# All Zeros of the Riemann Zeta Function in the Critical Strip Are Located on the Critical Line and Are Simple 

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#### Abstract

In this paper we study the function $G(z):=\int_{0}^{\infty} y^{z-1}(1+\exp (y))^{-1} \mathrm{~d} y$, for $z \in \mathbf{C}$. We derive a functional equation that relates $G(z)$ and $G(1-z)$ for all $z \in \mathbf{C}$, and we prove: 1 ) that $G$ and the Riemann zeta function $\zeta$ have exactly the same zeros in the critical region $D:=\{z \in \mathbf{C}: \mathfrak{R} z \in(0,1)\}$; 2) the Riemann hypothesis, i.e., that all of the zeros of $G$ in $D$ are located on the critical line $:=\{z \in D: \mathfrak{R z}=1 / 2\}$; and that 3 ) all the zeros of the Riemann zeta function located on the critical line are simple.


## Keywords

Riemann Hypothesis, Fourier Transforms, Schwarz Reflection Principle, Cauchy-Riemann Equations, Trapezoidal-Midordinate Quadrature

## 1. Introduction and Summary

The proof of the Riemann hypothesis is a problem that many mathematicians consider to be the most important problem of mathematics. Indeed, it is one of at most seven mathematics problems for which the Clay Institute has offered a million dollars for its solution. To this end, the pdf publication [1] of Bombieri presents an excellent summary-along with references, to papers and books, to connections with prime numbers, to Fermat's last theorem, and to the work of authors who have shown that the first 1.5 billion zeros of the zeta function listed with increasing imaginary parts are all simple-all of which are related to the mathematics of this subject. Similarly Wikipedia of the web [2] offers an excellent summary along with references about this subject. The magazine Nature re-
cently published a related article about a discovery by Y. Zhang, of a conjecture (see [1]) on the spacing of prime numbers $p_{n}$ with increasing size [3]. Physicists have also published results on the Riemann hypothesis: in [4], Meulens compares data about the Riemann hypothesis with solutions of two dimensional Navier Stokes equations, while others [5] have compared eigenvalues of self-adjoint operators with zeros of the Riemann Zeta function. Several papers about solutions to the Riemann hypothesis have also appeared. To this end, the papers of Violi [6], Coranson-Beaudu, [7], Garcia-Morales, [8], and Chen [9] are similar to ours, in that their proof of the Riemann hypothesis are for functions that are different from the zeta function, but which have the same zeros in $D$ as the zeta function.

Castelvecchi, author of the article [3] makes the comment: "The Riemann hypothesis will probably remain at the top of mathematicians wish lists for many years to come. Despite its importance, no attempts so far have made much progress."

We wish of course to disagree with Castelvecchi's comment at the end of the above paragraph, since we believe that we have indeed proved the Riemann hypothesis in this self-contained paper, in which we accomplish the following:

1) In $\S 2$, defining the function $G$ and showing in detail that $G$ has exactly the same zeros, in the critical strip, $D:=\{z \in \mathbf{C}: 0<\mathfrak{R z}<1\}$, including multiplicity, as the zeta function;
2) Proving the positivity of $\mathfrak{R} G^{[2 m]}(\sigma)$ for $\sigma \in(0,1)$ and the negativity of $\mathfrak{R} G^{[2 m+1]}(\sigma)$ for $\sigma \in(0,1 / 2]$, where

$$
\begin{align*}
& \mathfrak{R} G^{[2 m]}(\sigma+i t):=\mathfrak{R}\left(\frac{\partial}{\partial \sigma}\right)^{2 m} \mathfrak{R} G(\sigma+i t) \\
& \mathfrak{R}\left(-i G^{[2 m+1]}(\sigma, i t)\right):=\mathfrak{R}\left(-i\left(\frac{\partial}{\partial \sigma}\right)^{2 m+1} G(\sigma+i t)\right) \tag{1.1}
\end{align*}
$$

3) Introducing the Schwarz reflection principle in $\S 3$, which the functions $G^{[2 m]}$ and $-i G^{[2 m+1]}$ of Equation (1.1) satisfy in $D$;
4) In $\S 4$, proving the Riemann hypothesis by contradiction, by use of results developed in $\S 2$, and in $\S 3$, of this paper, and by use of the trapezoidal and midordinate rules, [10], i.e., by proving that $G$ (and $\zeta$ ) have no zeros in the region $D \backslash L$, where $L$ denotes the critical line, $L:=\{z \in \mathbf{C}: \mathfrak{R} z=1 / 2\}$; and
5) Proving by contradiction, via use of results developed in $\S 2$ and in $\S 3$ and by use of the trapezoidal and midordinate rules that all of the zeros of $G$, (i.e., all of the zeros of the zeta function) on the critical line $L$ are simple.

