Advances in Pure Mathematics, 2023, 13, 267-283
https://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

Some Characterizations of Upper and Lower M-Asymmetric Preirresolute Multifunctions

Levy Kahyata Matindih[1*], Elijah Bwanga Mpelele[1,2], Jimmy Hambulo[1,2], Davy Kabuswa Manyika[2]

[1]Department of Mathematics & Statistics, Mulungushi University, Kabwe, Zambia
[2]Department of Physics, Mulungushi University, Kabwe, Zambia
Email: *kahyamatindih99@gmail.com

How to cite this paper: Matindih, L.K., Mpelele, E.B., Hambulo, J. and Manyika, D.K. (2023) Some Characterizations of Upper and Lower M-Asymmetric Preirresolute Multifunctions. *Advances in Pure Mathematics*, **13**, 267-283.
https://doi.org/10.4236/apm.2023.135019

Received: April 24, 2023
Accepted: May 27, 2023
Published: May 30, 2023

Abstract

In this present paper, we introduce and investigate a new form of mappings namely; upper and lower M-asymmetric preirresolute multifunctions defined between M-structural asymmetric topological spaces. The relationships between the multifunctions in our sense and other types of precountinuous and preirresolute multifunctions defined on both symmetric and asymmetric topological structures are discussed.

Keywords

Asymmetric-Preopen Sets, M-Space, M-Asymmetric Preopen Sets, Upper (Lower) Preirresolute Multifunctions, Upper (lower) M-Asymmetric Preirresolute Multifunctions

1. Introduction

The notion of continuity and multifunctions, the basic concepts in the theory of classical point set topology that plays a vital role not only in the realm of functional analysis but also in other branches of applied science, such as; engineering, control theory, mathematical economics, and fuzzy topology has received considerable attention by many scholars. In this regard, there have been various generalizations of the notion of continuity for functions and multifunctions both in topological and bitopological spaces using the weaker forms of sets such as semiopen, preopen, α-open, β-open, γ-open, ω-open and δ-open sets.

In the realm of topological spaces, the concept of semiopen sets and semicontinuous functions was first introduced by Levine [1] and the concept was then extended by Maheshwari and Prasad [2] to the realm of bitopological spaces.

Further, Bose [3] investigated several properties of semi-open sets and semi-continuity in bitopological spaces. On the other hand, Berge [4] introduced and investigated the notion of upper and lower continuous multifunctions and lately, this notion was generalized to the settings of bitopological spaces by Popa [5], in which he studied how the conserving properties of connectedness, compactness and paracompactness are preserved by multifunctions between bitopological spaces. Noiri and Popa [6] in 2000, then introduced and studied the concept of upper and lower *M*-continuous multifunctions as an extension of upper (lower) continuous multifunction and *M*-continuous function deal to Berge [4] and, Popa and Noiri [7] respectively. They observed that, upper (lower) continuity of multifunctions has properties similar to those of upper (lower) continuous functions and continuous multifunctions on topological spaces. Recently, Matindih and Moyo [8] have generalized [6] ideas and studied *M*-asymmetric semicontinuous multifunctions and showed that, these kinds of mappings have properties similar to those of upper (lower) continuous functions and *M*-continuous multifunctions between topological spaces, with the difference that, the semiopen sets in use are asymmetric.

Mashhour *et al.* [9] in 1982, introduced and investigated a new form of open sets and continuity called preopen sets and precontinuous functions in the realm of topological spaces. They showed that, general openness and continuity implies preopeness and precontinuity and the reverse does not generally hold. This concept of preopen sets and precontinuity was then generalized to the setting of bitopological spaces by Jelić [10] and Khedr *et al.* [11] respectively. And, as an extension to the results in [9], Min and Kim [12] have recently introduced and investigated some basic properties of m-preopen sets and M-precontinuity on spaces with minimal structures. On the other hand, Boonpok *et al.* [13] have gone further to extend the results by studying a new form of mapping namely; $\left(\mathscr{T}_1,\mathscr{T}_2\right)$-precontinuous multifunctions in bitopological spaces and obtained several characterizations.

Irresolute functions and their fundamental properties on the other hand, were first introduced and investigated by Crossley and Hildebrand [14] in 1972. They observed that, irresolute functions are generally not continuous and neither are continuous functions necessarily irresolute. Ewert and Lipski [15], on the other hand, extended this concept to upper and lower irresolute multivalued mappings, followed by Popa [16] who investigated some characteristics of upper and lower irresolute multifunctions in topological spaces and, extended the results to study upper and lower preirresolute multifunctions in [17]. However, Matindih *et al.* [18] have recently generalized the results deal to Popa [16], and investigated a new form of mappings the upper and lower *M*-asymmetric irresolute multifunctions in bitopologgical spaces. They have shown that, upper and lower *M*-asymmetric irresolute multifunctions have properties similar to those of upper and lower irresolute multifunctions defined between topological spaces. Furthermore, they showed that, such mappings are respectively upper and lower *M*-asymmetric semicontinuous, but, the converse is not necessarily true.

In this paper, we generalize the idea deal to Popa and *et al.* [17] to introduce and investigate a new form of mappings namely; upper and lower *M*-asymmetric preirresolute multifunctions defined on bitopological spaces satisfying certain minimal conditions. Furthermore, the relationships between these multifunctions and other types of irresolute multifunctions will be discussed.

The organization of this paper is as follows. Section 2 presents necessary preliminaries concerning preopen sets, *m*-preopen sets and precontinuous and preirresulute multifunctions. In Section 3, we generalize the notions of upper and lower *M*-asymmetric irresolute multifunctions deal to Matindih *et al.* [18] and, upper and lower *M*-preirresolute multifunctions deal to Papa *et al.* [17] to minimal bitopological structured spaces. Section 4 outlines the concluding remarks.

2. Preliminaries and Basic Properties

We present in this section some important properties and notations to be used in this paper. For more details, we refer the reader to [2] [3] [8] [9] [10] [11] [16] [17] [19] [20] [21].

By a bitopological space $(X,\mathscr{T}_1,\mathscr{T}_2)$, in the sense of Kelly ([20]), we imply a nonempty set X on which are defined two topologies $\mathscr{T}_1$ and $\mathscr{T}_2$ and the left and right topologies respectively.

