node.*

4.2. Stochastic Differential Equations

Let us consider a stochastic differential equation for a slow-fast system with a Brownian motion $B(t)$ as the random noises modifying the slow fast System (1): For $t\in[0,T], T>0$

$$\begin{cases}\varepsilon \mathrm{d}x=h(x,y,\varepsilon)\mathrm{d}t\\ \mathrm{d}y=g(x,y,\varepsilon)\mathrm{d}t+\sigma \mathrm{d}B\end{cases} \tag{20}$$

where $B=\begin{pmatrix}B_1\\B_2\end{pmatrix}\in R^2$ is a 2-dimensional standard Brownian motion and $\sigma>0$ is a positive constant which gives a standard deviation for the Brownian motion $B(t)$.

When the system (12) is disturbed by $B(t)$, which gives small noises, the canard solution changes to another orbit.

Anderson [8] showed that the Brownian motion is described by step functions using non-standard analysis on a hyper finite time line by the following definition.

Definition 4. *Let* $N_t=\frac{t}{\Delta t}, 0\le t\le T$ *and* $N=N_T$. *Assume that a sequence of i.i.d. random variables* $\{\Delta B_k, k=1,\cdots,N\}$ *has the distribution*

$$P\left\{\Delta B_k=\sqrt{\Delta t}\right\}=P\left\{\Delta B_k=-\sqrt{\Delta t}\right\}=\frac{1}{2} \tag{21}$$

for each $k=1,\cdots,N$. *An extended Wiener process* $\{B(t),t\ge 0\}$ *is defined by*

$$B(t)=\sum_{k=1}^{N_t}\Delta B_k,\quad 0\le t\le T. \tag{22}$$

Rewriting the System (20) via step functions on the hyper finite time line, the following System (23) is obtained.

$$\begin{cases}\varepsilon\left\{x_1(t_k)-x_1(t_{k-1})\right\}=\left(x_2+y_1-\frac{x_1^3}{3}\right)\Delta t\\ \varepsilon\left\{x_2(t_k)-x_2(t_{k-1})\right\}=\left(x_1+y_2-\frac{x_2^3}{3}\right)\Delta t\\ y_1(t_k)-y_1(t_{k-1})=-\frac{1}{c}(x_1+by_1)\Delta t+\sigma_1\Delta B_{1k}\\ y_2(t_k)-y_2(t_{k-1})=-\frac{1}{c}(x_2+by_2)\Delta t+\sigma_2\Delta B_{2k}\end{cases} \tag{23}$$

where

$$B_1(t)=\sum_{k=1}^{N_t}\Delta B_{1k},\quad B_1(t)=\sum_{k=1}^{N_t}\Delta B_{1k}.$$

When $\sigma_1=\sigma_2=0$, the System (23) is the nonstandard form of (12). For more details of the stochastic slow fast system, see [9].

4.3. Simulation Results

In **Figures 1-8**, $\varepsilon=0.01$, $c=1$ and $\Delta t=0.0001$ in (23). The curves, which satisfy $x_1x_2=1$ and $x_1x_2=-1$, respectively, are Pli set.

Figure 1 $\sigma_1=\sigma_2=0$.

Figure 1 shows an orbit of $\{(x_1(t),x_2(t)),0\le t\le T=7\}$ satisfying Equation (23) with $\sigma_1=\sigma_2=0$, $b=-0.6$ and starting from $(1.1,-1.2)$ near the pseudo