Let $\mathbf{R}^{+}:=(0, \infty)$, and let $\mathbf{R}^{-}=(-\infty, 0)$. In this paper we thus derive results about the function $G$ defined by the integral,

$$
\begin{equation*}
G(z):=\int_{\mathrm{R}^{+}} \frac{y^{z-1}}{\mathrm{e}^{y}+1} \mathrm{~d} y, \mathfrak{R} z>0 \tag{1.2}
\end{equation*}
$$

which is related to the well-known integral for the Riemann zeta function, defined by

$$
\begin{equation*}
\zeta(z):=\frac{1}{\Gamma(z)} \int_{\mathbf{R}^{+}} \frac{y^{z-1}}{\mathrm{e}^{y}-1} \mathrm{~d} y, \mathfrak{R} z>1 \tag{1.3}
\end{equation*}
$$

where $\Gamma$ denotes the gamma function.
The operations of Schwarz reflection, the evaluation of $\mathfrak{R} G$ and $\mathfrak{J} G$ on important intervals of $\mathbf{R}$, and the operations of trapezoidal and midordinate quadrature can be readily applied to the Fourier transform representation of $G$, which is gotten from Equation (2) defined for $\mathfrak{R z}>0$, whereas an explicit Fourier transform of $\zeta$ defined by Equation (1.3) for $\mathfrak{R z}>1$ does not seem to be available.

## 2. Fourier Integral Representation of $G$, via $\kappa$

The function $\zeta$ has many other representations, with the best known of these given by:

$$
\begin{align*}
& \zeta(z):=\sum_{n=1}^{\infty} \frac{1}{n^{2}}, \mathfrak{M} z>1, \\
& \eta(z):=\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{2}}, \mathfrak{R z > 0 , \text { and }}  \tag{2.1}\\
& G(z):=\Gamma(z) \eta(z):=\left(1-2^{1-z}\right) \Gamma(z) \zeta(z), z \in \mathbf{C} .
\end{align*}
$$

By setting $y=\mathrm{e}^{x}$ and $z=\sigma+i t$ in (2.1), we get the Fourier integral representation of $G$, namely,

$$
\begin{equation*}
G(\sigma+i t):=\int_{\mathrm{R}} \kappa(\sigma, x) \mathrm{e}^{i x t} \mathrm{~d} x \tag{2.2}
\end{equation*}
$$

where $\sigma \in(0,1), t \in \mathbf{R}$, and where $\kappa$ is defined by

$$
\begin{equation*}
\kappa(\sigma, x):=\frac{\mathrm{e}^{\sigma x}}{1+\exp \left(\mathrm{e}^{x}\right)} \tag{2.3}
\end{equation*}
$$

### 2.1. Properties of $\kappa, \zeta$ and $G$

In this section we use the definition of $G$ given in Equation (1.2) and the identities of Equation (2.1) to derive a functional equation for $G$, and to derive additional properties of $\kappa$ and $G$. We also show in detail, that $\zeta$ and $G$ have exactly the same zeros in $D$, including multiplicity, that $\kappa(\sigma,-x)-\kappa(\sigma, x)$ is positive for all $(\sigma, x) \in(0,1 / 2] \times \mathbf{R}^{+}$and strictly decreasing as a function of $\sigma$, for $\sigma \in(0,1 / 2]$, and we determine ranges of values of $\mathfrak{R} G^{(m)}, \mathfrak{J} G^{(m)}$ and their derivatives on the real line.

Let us next assign notations for the left and right half of the complex plane, the critical strip(s), and the critical line.

Definition 2.1 Let $\mathbf{C}^{-}$denote the left half of the complex plane, i.e., $\mathbf{C}^{-}:=\{z \in \mathbf{C}: \mathfrak{R z}<0\}$, and let $\mathbf{C}^{+}:=\{z \in \mathbf{C}: \mathfrak{R z}>0\}$ denote the right half. Let the critical strip be defined by $D=\{z \in \mathbf{C}: 0<\mathfrak{R} z<1\}$, and let the negative and positive critical strips $D^{\mp}$ be defined as follows. $D^{-}:=\{z \in D: \Im z \leq 0\}$, and $D^{+}:=\{z \in D: \Im z \geq 0\}$. The critical line is defined by $L:=\{z \in D: \mathfrak{R} z=1 / 2\}$.