In sequel, $(X,\mathscr{T}_1,\mathscr{T}_2)$ or in shorthand X will denote a bitopological space unless where clearly stated. For a bitopological space $(X,\mathscr{T}_i,\mathscr{T}_j)$, $i,j=1,2$; $i\neq j$, the interior and closure of a subset E of X with respect to the topology $\mathscr{T}_i=\mathscr{T}_j$ shall be denote by $Int_{\mathscr{T}_i}(E)$ and $Cl_{\mathscr{T}_i}(E)$ respectively.

Definition 2.1. Let $(X,\mathscr{T}_i,\mathscr{T}_j)$, $i,j=1,2$; $i\neq j$ be a bitopological space and E be any subset of *X*.

1) E is said to be $\mathscr{T}_i\mathscr{T}_j$-open if $E\in\mathscr{T}_i\cup\mathscr{T}_j$; *i.e.*, $E=E_i\cup E_j$ where $E_i\in\mathscr{T}_i$ and $E_j\in\mathscr{T}_j$. The complement of an $\mathscr{T}_i\mathscr{T}_j$-open set is a $\mathscr{T}_i\mathscr{T}_j$-closed set.

2) The $\mathscr{T}_i\mathscr{T}_j$-interior of E denoted by $Int_{\mathscr{T}_i}\left(Int_{\mathscr{T}_j}(E)\right)$ (or $\mathscr{T}_i\mathscr{T}_j\text{-}Int(E)$) is the union of all $\mathscr{T}_i\mathscr{T}_j$-open subsets of X contained in *A*. Evidently, provided $E=Int_{\mathscr{T}_i}\left(Int_{\mathscr{T}_j}(E)\right)$, then E is $\mathscr{T}_i\mathscr{T}_j$-open.

3) The $\mathscr{T}_i\mathscr{T}_j$-closure of E denoted by $Cl_{\mathscr{T}_i}\left(Cl_{\mathscr{T}_j}(E)\right)$ is defined to be the intersection of all $\mathscr{T}_i\mathscr{T}_j$-closed subsets of X containing *A*. Note that asymmetrically, $Cl_{\mathscr{T}_i}\left(Cl_{\mathscr{T}_j}(E)\right)\subseteq Cl_{\mathscr{T}_i}(E)$ and $Cl_{\mathscr{T}_i}\left(Cl_{\mathscr{T}_j}(E)\right)\subseteq Cl_{\mathscr{T}_j}(E)$.

Definition 2.2. Let $(X,\mathscr{T}_i,\mathscr{T}_j)$, $i,j=1,2$; $i\neq j$ be a bitopological space and, E and D be any subsets of *X*.

1) A is said to be $\mathscr{T}_i\mathscr{T}_j$-preopen in X if there exists a $\mathscr{T}_i$-open set O such that $E\subseteq U\subseteq Cl_{\mathscr{T}_j}(E)$, equivalently $E\subseteq Int_{\mathscr{T}_i}\left(Cl_{\mathscr{T}_j}(E)\right)$. It's complement is said to be $\mathscr{T}_i\mathscr{T}_j$-preclosed. A subset E is $\mathscr{T}_i\mathscr{T}_j$-preclosed if $Cl_{\mathscr{T}_i}\left(Int_{\mathscr{T}_j}(E)\right)\subset E$.

2) The $\mathscr{T}_i\mathscr{T}_j$-preinterior of E denoted by $\mathscr{T}_i\mathscr{T}_j\text{-}pInt(E)$ is defined to be

the union of $\mathscr{T}_i\mathscr{T}_j$-propen subsets of X contained in E. The $\mathscr{T}_i\mathscr{T}_j$-preclosure of E denoted by $\mathscr{T}_i\mathscr{T}_j$-$pCl(E)$, is the intersection of all $\mathscr{T}_i\mathscr{T}_j$-preclosed sets of X containing E.

3) D is said to be a $\mathscr{T}_i\mathscr{T}_j$-pre-neighbourhood of $x \in X$ if there is some $\mathscr{T}_i\mathscr{T}_j$-preopen subset O of X such that $x \in O \subseteq D$.

The family of all $\mathscr{T}_i\mathscr{T}_j$-preopen and $\mathscr{T}_i\mathscr{T}_j$-preclosed subsets of X are will be denote by $\mathcal{T}_i\mathcal{T}_j - pO(X)$ and $\mathcal{T}_i\mathcal{T}_j - pC(X)$ respectively.

Definition 2.3. [6] [19] A subfamily m_X of a power set $\mathscr{P}(X)$ of a set $X \neq \varnothing$ is said to be a minimal structure (briefly m-structure) on X if both $\varnothing$ and X lies in m_X. The pair (X, m_X) is called an m-space and the members of (X, m_X) is said to be m_X-open.

Definition 2.4. Let $(X, \mathscr{T}_i, \mathscr{T}_j)$, $i, j = 1,2$; $i \neq j$ be a bitopological space and m_X a minimal structure on X generated with respect to m_i and m_j. An ordered pair $((X, \mathscr{T}_i, \mathscr{T}_j), m_X)$ is called a minimal bitopological space.

Since the minimal structure m_X is determined by the left and right minimal structures m_i and m_j, $i, j = 1,2$; $i \neq j$, we shall denote it by $m_{ij}(X)$ (or simply $m_{ij}(X)$ in the sense of Matindih and Moyo [8], and call the pair (X, m_{ij})) a minimal bitopological space unless explicitly defined.

Definition 2.5. A minimal structure $m_{ij}(X)$, $i, j = 1,2$; $i \neq j$, on on X is said to have property ($\mathscr{B}$) of Maki [19] if the union of any collection of $m_{ij}(X)$-open subsets of X belongs to $m_{ij}(X)$.

Definition 2.6. Let (X, m_{ij}), $i, j = 1,2$; $i \neq j$ be a bitopological space having minimal condition. The , E a subset of X is said to be:

1) $m_{ij}(X)$-preopen if there exists an m_i-open set O such that $E \subseteq O \subseteq Cl_{m_j}(E)$ or equivalently, $E \subseteq Int_{m_i}(Cl_{m_j}(E))$.

2) $m_{ij}(X)$-preclosed if there exists an m_i-open set O such that $Cl_{m_j}(O) \subseteq E$ whenever $E \subseteq O$, that is, $Cl_{m_i}(Int_{m_j}(E)) \subseteq E$.