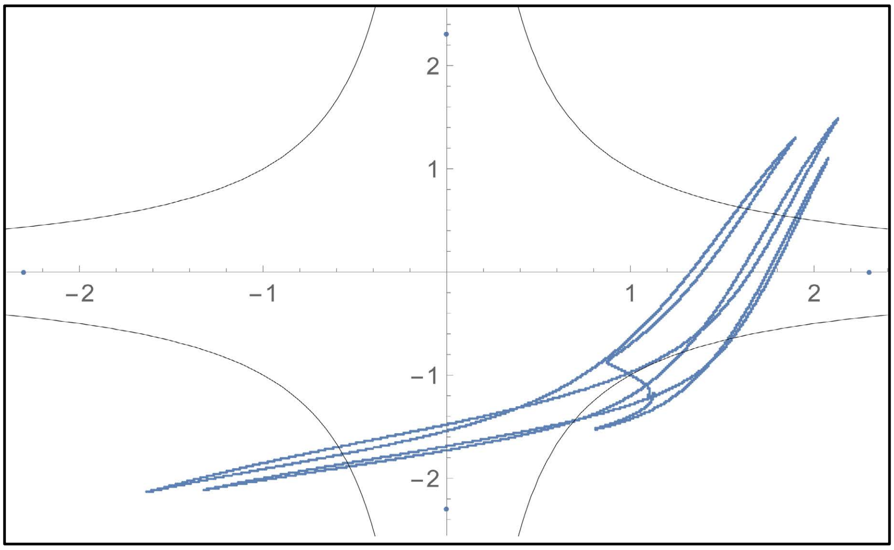

Figure 1. $b=-0.6$, $(x_1(0), x_2(0))=(1.1,-1.2)$.

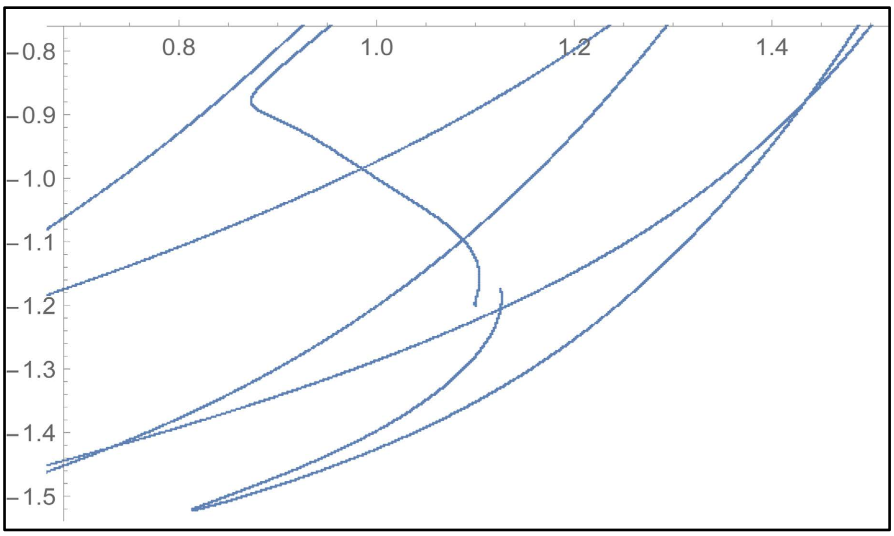

Figure 2. $b=-0.6$, $(x_1(0), x_2(0))=(1.1,-1.2)$. Enlarged orbit of **Figure 1**.

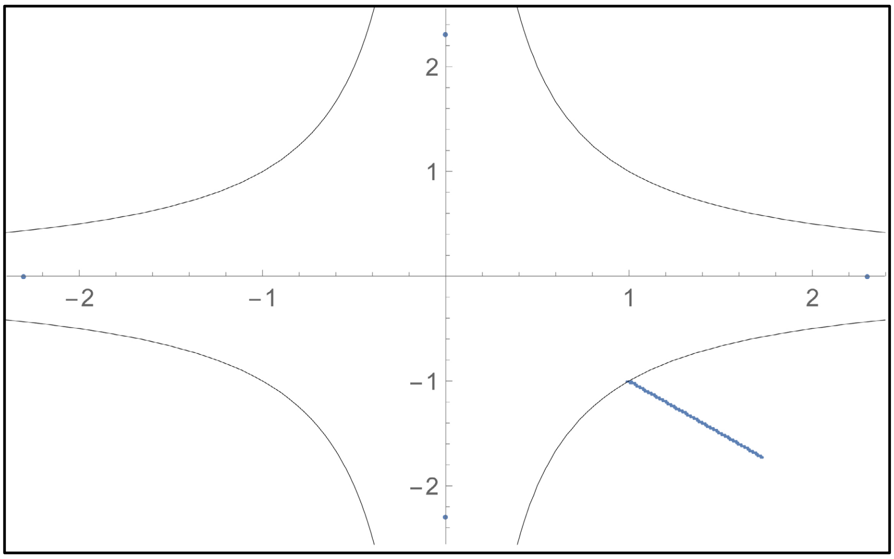

Figure 3. $b = -0.8$, $(x_1(0), x_2(0)) = (0.99, -1)$.