### 2.2. Relevant Gamma Function Relations

We shall require the use of the following lemma:
Lemma 2.2 (i.) Replacement of $z$ with $z / 2$ in the duplication formula for the Gamma function, to get:

$$
\Gamma(z)=(2 \pi)^{-1 / 2} 2^{z-1 / 2} \Gamma((z+1) / 2) \Gamma(z / 2) ;
$$

(ii.) Both $\Gamma(1 / 2+i x)$ and $\Gamma(1+i x)$ are bounded by $\pi^{1 / 2}$ for all $x \in \mathbf{R}$, by Equations (6.1.30) and (6.1.31) of [11]; and
(iii.) That the function $1 / \Gamma(z)$ is an entire function [11]; $\Gamma$ is analytic in C except for simple poles at $z=-n \quad(n=0,1,2, \cdots)$.

Proof. Item (i.) is just Equation (6.1.18) of [11] with $z$ replaced by $z / 2$;
Items (ii.) follow from Equations (6.1.30) and (6.1.31) of [11]; and Item (iii.) is just a restatement of a result found in Chapter 16. of [11].

### 2.3. Bounds on $\kappa$

The next lemma describes some asymptotic bounds on the function $\kappa$, which are obtained by inspection of Equation (2.2).

Lemma 2.3 For any $\varepsilon \in(0, \sigma) \subset(0,1) f$ and for $x$ real, we have

$$
\kappa(\sigma, x)=\left\{\begin{array}{l}
\mathcal{O}\left(\mathrm{e}^{(\sigma-\varepsilon) x}\right), x \rightarrow-\infty,  \tag{2.4}\\
\mathcal{O}\left(\exp \left(\sigma x-\mathrm{e}^{x(1-\varepsilon)}\right)\right), x \rightarrow \infty
\end{array}\right.
$$

Hence the integral $\int_{\mathrm{R}} Q(x) \kappa(\sigma, x) \mathrm{d} x$ is finite for any polynomial $Q$.
Proof. The bounds of $\kappa$ given in (2.4) follow by inspection of the function $\kappa$ as defined in (2.2).

### 2.4. Analyticity Definition of Multiplicity

Definition 2.4 Let $z_{0} \in \mathbf{C}$, let $m$ denote an integer, and let $f$ be analytic in a neighborhood of $z_{0}$.
(a.) The function fis said to have multiplicity $m$ at $z_{0}$ if
$\lim _{z \rightarrow z_{0}} f(z) /\left(z-z_{0}\right)^{m}=c$, with finite $c$,
(b.) If the multiplicity of $f$ at $z_{0}$ is $m$, and if $c \neq 0$, then we shall more specifically say that fis of exact multiplicity $m$ at $z_{0}$;
(c.) If $f$ is of exact multiplicity $m$ at $z_{0}$, then $z_{0}$ is said to be a zero (resp., a pole) of $f$ of multiplicity $m$ if $m>0$ (resp., if $m<0$ ). In particular, if $m=1$, (resp., if $m=-1$,) then $z_{0}$ is said to be a simple zero (resp., a simple pole) of $f$.

### 2.5. Functional Equations for $\zeta$ and $G$

An important identity of the Riemann zeta function is the well known functional equations for $\zeta$ :

$$
\begin{equation*}
\pi^{-(1-z) / 2} \Gamma((1-z) / 2) \zeta(1-z)=\pi^{-z / 2} \Gamma(z / 2) \zeta(z) \tag{2.5}
\end{equation*}
$$

This functional equation for the Riemann zeta function has many important uses, including, e.g., the analytic continuation of the zeta function to all of $\mathbf{C}$.