We shall denote the collection of all m_{ij}-preopen and m_{ij}-preclosed sets in $(X, m_{ij}(X))$ by $m_{ij}pO(X)$ and $m_{ij}pC(X)$ respectively.

Remark 2.7. Let $(X, m_{ij}(X))$, $i, j = 1,2$; $i \neq j$ be a bitopological space having a minimal condition.

1) If $m_i = \mathscr{T}_i$ and $m_j = \mathscr{T}_j$, the any $m_{ij}(X)$-preopen set is $\mathscr{T}_i\mathscr{T}_j$-preopen.

2) Every $m_{ij}(X)$-open set is $m_{ij}(X)$-preopen, however, the converse is not necessarily true.

It should be understood that, m_{ij}-open sets and the m_{ij}-preopen sets are not stable for the union. However, for certain m_{ij}-structures, the class of m_{ij}-preopen sets are stable under union of sets, as in the Lemma below.

Lemma 2.8. *Let* $(X, m_{ij}(X))$, $i, j = 1,2$; $i \neq j$ *be an* m_{ij}*-space and* $\{E_\gamma : \gamma \in \Gamma\}$ *be a family of subsets of X. Then, the properties below hold*:

1) $\bigcup_{\gamma \in \Gamma} E_\gamma \in m_{ij}pO(X)$ *provided for all* $\gamma \in \Gamma$, $E_\gamma \in m_{ij}pO(X)$.

2) $\bigcap_{\gamma \in \Gamma} E_\gamma \in m_{ij}pC(X)$ *provided for all* $\gamma \in \Gamma$, $E_\gamma \in m_{ij}pC(X)$.

Remark 2.9. It should generally be noted that, the intersection of any two m_{ij}-preopen sets may not be m_{ij}-preopen in a minimal bitopological space $(X, m_{ij}(X))$.

Definition 2.10. Let $(X, m_{ij}(X))$, $i, j = 1,2$; $i \neq j$ be an m_{ij}-space. A subset:

1) N of X is an m_{ij}-preneighborhood of a point x of X if there exists an m_{ij}-preopen subset O of X such that $x \in O \subseteq N$.

2) U of X is an m_{ij}-preneighborhood of a subset E of X if there exists an m_{ij}-preopen subset O of X such that $A \subseteq O \subseteq N$.

Definition 2.11. Let $(X, m_{ij}(X))$, $i, j = 1,2$; $i \neq j$ be an m_{ij}-space and E a non-empty subset of X. Then, we denoted and defined the m_{ij}-preinterior and m_{ij}-preclosure of E respectively by:

1) $m_{ij}(X)pInt(E) = \bigcup\{U : U \subseteq A \text{ and } U \in m_{ij}pO(X)\}$,

2) $m_{ij}(X)pCl(E) = \bigcap\{F : E \subseteq F \text{ and } X \setminus F \in m_{ij}pO(X)\}$.

Remark 2.12. For any bitopological spaces $(X, \mathcal{T}_1, \mathcal{T}_2)$:

1) $\mathcal{T}_i\mathcal{T}_j pO(X)$ is a minimal structure of X.

2) In the following, we denote by m_{ij} a minimal structure on X as a generalization of $\mathcal{T}_i$ and $\mathcal{T}_j$. For a nonempty subset A of X, if $m_{ij}(X) = \mathcal{T}_i\mathcal{T}_j pO(X)$, then by Definition 2.11:

a) $m_{ij}Int(E) = \mathcal{T}_i\mathcal{T}_j pInt(E)$,

b) $m_{ij}Cl(E) = \mathcal{T}_i\mathcal{T}_j pCl(E)$.

Lemma 2.13. *Let $(X, m_{ij}(X))$, $i, j = 1,2$; $i \neq j$ be an m_{ij}-space and A and B be subsets of X. The following properties of m_{ij}-preinterior and m_{ij}-preclosure holds*:

1) $m_{ij}(X)pInt(E) \subseteq A$ and $m_{ij}pCl(E) \supseteq E$.

2) $m_{ij}(X)pInt(E) \subseteq m_{ij}pInt(B)$ and $m_{ij}(X)pCl(E) \subseteq m_{ij}pC(B)$ provided $E \subseteq B$.

3) $m_{ij}(X)pInt(\varnothing) = \varnothing$, $m_{ij}(X)pInt(X) = X$, $m_{ij}(X)pCl(\varnothing) = \varnothing$ and $m_{ij}(X)pCl(X) = X$.

4) $A = m_{ij}(X)pInt(E)$ provided $A \in m_{ij}pO(X)$.

5) $A = m_{ij}(X)pCl(E)$ provided $X \setminus E \in m_{ij}pO(X)$.

6) $m_{ij}(X)pInt(m_{ij}(X)pInt(E)) = m_{ij}(X)pInt(E)$.

And $m_{ij}(X)pCl(m_{ij}(X)pCl(E)) = m_{ij}(X)pCl(E)$.

Lemma 2.14. *Let $(X, m_{ij}(X))$, $i, j = 1,2$; $i \neq j$ be an m_{ij}-space and E a non-empty subset of X. For each $U \in m_{ij}pO(X)$ containing x_o, $U \cap E \neq \varnothing$ if and only if $x_o \in m_{ij}pCl(E)$.*

Lemma 2.15. *Let $(X, m_{ij}(X))$, $i, j = 1,2$; $i \neq j$ be an m_{ij}-space and E be a nonempty subset of X. The properties below holds*:

1) $m_{ij}(X)pCl(X \setminus E) = X \setminus (m_{ij}(X)pInt(E))$,

2) $m_{ij}(X)pInt(X \setminus E) = X \setminus (m_{ij}(X)pCl(E))$.

Lemma 2.16. *Let $(X, m_{ij}(X))$, $i, j = 1,2$; $i \neq j$ be an m_{ij}-space and E be a nonempty subset of X. The properties below are true*:

1) $m_{ij}(X)pCl(E) = Cl_{m_i}(Int_{m_j}(E)) \cup E$.

2) $m_{ij}(X)pCl(E) = Cl_{m_i}(Int_{m_j}(E))$ provided $E \in m_{ij}O(X)$. The converse to

this assertion is not necessarily true.