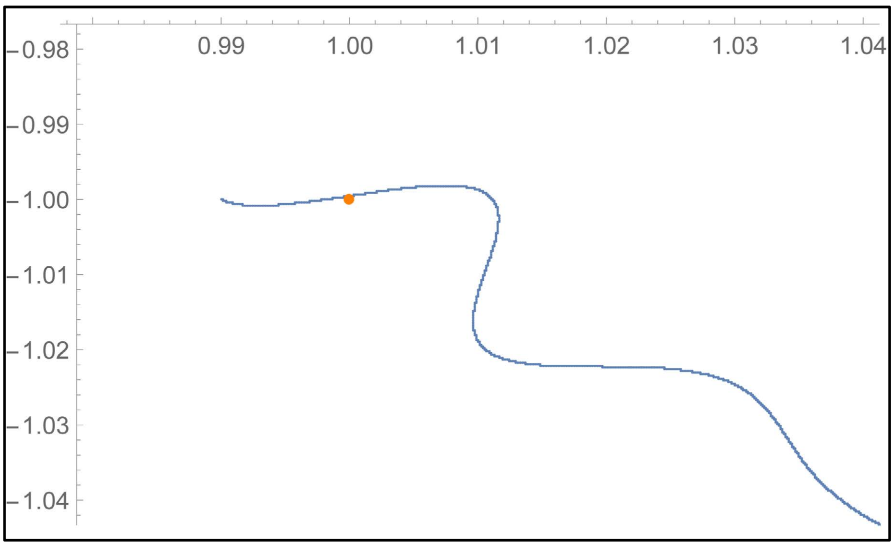

Figure 4. $b = -0.8$, $(x_1(0), x_2(0)) = (0.99, -1)$. Enlarged orbit of Figure 3.

singular point (1, −1). At the pseudo singular point (1, −1), the eigenvalue λ in (18) is negative and it is node.

Figure 2 *Enlarged orbit of* Figure 1.

Figure 2 shows an enlarged orbit of Figure 1. The orbit starts at $(1.1, -1.2)$ and goes up near the pseudo singular point (1, −1). After that, it jumps out towards right direction from the neighborhood of (0.87, −0.9).

Figure 3 $\sigma_1 = \sigma_2 = 0$.

Figure 3 shows an orbit of $\{(x_1(t), x_2(t)), 0 \le t \le T = 7\}$ satisfying Equation (23) with $\sigma_1 = \sigma_2 = 0$, $b = -0.8$ and starting from $(0.99, -1)$ near the pseudo singular point (1, −1). At the pseudo singular point (1, −1), the eigenvalue λ in (18) is positive and it is saddle.

Figure 4 *Enlarged orbit of* Figure 3.

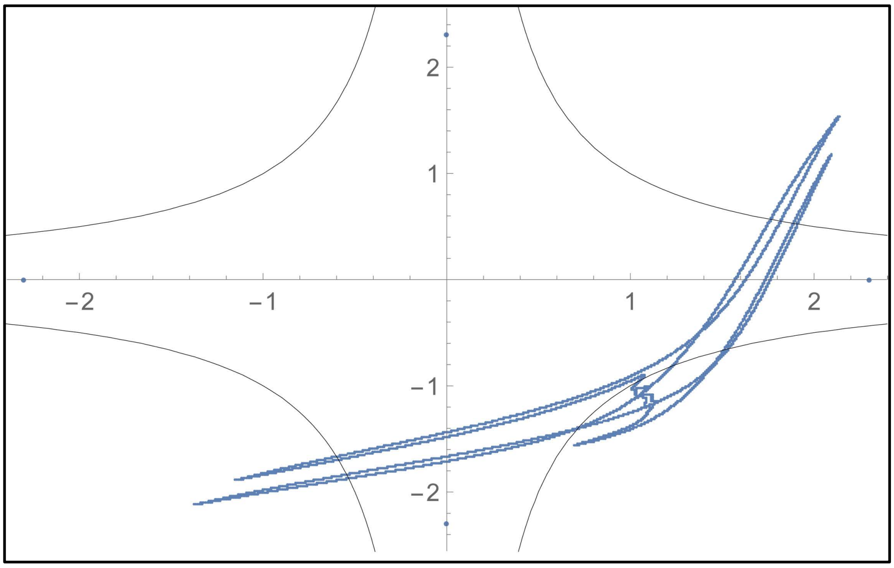

Figure 5. $b = -0.6$, $(x_1(0), x_2(0)) = (1.1, -1.2)$, $\sigma_1 = \sigma_2 = 0.001$.