The function $G$ also possesses a functional equation which is given in Lemma 2.5 below, which plays a similar role as the functional equation for $\zeta$. the functional equation for $G$ is gotten by substituting the right-hand-side of the third equation of (2.1) into (2.5), and by use of Lemma 2.2:

Lemma 2.5 Let $z \in \mathbf{C}$, and let $G$ be defined as in (2.2). Then, a functional equation for the function $G$, valid for all $z \in \mathbf{C}$ is.

$$
\begin{equation*}
\frac{2^{1-z}-1}{(4 \pi)^{\frac{1-z}{2}} \Gamma\left(\frac{1+(1-z)}{2}\right)} G(1-z):=\frac{2^{z}-1}{(4 \pi)^{\frac{z}{2}} \Gamma\left(\frac{1+z}{2}\right)} G(z) \tag{2.6}
\end{equation*}
$$

This equation can also be written in the form:

$$
\begin{equation*}
G(1-z):=K(z) G(z) \tag{2.7}
\end{equation*}
$$

where $K$ is given by

$$
\begin{equation*}
K(z):=(4 \pi)^{1 / 2-z} \frac{2^{z}-1}{2^{1-z}-1} \frac{\Gamma\left(\frac{1+(1-z)}{2}\right)}{\Gamma\left(\frac{1+z}{2}\right)} \tag{2.8}
\end{equation*}
$$

and where $K$ is non-vanishing in $D$.
Proof. That $K$ is non-vanishing on $D$ follows from Lemma 2.2.

### 2.6. Zeros of $G$ and $\zeta$ in $D$

We prove here the $G$ and $\zeta$ have the same zeros with the same multiplicity in $D$ and that these zeros are isolated.

Lemma 2.6 (i.) The functions $G$ and $\zeta$ have exactly the same zeros in $D$, including multiplicity, and
(ii.) All zeros of $G$ in $\mathbf{C}$ are isolated.

Proof. (i.) By inspection if the third equation of (2.1) we get, if $z_{0} \in D$ is a zero of $G$ of multiplicity $k \geq 1$, then we have the identity $(*)$
$G(z)=w(z) \zeta(z)$, where the function $w(z):=\left(1-2^{1-z}\right) \Gamma(z)$ is analytic and non-vanishing in $D$, so that $z_{0}$ is also a zero of $\zeta$ of multiplicity $k \geq 1$. In addition, by taking the $n^{\text {th }}$ derivative of $(*)$, with non-negative integer, $n$, we get,

$$
\begin{equation*}
G^{(n)}(z)=(-1)^{n} \sum_{j=0}^{n}\binom{n}{j} w^{(n-j)}(z) \zeta^{(j)}(z) \tag{2.9}
\end{equation*}
$$

Hence, if $z_{0} \in \Omega^{+}$is a zero of $\zeta$ of multiplicity $m \geq 1$, then by applying induction with respect to $n=0,1,2, \cdots, m$ to Equation (2.9), we conclude that the multiplicity of the zero $z_{0}$ of $G$ is also $m$;
(ii.) Suppose that there exists a cluster of zeros $\left\{z_{j}\right\}_{j=1}^{\infty}$ of $G$ in $D$ with a sub-sequence that has a limit point $z$. If $z \in D$, then $G$ would have to vanish, by Vitali's theorem. If $z$ is on the line $\{\mathfrak{R} z=0\}$, then, since $D \subset \mathbf{C}^{+}$, and since $G$ is analytic in $\mathbf{C}^{+}$, it follows by use of the functional equation of $G$, that $0=G(z)=G(1-z)$ where the point $1-z$ is now located on the line $\{z \in \mathbf{C}: \mathfrak{R z}=1\}$, i.e., we are back to the previous case of the convergence of such
a sub-sequence to a point on the interior of the right half plane, where $G(z)$ is analytic and bounded, so that $G(z)$ would again have to vanish identically in C.

### 2.7. Definitions of $\kappa^{\mp}, G^{(m)}$ and $G^{[m]}$

Definition 2.7 Let $G$ and $\kappa$ be defined as in Equation (2.2), and let us define $\kappa^{\mp}(\sigma, x)$ as follows:

$$
\kappa^{\mp}(\sigma, x):=\left\{\begin{array}{l}
\kappa(\sigma, \mp x), x \in \mathbf{R}^{+}  \tag{2.10}\\
0, x \in \mathbf{R}^{-}
\end{array}\right.
$$

If for brevity, we write $\kappa^{\mp}$ for $\kappa^{\mp}(\sigma, x)$, C and S for $\cos (x t)$ and $\sin (x t)$, and $\int$ for $\int_{\mathbb{R}^{+}}$, then Equation (2) yields the following definitions for $G^{(2 n)}$ and for $G^{(2 n+1)}$, where $n$ denotes a non-negative integer:

$$
\begin{align*}
& G^{(2 n)}(\sigma+i t)=(-1)^{n} \int x^{2 n}\left(\left(\kappa^{-}+\kappa^{+}\right) C-i\left(\kappa^{-}-\kappa^{+}\right) S\right) \mathrm{d} x  \tag{2.11}\\
& G^{(2 n+1)}(\sigma+i t)=(-1)^{n+1} \int x^{2 n+1}\left(\left(\kappa^{-}+\kappa^{+}\right) S+i\left(\kappa^{-}-\kappa^{+}\right) C\right) \mathrm{d} x .
\end{align*}
$$