Remark 2.17. For a bitopological space $(X, \mathscr{T}_i, \mathscr{T}_j)$, $i, j = 1,2$; $i \neq j$ the families $\mathscr{T}_i\mathscr{T}_j O(X)$ and $m_{ij} pO(X)$ are all m_{ij}-structures of X satisfying property $\mathscr{B}$.

Lemma 2.18. *Let* $(X, m_{ij}(X))$, $i, j = 1,2$; $i \neq j$ *be an* m_{ij}*-space satisfying property* $\mathscr{B}$ *and E and F be subsets of X. Then, the properties below holds*:

1) $m_{ij}(X) pInt(E) = E$ provided $E \in m_{ij}(X) pO(X)$.
2) $X \setminus F \in m_{ij}(X) pO(X)$ provided $m_{ij}(X) pCl(F) = F$.

Lemma 2.19. *Let* $(X, m_{ij}(X))$, $i, j = 1,2$; $i \neq j$ *be an* m_{ij}*-space satisfying property* $\mathscr{B}$ *and A be any nonempty subset of X. Then, the properties below holds*:

1) $E = m_{ij}(X) pInt(E)$ if and only if A is an $m_{ij}(X)$-preopen set.
2) $E = m_{ij}(X) pCl(E)$ if and only if $X \setminus E$ is an $m_{ij}(X)$-preopen set.
3) $m_{ij}(X) pInt(E)$ is $m_{ij}(X)$-preopen.
4) $m_{ij}(X) pCl(E)$ is $m_{ij}(X)$-preclosed.

Lemma 2.20. *Let* $(X, m_{ij}(X))$, $i, j = 1,2$; $i \neq j$ *be an* m_{ij}*-space satisfying satisfying the property* $\mathscr{B}$ *and let* $\{E_\gamma : \gamma \in \Gamma\}$ *be an arbitrary collection of subsets of X. Then,* $\bigcup_{\gamma \in \Gamma} E_\gamma \in m_{ij} pO(X)$ *provided* $E_\gamma \in m_{ij} pO(X)$ *for every* $\gamma \in \Gamma$.

Lemma 2.21. *Let* $(X, m_{ij}(X))$, $i, j = 1,2$; $i \neq j$ *be an* m_{ij}*-space with* m_{ij}*-satisfy property* $\mathscr{B}$ *and let A be a nonempty subset of X. Then*:

1) $m_{ij}(X) pInt(E) = E \cap Int_{m_i}\left(Cl_{m_j}(E)\right)$, and
2) $m_{ij}(X) pCl(E) = E \cup Cl_{m_i}\left(Int_{m_j}(E)\right)$ holds.

And the equality does not necessarily hold if the property $\mathscr{B}$ of Make is removed.

Lemma 2.22. *Let* (X, m_{ij}), $i, j = 1,2$; $i \neq j$ *be an* m_{ij}*-space and U be any subset of X. Then, the properties below holds*:

1) $m_{ij}(X) pInt(U) \subseteq Int_{m_i}\left(Cl_{m_j}\left(m_{ij} pInt(U)\right)\right) \subseteq Int_{m_i}\left(Cl_{m_j}(U)\right)$.
2) $Cl_{m_i}\left(Int_{m_j}(U)\right) \subseteq Cl_{m_i}\left(Int_{m_j}\left(m_{ij}(X) pCl(U)\right)\right) \subseteq m_{ij}(X) pCl(U)$.

Definition 2.23. [6] A multifunction is a point-to-set correspondence $F : X \to Y$ between two topological spaces X and Y such that for each point x of X, $F(x)$ is a none-void subset of Y.

In the sense of Berge [4], we shall denote and define the upper and lower inverse of a non-void subset G of Y with respect to a multifunction F respectively by:

$$F^+(G) = \{x \in X : F(x) \subseteq G\} \text{ and } F^-(G) = \{x \in X : F(x) \cap G \neq \varnothing\}.$$

Generally, F^- and F^+ between Y and the power set $\mathscr{P}(X)$, $F^-(y) = \{x \in X : y \in F(x)\}$ provided $y \in Y$. Clearly for a nonempty subset G of Y, $F^-(G) = \bigcup\{F^-(y) : y \in G\}$ and also,

$$F^+(G) = X \setminus F^-(Y \setminus G) \text{ and } F^-(G) = X \setminus F^+(Y \setminus G)$$

For any non-void subsets E and G of X and Y respectively, $F(E)=\bigcup_{x\in E} F(x)$ and $E\subseteq F^{+}(F(E))$ and also, $F(F^{+}(G))\subseteq G$.

Definition 2.24. [15] [16] A multifunction $F:(X,\mathscr{T})\to(Y,\mathscr{O})$, between topological spaces X and Y is said to be:

1) Upper irresolute at a point x_o of X provided for any semiopen subset G of Y such that $F(x_o)\subseteq G$, there exists a semiopen subset O of X with $x_o\in O$ such that $F(O)\subseteq G$ (or $O\subseteq F^{+}(G)$).

2) Lower irresolute at a point x_o of X provided for any semiopen subset G of Y such that $F(x_o)\cap G\neq\varnothing$, there exists a semiopen subset O of X with $x_o\in O$ such that $F(x)\cap G\neq\varnothing$ for all $x\in O$ (or $O\subseteq F^{-}(G)$).

3) Upper (resp lower) irresolute provided it is upper (resp lower) irresolute at all points x_o of X.

Definition 2.25. [17] A multifunction $F:(X,\mathscr{T})\to(Y,\mathscr{O})$, between topological spaces X and Y is said to be:

1) Upper preirresolute at a point x_o of X if for any preopen subset G of Y such that $F(x_o)\subseteq G$, there exists a preopen subset O of X with $x_o\in O$ such that $F(O)\subseteq G$ (or $O\subseteq F^{+}(G)$).

2) Lower preirresolute at a point x_o of X provided for any preopen subset G of Y such that $F(x_o)\cap G\neq\varnothing$, there exists a preopen subset O of X with $x_o\in O$ such that $F(x)\cap G\neq\varnothing$ for all $x\in O$ (or $O\subseteq F^{-}(G)$).

3) Upper (resp lower) preirresolute provided it is upper (resp lower) preirresolute at all points x_o of X.