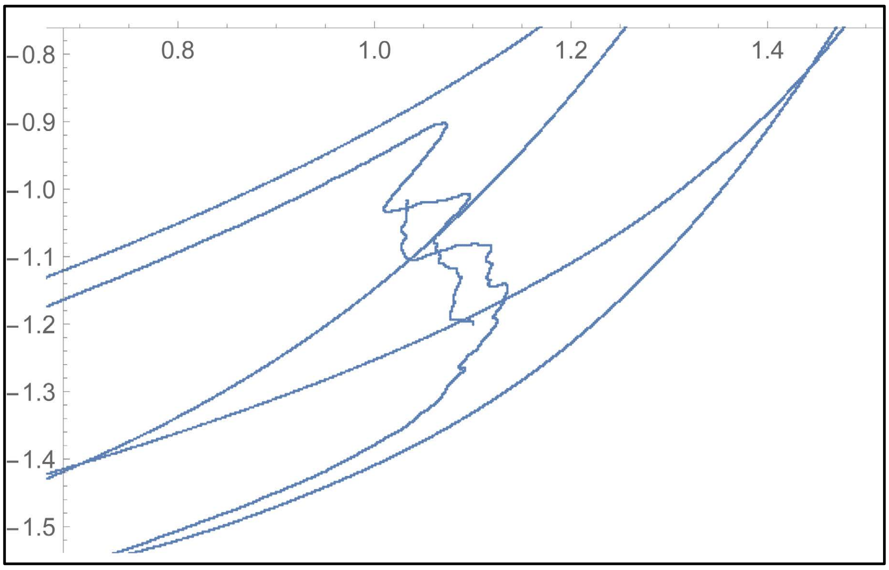

Figure 6. $b = -0.6$, $(x_1(0), x_2(0)) = (1.1, -1.2)$, $\sigma_1 = \sigma_2 = 0.001$, Enlarged orbit of Figure 1.

Figure 4 shows an enlarged orbit of Figure 1. The orbit starts at $(0.99, -1)$ and passes through near the pseudo singular point (1, −1). After that, it jumps out towards right direction from the neighborhood of (1.012, −1).

Figure 5 $\sigma_1 = \sigma_2 = 0.001$.

Figure 5 starting at $(1.1, -1.2)$ shows an orbit of $\{(x_1(t), x_2(t)), 0 \le t \le T = 7\}$ satisfying Equation (23) with $\sigma_1 = \sigma_2 = 0.001$, $b = -0.6$. The orbit of Figure 5 is similar to Figure 1. But the orbit starts at $(1.1, -1.2)$ and goes up near the pseudo singular point (1, −1) according to the Brownian motion B.

Figure 6 *Enlarged orbit of* Figure 5.

Figure 6 shows an enlarged orbit of Figure 5.

Figure 7 $\sigma_1 = \sigma_2 = 0.001$.