Let $G^{[m]}$ be defined for any non-negative integer $m$ by $G^{[m]}(\sigma+i t)=\left(\frac{\partial}{\partial \sigma}\right)^{m} G(\sigma+i t)$, so that by the Cauchy-Riemann equations, $G^{(n)}(\sigma+i t)=i^{m}\left(\frac{\partial}{\partial(i t)}\right)^{m} G(\sigma+i t)=i^{m} G^{[m]}(\sigma+i t)$, where these functions are readily shown to exist, by Lemma 2.3.

In addition, by Equation (2.1), the functions $K^{[m]}$ and $G^{[m]}$ are related by the following identity:

$$
\begin{equation*}
G^{[m]}(1-z)=(-1)^{m} \sum_{j=0}^{m}\binom{m}{j} K^{[m-j]}(z) G^{[j]}(z) \tag{2.12}
\end{equation*}
$$

Lemma 2.8 Let the functions $G^{[m]}$ be defined as in Definition 2.7. Then, for all $m=0,1,2, \cdots$, and for all $\sigma \in(0,1), G^{(m)}$ is analytic on the right half plane, and hence also in $D$. In particular given any $\varepsilon>0, G^{(m)}(z)$ is uniformly bounded in the region $\{z \in D: \mathfrak{R} z \geq \varepsilon\}$.

Proof. This result follows directly by inspection of Equation (2.2 and Lemma 2.3. We omit the straight-forward proofs.

### 2.8. Restricting the Domain of $\mathfrak{J} G^{(2 m+1)}$

The following lemma restricts the domain of some of our inequalities:
Lemma 2.9 Let $\Delta$ be defined by $\Delta(\sigma, x):=\kappa^{-}(\sigma, x)-\kappa^{+}(\sigma, x)$, where the functions $\kappa^{\mp}$ are defined in Definition 2.7. Then $\Delta(\sigma, x)>0$ for all $(\sigma, x) \in(0,1 / 2] \times \mathbf{R}^{+}$, and moreover, $\Delta(\sigma, x)$ is a strictly decreasing function of $\sigma \in(0,1 / 2]$ for any fixed $x \in \mathbf{R}^{+}$.

Proof. We have

$$
\frac{\partial}{\partial \sigma} \Delta(\sigma, x)=-x\left(\kappa^{-}(\sigma, x)+\kappa^{+}(\sigma, x)\right)
$$

which shows $\Delta(\sigma, x)$ is a strictly decreasing function of $\sigma \in(0,1 / 2]$ for all fixed $x \in \mathbf{R}^{+}$. By making the one-to-one transformation $y=\mathrm{e}^{-x / 2}$ of $\mathbf{R}^{+} \rightarrow(0,1)$ in the above expressions for $\Delta(\sigma, x)$, and setting $W:=\left(1+\mathrm{e}^{y^{2}}\right)\left(1+\mathrm{e}^{1 / y^{2}}\right)$, we get

$$
\begin{aligned}
& \Delta(\sigma, x):=\delta(\sigma, y) / W \\
& \delta(\sigma, y):=y^{2 \sigma}\left(1+\exp \left(1 / y^{2}\right)\right)-1 / y^{2 \sigma}\left(1+\exp \left(y^{2}\right)\right)
\end{aligned}
$$

Since $1 / W$ is positive on $(0,1) \times \mathbf{R}^{+}$, and since $\Delta(\sigma, x)$ is a strictly decreasing function of $\sigma \in(0,1 / 2]$ for all fixed $x \in \mathbf{R}^{+}$, we need only prove that $\delta(1 / 2, y)>0$ for all $y \in(0,1)$. To this end we have, by use of Taylor series expansions, that

$$
\begin{align*}
\delta(1 / 2, y) & =y\left(1+\mathrm{e}^{1 / y^{2}}\right)-1 / y\left(1+\mathrm{e}^{y^{2}}\right), y \in(0,1) \\
& =(1 / 2)(1 / y-y)\left(1 / y^{2}+1+y^{2}-2\right)+\sum_{n=3}^{\infty} \frac{1 / y^{2 n-1}-y^{2 n-1}}{n!}>0, y \in(0,1) \tag{2.13}
\end{align*}
$$

By way of proceeding from the first to the second line of Equation (2.13) we used the following relations, which are valid for all $y \in(0,1): 0<y<1<1 / y$. The right hand side of Equation (13) then shows that $\delta(1 / 2, y)>0$ for all $y \in(0,1)$, i.e., that $\Delta(\sigma, x)>0$ for all $(\sigma, x) \in(0,1 / 2] \times \mathbf{R}^{+}$.