3. Upper and Lower *M*-Asymmetric Preirresolute Multifunctions

In this section, we introduce and investigate a new form of multifunctions with the property that the inverse of an *M*-asymmetric preopen set is an *M*-asymmetric preopen set.

Definition 3.1. A multifunction $F:(X,m_{ij}(X))\to(Y,m_{ij}(Y))$, $i,j=1,2$; $i\neq j$ between bitopological spaces satisfying certain minimal conditions, shall be called:

1) Upper *M*-asymmetric preirresolute at a point $x_o\in X$ provided for any $m_{ij}(Y)$-preopen subset G such that $F(x_o)\subseteq G$, there exists an $m_{ij}(X)$-preopen set O with $x_o\in O$ such that $F(O)\subseteq G$ whence $O\subseteq F^{+}(G)$.

2) Lower *M*-asymmetric preirresolute at a point $x_o\in X$ provided for any $m_{ij}(Y)$-preopen set G such that $G\cap F(x_o)\neq\varnothing$, there exists a $m_{ij}(X)$-preopen set O with $x_o\in O$ such that $F(x)\cap G\neq\varnothing$ for all $x\in O$ whence $O\subseteq F^{-}(G)$.

3) Upper (resp lower) *M*-Asymmetric irresolute provided it is upper (resp lower) *M*-Asymmetric irresolute at each and every point x_o of *X*.

Remark 3.2. It should be understood that, upper *M*-asymmetric preirresolute and lower *M*-asymmetric preirresolute multifunctions are independent of each other.

We begin by investigating some characterizations for upper *M*-asymmetric preirresolute multifunctions.

Theorem 3.3. A multifunction $F:\left(X,m_{ij}\left(X\right)\right)\to\left(Y,m_{ij}\left(Y\right)\right)$, $i,j=1,2$; $i\neq j$ with Y satisfies property $\mathscr{B}$, is upper *M*-asymmetric preirresolute at a point x_o in X if and only if $x_o\in Int_{m_i}\left(Cl_{m_j}\left(F^+\left(G\right)\right)\right)$ for every $m_{ij}\left(Y\right)$-preopen set G with $F\left(x_o\right)\subseteq G$.

Proof. Suppose F is upper *M*-asymmetric preirresolute at a point x_o in *X*. Let G be any $m_{ij}\left(Y\right)$-preopen set such that $F\left(x_o\right)\subseteq G$. Then, there is some $m_{ij}\left(X\right)$-preopen set O with $x_o\in O$ such that $F\left(O\right)\subseteq G$ and, giving $Int_{m_i}\left(Int_{m_j}\left(G\right)\right)\supseteq G\supseteq F\left(O\right)$. Since Y satisfies property $\mathscr{B}$ and $O=m_{ij}pInt\left(O\right)$ by Lemma 2.18 (1), then we have from Lemma 2.19 (3) that

$$Int_{m_i}\left(Cl_{m_j}\left(F^+\left(G\right)\right)\right)\supseteq Int_{m_i}\left(Cl_{m_j}\left(O\right)\right)\supseteq m_{ij}pInt\left(O\right)=O\ni x_o$$

Conversely, assume for any $m_{ij}\left(Y\right)$-preopen set G such that $F\left(x_o\right)\subseteq G$, $x_o\in Int_{m_i}\left(Cl_{m_j}\left(F^+\left(G\right)\right)\right)$. Then, by Lemma 2.14, we can find some $m_{ij}\left(X\right)$-preopen neighborhood O of x_o such that $O\subseteq F^+\left(G\right)$. Since G is $m_{ij}\left(Y\right)$-preopen, we then have, $Int_{m_i}\left(Cl_{m_j}\left(G\right)\right)\supseteq G\supseteq F\left(O\right)$ and so, F is an upper *M*-asymmetric preirresolute at a point x_o in *X*. □

Theorem 3.4. A Multifunction $F:\left(X,m_{ij}\left(X\right)\right)\to\left(Y,m_{ij}\left(Y\right)\right)$, $i,j=1,2$; $i\neq j$ having Y satisfying property $\mathscr{B}$ is upper *M*-asymmetric preirresolute at a point x_o in X if and only if for any $m_{ij}\left(X\right)$-preopen neighbourhood O of x_o and any $m_{ij}\left(Y\right)$-preopen set *G*, with $F\left(x_o\right)\subset G$, there is some $m_{ij}\left(X\right)$-open set O_G such that $O_G\subseteq O$ and $F\left(O_G\right)\subseteq G$.

Proof. Suppose that, $\left\{O_{x_o}\right\}$ is a family of $m_{ij}\left(X\right)$-preopen neighbourhoods of a point x_o. Then, for any $m_{ij}\left(X\right)$-preopen set O with $x_o\in O$ and any $m_{ij}\left(Y\right)$-preopen set G such that $F\left(x_o\right)\subseteq G$, there exists an $m_{ij}\left(X\right)$-open subset O_G of O such that $F\left(O_G\right)\subseteq G$. Put $U=\bigcup_{O\in\left\{O_{x_o}\right\}}O_G$, then U is m_{ij}-open, $x_o\in Cl_{m_i}\left(Cl_{m_j}\left(U\right)\right)$ by Theorem 3.3 and $F\left(U\right)\subseteq G$. Put $W=\left\{x_o\right\}\cup U$, then $U\subseteq W\subseteq Cl_{m_i}\left(Cl_{m_j}\left(U\right)\right)$. As a result, U is $m_{ij}\left(X\right)$-preopen, $x_o\in W$, W is $m_{ij}\left(X\right)$-preopen and $F\left(W\right)\subseteq G$ whence, $W\subseteq F^+\left(G\right)$. Consequently, at the point x_o in *X*, the multifunctions F upper *M*-asymmetric preirresolute.