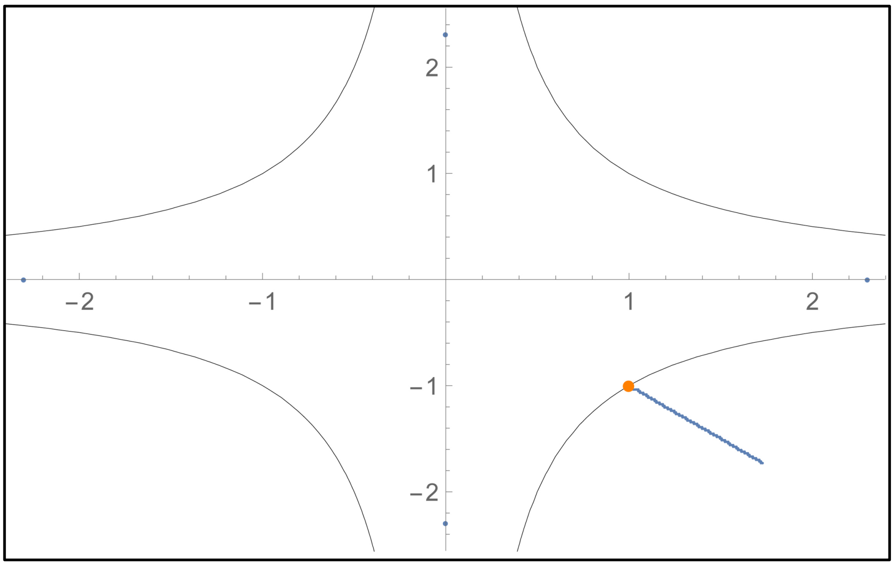

Figure 7. $b = -0.8$, $(x_1(0), x_2(0)) = (0.99, -1)$, $\sigma_1 = \sigma_2 = 0.001$.

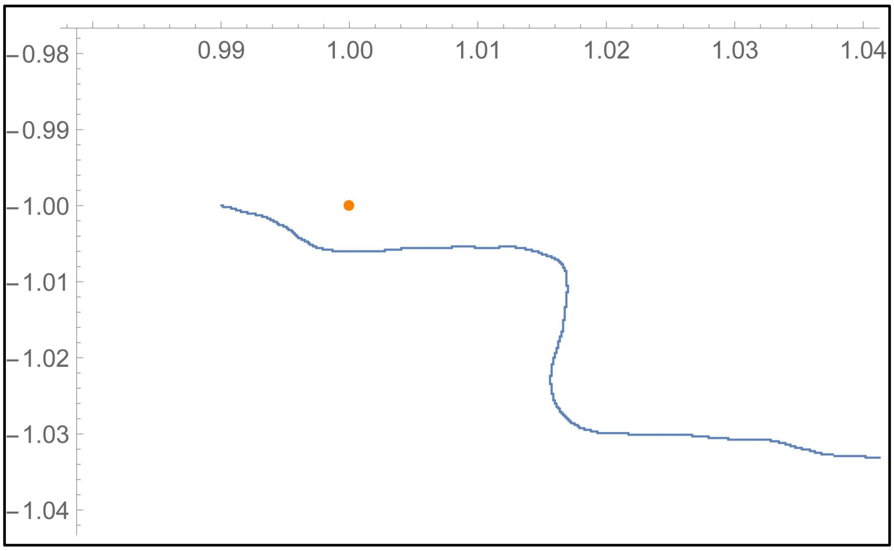

Figure 8. $b = -0.8$, $(x_1(0), x_2(0)) = (0.99, -1)$, $\sigma_1 = \sigma_2 = 0.001$, Enlarged orbit of Figure 3.

Figure 7 starting at $(0.99, -1)$ shows an orbit of $\{(x_1(t), x_2(t)), 0 \le t \le T = 7\}$ satisfying Equation (23) with $\sigma_1 = \sigma_2 = 0.001$, $b = -0.8$.

Figure 8 *Enlarged orbit of* Figure 8.

Figure 7 shows an enlarged orbit of Figure 8.

5. Conclusions

The slow-fast system with a bifurcation parameter gives us structural stability under the key notion "symmetry". In our previous paper published in Advances in Pure Mathematics, vol.12 (2022), it has been confirmed, then we emphasize that the stability for the pseudo singular point plays an important role.

In economic models [9] [10], regarding [9] the second fast equation in (18),

(19) is mistyped: $(1-n_1)$ is n_1. In that system, when n_1, n_2, m_1, m_2 taking as parameters, one of the pseudo singular points is structurally stable. Then corresponding multi-variable functions are complicated ones. Some special cases may satisfy Thom's functions classified.

When being $\varepsilon = 0$, the constrained surface describes something like a potential function because of holding (A4). On the invariant manifold, it is "fold" in Thom's function: only domestic case has such a function, and the two-region model has a multi-variable one much complicated. Especially, take a note that the pseudo singular point has no structural stability due to no bifurcation parameter in the original system [10]. It may be "elementary catastrophe", which depends on a parameter not on time. In general, however, it is not established.

The orbit (ε tends to "zero") after passing through near the pseudo singular point jumps out with delay, that is, the state itself is jumping and the behavior is complicated as "dynamical catastrophe".

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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