### 2.9. Inequalities for $G^{(m)}(\sigma)$

The following lemma summarizes values of $G^{(m)}(\sigma)$ that have been established.

Lemma 2.10 Let $m$ denote a non-negative integer. Then:
(i.) $(-1)^{m} \mathfrak{R} G^{(2 m)}(\sigma)=\mathfrak{R} G^{[2 m]}(\sigma)>0$ for all $\sigma \in(0,1)$;
(ii.) $(-1)^{m} \mathfrak{I} G^{(2 m+1)}(\sigma)=\mathfrak{R} G^{[2 m+1]}(\sigma)<0$ for all $\sigma \in(0,1 / 2]$; and
(iii.) $\mathfrak{I} G^{(2 m)}(\sigma)=\mathfrak{J} G^{[2 m]}(\sigma)=\mathfrak{J} G^{[2 m+1]}(\sigma)=\mathfrak{R} G^{(2 m+1)}(\sigma)=0$ for all $\sigma \in(0,1)$.

Proof. Item (i.) The proof of this Item follows by inspection of Equation (2.11);

Item (ii.) The proof of this Item follows by Lemma 2.9 and by inspection of Equation (2.11); and

Item (iii.) The proof of this item follows by inspection of Equation (2.11).

## 3. Schwarz Reflection

We present the Schwarz reflection principle, which we define as follows:
Definition 3.1 Let $f$ be analytic in $D$, and real on $(0, a)$, for some $a \in[1 / 2,1)$. Then $f$ can be continued analytically (i.e., reflected) across $(0, a)$ from $D^{\mp}$ to $D^{ \pm}$by means of the formula

$$
\begin{equation*}
f(\bar{z})=\overline{f(z)} \tag{3.1}
\end{equation*}
$$

Remark 3.2 The Schwarz reflection principle enables analytic continuation from $D^{\mp}$ to all of $D$. For example, if $n$ denotes a non-negative integer, so that the functions $f\left(z^{+}\right):=G^{(2 n)}\left(z^{+}\right)$and $g\left(z^{+}\right):=-i G^{(2 n+1)}\left(z^{+}\right)$are given for
$z^{+}=\sigma+i t^{+} \in D^{+}$, then by Lemma 2.10, and by Equation (2.8), $\mathfrak{R} f(\sigma)$ is a non-vanishing function of $\sigma$ on $(0,1)$, while if $\sigma \in(0,1 / 2]$, then $\mathfrak{R g}(\sigma)$ is a non-vanishing function of $\sigma, \mathfrak{J} G(\sigma+i t)$ changes sign as $t$ changes sign, while $\mathfrak{R} G(\sigma+i t)$ does not change sign as $t$ changes sign.

## 4. Proof of the Riemann Hypothesis

Short proofs of all of all of the results which stated in the abstract of this paper are made possible by means of two well-known methods of quadrature (see e.g., [10]), which are defined by the following lemma.

Lemma 4.1 Let $f$ be a real-valued function that is continuous on a finite interval $[a, b]$ of $\mathbf{R}$, and twice differentiable in $(a, b)$, let $h:=(a+b) / 2$, and let us set

$$
\begin{equation*}
I(f):=\int_{a}^{b} f(x) \mathrm{d} x, T(f):=h(f(a)+f(b)), M(f):=2 h f((a+b) / 2) \tag{4.1}
\end{equation*}
$$

Then there exist points $\xi_{T}$ and $\xi_{M}$ in $(a, b)$, such that

$$
\begin{align*}
& I(f)-T(f):=-\frac{(2 h)^{3}}{12} f^{\prime \prime}\left(\xi_{T}\right)  \tag{4.2}\\
& I(f)-M(f):=\frac{(2 h)^{3}}{24} f^{\prime \prime}\left(\xi_{M}\right)
\end{align*}
$$

The first equation of (4.2) denotes the simplest trapezoidal rule, while the second denotes the simplest midordinate rule.