Conversely, suppose F is upper *M*-asymmetric preirresolute at a point x_o in *X*. Let G be an $m_{ij}\left(Y\right)$-preopen set satisfying $F\left(x_o\right)\subseteq G$, then by Theorem 3.3, $x_o\in F^+\left(G\right)\subseteq Int_{m_i}\left(Cl_{m_j}\left(F^+\left(G\right)\right)\right)$. Thus, for any $m_{ij}\left(X\right)$-preopen neighbourhood O of x_o, $F\left(O\right)\subseteq G$, giving $O\subseteq F^+\left(G\right)$ so that, $Int_{m_i}\left(Int_{m_j}\left(F^+\left(G\right)\right)\right)\cap O\neq\varnothing$. But, $Int_{m_i}\left(Int_{m_j}\left(F^+\left(G\right)\right)\right)\subseteq F^+\left(G\right)\subseteq Int_{m_i}\left(Cl_{m_j}\left(F^+\left(G\right)\right)\right)$ and so, Lemma 2.14 implies $Int_{m_i}\left(Cl_{m_j}\left(F^+\left(G\right)\right)\right)\cap O\neq\varnothing$. Put $Int_{m_i}\left(Int_{m_j}\left(F^+\left(G\right)\right)\right)\cap O=O_G$. Then, $O\supseteq O_G$, $F^+\left(G\right)\supseteq Int_{m_i}\left(Int_{m_j}\left(F^+\left(G\right)\right)\right)\supseteq O_G$ whence, $G\supseteq F\left(O_G\right)$. Thus,

O_G is $m_{ij}(X)$-open. □

Remark 3.5. The preceding Theorem 3.4 generally states that, every upper *M*-asymmetric preirresolute multifunction is upper *M*-asymmetric precontinuous, however, the converse is not necessarily true, as we shall clearly illustrates in Example 3.7.

Theorem 3.6. Let $F:(X,m_{ij}(X))\to(Y,m_{ij}(Y))$, $i,j=1,2$; $i\neq j$ with Y satisfying property $\mathscr{B}$ be a multifunction and a point x_o in *X*. Then, the properties are equivalent:

1) *F* is upper *M*-asymmetric preirresolute;

2) The set $F^+(G)$ is $m_{ij}(X)$-preopen for any $m_{ij}(Y)$-preopen set *G*;

3) The set $F^-(K)$ is $m_{ij}(X)$-preclosed, for any $m_{ij}(Y)$-preclosed set *K*;

4) The set inclusion $F(m_{ij}(X)pCl(E))\subseteq m_{ij}(Y)pCl(F(E))$ is true for any subset *E* of *X*;

5) The set inclusion $F^-(m_{ij}(Y)pCl(V))\supseteq m_{ij}(X)pCl(F^-(V))$ holds true given any subset *V* of *Y*;

6) The results $F^+(m_{ij}(Y)pInt(R))\subseteq m_{ij}(X)pInt(F^+(R))$ holds, for any subset *R* of *Y*.

Proof. (1) $\Rightarrow$ (2): Assume (1) holds. Let x_o be some point in *X* and *G* be a $m_{ij}(Y)$-preopen set such that $G\supseteq F(x_o)$, whence $x_o\in F^+(G)$. By hypothesis, there exists $m_{ij}(X)$-preopen set *O* with $x_o\in O$ such that $F(O)\subseteq G$, whence $O\subseteq F^+(G)$. Thus, Theorem 3.3 implies $x_o\in Int_{m_i}(Cl_{m_j}(F^+(G)))$ and as a consequence,

$$F^+(G)\subseteq Int_{m_i}(Cl_{m_j}(F^+(G)))=m_{ij}(X)pInt(F^+(G)).$$

Therefore, $F^+(G)$ is $m_{ij}(X)$-preopen by Lemma 2.13 and 2.18.

(2) $\Rightarrow$ (3): If (2) holds, let *K* be an $m_{ij}(Y)$-preclosed set. Then $F^+(Y\setminus K)=X\setminus F^-(K)$ and $F^-(Y\setminus K)=X\setminus F^+(K)$ since $Y\setminus K$ is $m_{ij}(Y)$-preopen. By Lemma 2.15 and Lemma 2.18, we have,

$$\begin{aligned}X\setminus F^-(K)=F^+(Y\setminus K)&=m_{ij}(X)pInt(F^+(Y\setminus K))\\&=m_{ij}(X)pInt(X\setminus F^-(K))\\&=X\setminus m_{ij}(X)pCl(F^-(K))\end{aligned}$$

As a consequence, $F^-(K)=m_{ij}(X)pCl(F^-(K))$ and so, $F^-(K)$ is $m_{ij}(X)$-preclosed.

(3) $\Rightarrow$ (4): Suppose (3) holds. Then by the closure law, we have any subset *E* of *X* that,

$$\begin{aligned}m_{ij}(X)pCl(E)&=\bigcap\{N:E\subseteq N\text{ and }X\setminus N\in m_{ij}pO(X)\}\\&=\bigcap\{F^-(K):E\subseteq F^-(K)\text{ and }X\setminus F^-(K)\in m_{ij}pO(X)\}\\&=\bigcap\{F^-(K):E\subseteq F^-(K)\text{ and }F^+(Y\setminus K)\in m_{ij}pO(X)\}\\&\subseteq F^-\left(\bigcap\{K:F(E)\subseteq K\text{ and }Y\setminus K\in m_{ij}pO(Y)\}\right)\\&=F^-(m_{ij}(Y)pCl(F(E))).\end{aligned}$$

Hence, $F\left(m_{ij}\left(X\right)pCl\left(E\right)\right)\subseteq m_{ij}\left(Y\right)pCl\left(F\left(E\right)\right)$.

(4) $\Rightarrow$ (5): Suppose (4) holds. Since $m_{ij}\left(Y\right)pCl\left(V\right)\in m_{ij}pC\left(Y\right)$, for any subset V of Y, the closure definition of sets implies;

$$\begin{aligned}
&F^{-}\left(m_{ij}\left(Y\right)pCl\left(V\right)\right)\\
&=F^{-}\left(\bigcap\left\{K:V\subseteq K\text{ and }Y\setminus K\in m_{ij}pO\left(Y\right)\right\}\right)\\
&\supseteq\bigcap\left\{F^{-}\left(K\right):F^{-}\left(V\right)\subseteq F^{-}\left(K\right)\text{ and }X\setminus F^{-}\left(K\right)\in m_{ij}pO\left(X\right)\right\}\\
&=\bigcap\left\{R:F^{-}\left(V\right)\subseteq R\text{ and }X\setminus R\in m_{ij}pO\left(X\right)\right\}\\
&=m_{ij}\left(X\right)pCl\left(F^{-}\left(V\right)\right).
\end{aligned}$$

And the implication follows.