Theorem 4.2 Every zero of $G$ in the critical strip $D=\{z \in \mathbf{C}: 0<\mathfrak{R} z<1\}$ is located on the critical line $\{z \in D: \mathfrak{R z}=1 / 2\}$, and it is a simple zero.

Proof. We shall now carry out the proof of Theorem 4.2 by means of the proof of the two lemmas, which follow.

Lemma 4.3 If $z_{1}=\sigma_{1}+i t_{1} \in D$, with $\sigma_{1} \neq 1 / 2$ and with $t_{1} \in \mathbf{R}$, then $G\left(z_{1}\right) \neq 0$.

Proof. Let $z_{1}:=\sigma_{1}+i t_{1}$, with $\sigma_{1} \in(0,1 / 2)$ and with $t_{1} \in \mathbf{R}$, and let us set $z_{2}:=\overline{z_{1}}, \quad z_{3}:=1-z_{1}$, and $z_{4}:=\overline{z_{3}}$. If for $j=1,2,3$, and $4, G\left(z_{j}\right)$ vanishes at one of these points, then by the functional equation of Lemma 2.5, and by Schwarz reflection, $G\left(z_{j}\right)$ vanishes at all of them.

If we assume that $G\left(z_{1}\right)$ vanishes then $\mathfrak{R} G\left(z_{1}\right)$ and $\mathfrak{R} G\left(z_{4}\right)$ vanish, so that we have, by integration of $\mathfrak{R} G$ over $\ell\left(z_{1}, z_{4}\right)$, that

$$
\begin{equation*}
I(\Re G):=\int_{\ell\left(z_{1}, z_{4}\right)} \mathfrak{R} G(z) \mathrm{d} z=\int_{\sigma_{1}}^{1-\sigma_{1}} \mathfrak{R} G\left(s+i t_{1}\right) \mathrm{d} s \tag{4.3}
\end{equation*}
$$

Thus, in the notation of Definition 2.7 and Lemma 4.1, with $h=\left(1-2 \sigma_{1}\right) / 2$, and for some $\sigma_{T}$ and $\sigma_{M} \in\left(\sigma_{1}, 1-\sigma_{1}\right)$, we get,

$$
\begin{align*}
& I(\Re G)-T(\Re G)=-\frac{(2 h)^{3}}{12} \mathfrak{\Re} G^{[2]}\left(\sigma_{M}+i t_{1}\right)=0  \tag{4.4}\\
& I(\Re G)-M(\Re G)=\frac{(2 h)^{3}}{24} \mathfrak{R} G^{[2]}\left(\sigma_{M}+i t_{1}\right)=0
\end{align*}
$$

By taking the difference between the two equations of (4.4), forming the aver-
age of this difference and its complex conjugate, noting that $T(\Re G)=0$, and that $-1 / 3\left(2 \Re G^{[2]}\left(\sigma_{T}\right)+\Re G^{[2]}\left(\sigma_{M}\right)\right)=-\Re G^{[2]}\left(\sigma_{A}\right)$, a weighted average for some $\sigma_{A} \in\left(\sigma_{1}, 1-2 \sigma_{1}\right)$, we get,

$$
\begin{equation*}
2 h \Re G(1 / 2)=-h^{3} \mathfrak{R} G^{[2]}\left(\sigma_{A}\right) \tag{4.5}
\end{equation*}
$$

However, this is a contradiction, since the left and right hand side of Equation (4.5) are different in sign, by Lemma 2.10.

Hence we cannot allow the vanishing of $\mathfrak{R} G\left(z_{1}\right)$, i.e., of $G\left(z_{1}\right)$, so that our above assumption of the vanishing of $G\left(z_{1}\right)$ is false.

This completes the proof of Lemma 4.3.
Remark 4.4 The Riemann hypothesis is true, by Lemma 4.3.
Lemma 4.5 If $z_{0}=1 / 2+i t_{0}$ is a zero of $G$ on the critical line, then $z_{0}$ is a simple zero of $G$.

Proof. (i.) Let us assume that $G\left(z_{0}\right)$ vanishes with multiplicity $2 n+2$, where $n$ denotes an arbitrary finite non-negative integer, and let us apply the trapezoidal rule of Lemma 4.1 to the integration of $\mathfrak{J} G^{(2 n+1)}$ over $\ell\left(\overline{z_{0}}, z_{0}\right)$. We then have $h=t_{0}$, and since by Definition 2.4, both $G^{(2 n)}\left(z_{0}\right)$ and $G^{(2 n+1)}\left(z_{0}\right)$ must vanish, $I\left(\mathfrak{J} G^{(2 n+1)}\left(z_{0}\right)\right)$ and $T\left(\mathfrak{I} G^{(2 n+1)}\left(z_{0}\right)\right)$ must also vanish. We thus get, for some $t_{T} \in\left(-t_{0}, t_{0}\right)$, that

$$
\begin{equation*}
0=-\frac{(2 h)^{3}}{12} \Im G^{(2 n+3)}\left(1 / 2+i t_{T}\right) \tag{4.6}
\end{equation*}
$$

By averaging of this equation and it's complex conjugate, thus eliminating the possible imaginary part on the right hand side, we get

$$
\begin{equation*}
0=-\frac{(2 h)^{3}}{12} \mathfrak{J} G^{(2 n+3)}(1 / 2) \tag{4.7}
\end{equation*}
$$

Since the left hand side of Equation (4.7) vanishes whereas, by Lemma 2.10, the right hand side does not, this equation provides a contradiction, which tells us that we cannot allow the vanishing of $\mathfrak{R} G\left(z_{0}\right)$, i.e., of $G\left(z_{0}\right)$, with multiplicity $2 n+2$, where $n$ denotes an arbitrary finite non-negative integer.
(ii.) Similarly, if $G\left(z_{0}\right)$ vanishes with multiplicity $2 n+1$, with arbitrary finite positive integer $n$, then both $G^{(2 n-1)}\left(z_{0}\right)$ and $G^{(2 n)}\left(z_{0}\right)$ must vanish, so that, by proceeding as we did to arrive at Equation (7), we get

$$
\begin{equation*}
0=-\frac{(2 h)^{3}}{12} \mathfrak{R} G^{(2 n+2)}(1 / 2) \tag{4.8}
\end{equation*}
$$

The left hand side this equation vanishes whereas by Lemma 2.10, the right hand side does not, so that Equation (4.8) presents a contradiction. Together with our conclusion for Equation (4.7), this proves that we cannot allow the vanishing of $G\left(z_{0}\right)$ with multiplicity $m>1$, with $m$ denoting an arbitrary positive integer.

It was shown in [12] that the Riemann Zeta function has an infinite number of zeros on the critical line $L$, and by Lemma 4.3 above there are no zeros of $G$ in $D L$. It thus follows, that every zero $z_{0}$ of $D$ must be a simple zero on $L$.

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## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

## References

[1] Bombieri, E. (2014) The Riemann Hypothesis-Official Problem Description, Clay Mathematical Institute.
[2] https://en.wikipedia.org/wiki/riemann_zeta_function
[3] Castelvetcchi, D. (2022) Mathematician Claims Prime Number Riddle Breakthrough. Nature, 611, 645-646. https://doi.org/10.1038/d41586-022-03689-2
[4] Meulens, R. (2019) The Proof of the Riemann Hypothesis and an Application to Physics. Applied Mathematics, 10, 967-988. https://doi.org/10.4236/am.2019.1011068
[5] Endres, S. and Steiner, F. (2009) The Berry-Keating Operator on $L^{2}(R)$ and on Compact Quantum Graphs with General self-Adjoint Realizations. Journal of Physics A: Mathematical and Theoretical, 43, 095204.
[6] Violi, R. (2020) All Complex Zeros of the Riemann Zeta Function Are on the Critical Line. https://arxiv.org/pdf/2010.05335.pdf
[7] Coranson-Beaudu, J.-M. (2021) A Speedy New Proof of the Riemann Hypothesis. Pure and Applied Mathematics Journal, 10, 62-67.
[8] Garcia, M. (2007) On the Non-Trivial Zeros of the Dirichlet eta Function. arXiv 2007.04317v1(220).
[9] Chen, C. (2022) Proof of Riemann Conjecture. APM, 12, 374-391. https://doi.org/10.4236/apm.2022.125028
[10] https://mathweb.ucsd.edu/ebender/20B77_Trap.pdf
[11] Abramowitz, A. and Stegun, I.A. (1964) Handbook of Mathematical Functions. National Bureau of Standards Applied Mathematical Series, 55, 1964.
[12] Hardy, G.H. and Littlewood, J.E. (1917) Contributions to the Theory of the Riemann Zeta-Function and the Theory of the Distribution of Primes. Acta Mathematica, 41, 119-196.