(5) $\Rightarrow$ (6): Assume (5) holds. Since $Y\setminus m_{ij}\left(Y\right)pCl\left(Y\setminus R\right)=m_{ij}\left(Y\right)pInt\left(R\right)$ for any subset R of Y, Lemma 2.13 and 2.15 gives

$$\begin{aligned}
X\setminus F^{+}\left(m_{ij}\left(Y\right)pInt\left(R\right)\right)&=F^{-}\left(Y\setminus m_{ij}\left(Y\right)pInt\left(R\right)\right)\\
&=F^{-}\left(m_{ij}\left(Y\right)pCl\left(Y\setminus R\right)\right)\\
&\supseteq m_{ij}\left(X\right)pCl\left(F^{-}\left(Y\setminus R\right)\right)\\
&=m_{ij}\left(X\right)pCl\left(X\setminus F^{+}\left(R\right)\right)\\
&=X\setminus m_{ij}\left(X\right)pInt\left(F^{+}\left(R\right)\right)
\end{aligned}$$

Consequently, the implication follows.

(6) $\Rightarrow$ (1): Let G be any $m_{ij}\left(Y\right)$-preopen neighborhood of $F\left(x_o\right)$ for some point x_o in X. If (6) holds, then (2) implies $F^{+}\left(G\right)$ is an $m_{ij}\left(X\right)$-preopen neighborhood of x_o. Put $F^{+}\left(G\right)=O$, then $F\left(O\right)\subseteq G$. Consequently, F is upper M-asymmetric preirresolute at a point x_o. □

Example 3.7. Define the asymmetric minimal structures on $X=\{a,b,c,d\}$ by $m_1\left(X\right)=\{\varnothing,X,\{a\},\{c\},\{a,d\},\{b,c,d\}\}$, $m_2\left(X\right)=\{\varnothing,X,\{b\},\{d\},\{b,d\},\{a,b,d\}\}$ and on $Y=\{-2,-1,0,1\}$ by $m_1\left(Y\right)=\{Y,\varnothing,\{-2\},\{1\},\{-2,1\},\{-1,0,1\}\}$ and $m_2\left(Y\right)=\{\varnothing,Y,\{1\},\{-2,0\},\{-2,1\},\{-2,0,1\}\}$. Let the multifunctions $F,F^{*}:\left(X,m_{ij}\right)\to\left(Y,m_{ij}\right)$ be defined by:

$$F\left(a\right)=\{-1\},\quad F\left(b\right)=\{-2,1\},\quad F\left(d\right)=\{0,1\}$$

and

$$F^{*}\left(a\right)=\{1\},\quad F^{*}\left(c\right)=\{-2,1\},\quad F^{*}\left(d\right)=\{0,1\}.$$

Then, F is upper M-asymmetric preirresolute and so upper M-asymmetric precontinuous, but, even thought F' is upper M-asymmetric precontinuous it is not upper M-asymmetric preirresolute.

Theorem 3.8. For an upper M-asymmetric preirresolute multifunction $F:\left(X,m_{ij}\left(X\right)\right)\to\left(Y,m_{ij}\left(Y\right)\right)$, $i,j=1,2$; $i\neq j$, at a point x_o in X with Y satisfy property $\mathscr{P}$, the following properties hold:

1) The set $F^{+}\left(R\right)$ is an $m_{ij}\left(X\right)$-preneighbourhood of x_o for any arbitrary $m_{ij}\left(Y\right)$-preneighbourhood R of $F\left(x_o\right)$.

2) There is some $m_{ij}(X)$-preneighbourhood T of x_o such that $F(T)\subseteq R$ for any $m_{ij}(Y)$-preneighbourhood R of $F(x_o)$.

Proof.

1) Let R be an $m_{ij}(Y)$-preneighbourhood of $F(x_o)$, with x_o being a point in *X*. There exits an $m_{ij}(Y)$-preopen set G such that $F(x_o)\subseteq G\subseteq R$. Since F is upper irresolute, $x_o\in F^{+}(G)\subseteq F^{+}(R)$. Consequently, $F^{+}(R)$ is an $m_{ij}(X)$-preneighbourhood of x_o as $F^{+}(G)\in m_{ij}pO(X)$.

2) Let R be any $m_{ij}(Y)$-preneighbourhood of $F(x_o)$ with x_o being a point in *X*. Set $T=F^{+}(R)$, then from (i), T is an $m_{ij}(X)$-preneighbourhood of x_o and by the hypothesis, $F(T)\subseteq R$. □

We next investigate some properties for lower *M*-asymmetric preirresolute multifunctions.

Theorem 3.9. A multifunction $F:(X,m_{ij}(X))\to(Y,m_{ij}(Y))$, $i,j=1,2$; $i\neq j$ with Y satisfies property $\mathscr{R}$, is lower *M*-asymmetric preirresolute at a point x_o in X if and only if $x_o\in Int_{m_i}\left(Cl_{m_j}\left(F^{-}(G)\right)\right)$ for every $m_{ij}(Y)$-preopen set G for which $G\cap F(x_o)\neq\varnothing$.

Proof. Suppose that $F(x_o)\cap G\neq\varnothing$ for an $m_{ij}(Y)$-preopen set *G*. By assumption, Lemma 2.14 and 2.16, $x_o\in Int_{m_i}\left(Cl_{m_j}\left(F^{-}(G)\right)\right)$. By Definition 3.1, we can find some $m_{ij}(X)$-preopen neighborhood O of x_o such that for each $x\in O$, $F(x)\cap G\neq\varnothing$ and, $F^{-}(G)\supseteq O$. Since $G\in m_{ij}pO(Y)$ then, $F^{-}(G)\in m_{ij}pO(X)$ and so, we infer that, the multifunction F is lower *M*-asymmetric preirresolute at a point x_o in *X*.

On the other hand, suppose the multifunction F is a lower *M*-asymmetric preirresolute at a point x_o in *X*. Then by the hypothesis, there exists an $m_{ij}(X)$-preopen neighborhood O of x_o such that for any $m_{ij}(Y)$-preopen set G with $F(x_o)\cap G\neq\varnothing$, $F(x)\cap G\neq\varnothing$ for x in O whence, $x\in O\subseteq F^{-}(G)$. Since $O\in m_{ij}pO(X)$, we consequently have by Lemma 2.18 and 2.19 that $x\in O=m_{ij}(X)pInt(O)=Int_{m_i}\left(Cl_{m_j}(O)\right)\subseteq Int_{m_i}\left(Cl_{m_j}\left(F^{-}(G)\right)\right)$. □

Theorem 3.10. A multifunction $F:(X,m_{ij}(X))\to(Y,m_{ij}(Y))$, $i,j=1,2$; $i\neq j$ having Y satisfy property $\mathscr{R}$ is lower *M*-asymmetric preirresolute at a point x_o of X if and only if for any $m_{ij}(X)$-preopen neighbourhood O of a point x_o and any $m_{ij}(Y)$-preopen set G with $G\cap F(x_o)\neq\varnothing$, there is some $m_{ij}(X)$-open set O_G such that $O_G\subseteq O$ and for any other point $x\in O_G$, $F(x)\cap G\neq\varnothing$.

Proof. Let $T=\{O_{x_o}\}$ be a family of $m_{ij}(X)$-preopen neighbourhoods of a point x_o in *X*. Then, for any $O\in T$ with $x_o\in O$ and $m_{ij}(Y)$-preopen set G satisfying $F(x_o)\cap G\neq\varnothing$, we can find an $m_{ij}(X)$-open set O_G such that $O_G\subseteq O$ and for each $x\in O_G$, $F(x)\cap G\neq\varnothing$. Put $U=\bigcup_{O\in T}O_G$, then U is $m_{ij}(X)$-open, by Theorem 3.9 $x_o\in Cl_{m_i}\left(Cl_{m_j}(U)\right)$ and for each $x\in U$, $F(x)\cap G\neq\varnothing$. Let $Z=U\cup\{x_o\}$, then $U\subseteq Z\subseteq Cl_{m_i}\left(Cl_{m_j}(U)\right)$. Henceforth, $U\in m_{ij}pO(X)$, $x_o\in Z$ and for all $x\in Z$, $F(x)\cap G\neq\varnothing$, whence $Z\subseteq F^{-}(G)$. Consequently, the multifunction F is lower *M*-asymmetric preirresolute a x_o in *X*.

Suppose the multifunction F is lower M-asymmetric preirresolute at x_o in X. Let O be an $m_{ij}(X)$-preopen neighbourhood of x_o and $G \in m_{ij}pO(Y)$ be such that $F(x_o) \cap G \neq \varnothing$. Then, $x_o \in F^{-}(G) \subseteq Int_{m_i}\left(Cl_{m_j}\left(F^{-}(G)\right)\right)$ by Theorem 3.9. Since, $O \subseteq F^{-}(G)$, $O \cap Int_{m_i}\left(Int_{m_j}\left(F^{-}(G)\right)\right) \neq \varnothing$. But then, $Int_{m_i}\left(Int_{m_j}\left(F^{-}(G)\right)\right) \subseteq Int_{m_j}\left(Cl_{m_i}\left(F^{-}(G)\right)\right)$, and so,

$O \cap Int_{m_i}\left(Cl_{m_j}\left(F^{+}(G)\right)\right) \neq \varnothing$. Put $O_G = O \cap Int_{m_i}\left(Int_{m_j}\left(F^{-}(G)\right)\right)$, then

$O_G \subseteq O$, $O_G \neq \varnothing$ and $F(x) \cap G \neq \varnothing$ for every point $x \in O_G$. As a result, O_G is an $m_{ij}(X)$-open set. □

Theorem 3.11. For a multifunction $F:(X, m_{ij}(X)) \to (Y, m_{ij}(Y))$ $i,j = 1,2$; $i \neq j$, with Y satisfying property $\mathscr{B}$, the following properties are equivalent:

1) F is lower M-asymmetric preirresolute;

2) The set $F^{-}(G)$ is $m_{ij}(X)$-preopen for every $m_{ij}(Y)$-preopen set G;

3) The set $F^{+}(K)$ is $m_{ij}(X)$-preclosed for any $m_{ij}(Y)$-preclosed set K;

4) For any subset V of Y, the inclusion $F^{+}\left(m_{ij}(Y)pCl(V)\right) \supseteq m_{ij}(X)pCl\left(F^{+}(V)\right)$ holds;

5) The set inclusion $F\left(m_{ij}(X)pCl(U)\right) \subseteq m_{ij}(Y)pCl\left(F(U)\right)$ holds for any subset U of X;

6) Given any subset W of Y, $F^{-}\left(m_{ij}(Y)pInt(W)\right) \subseteq m_{ij}(X)pInt\left(F^{-}(W)\right)$ holds true.

Proof.

(1) $\Rightarrow$ (2): Assume (1) holds. Let $F(x_o) \cap G \neq \varnothing$ for an $m_{ij}(Y)$-preopen set G and point $x_o \in X$. Then, $x_o \in F^{-}(G)$ and by Theorem 3.9, $x_o \in Int_{m_i}\left(Cl_{m_j}\left(F^{-}(G)\right)\right)$. Since $x_o \in F^{-}(G)$ was arbitrarily chosen, it follows that $F^{-}(G) \subseteq Int_{m_i}\left(Cl_{m_j}\left(F^{-}(G)\right)\right)$ as a result, $F^{-}(G) \in m_{ij}pO(X)$ by Definition 2.6 (1).

(2) $\Rightarrow$ (3): Supposed (2) holds. Let K be an $m_{ij}(Y)$-preclosed set, then $Y \setminus K$ is $m_{ij}(Y)$-preopen. Applying Lemma 2.13 and Lemma 2.15 we have,

$$\begin{aligned} X \setminus F^{+}(K) = F^{-}(Y \setminus K) &= m_{ij}(X)pInt\left(F^{-}(Y \setminus K)\right) \\ &= m_{ij}(X)pInt\left(X \setminus F^{+}(K)\right) \\ &= X \setminus m_{ij}(X)pCl\left(F^{+}(K)\right) \end{aligned}$$

By Lemma 2.19, $m_{ij}(X)pCl\left(F^{+}(K)\right) \in m_{ij}(X)pC(X)$, as a result $F^{+}(K) \in m_{ij}(X)pC(X)$.

(3) $\Rightarrow$ (4): Assume (3) holds. By Lemma 2.19, $m_{ij}(Y)pCl(V) \in m_{ij}pC(Y)$ for any subset V of Y. By the assumption, $F^{+}\left(m_{ij}(Y)pCl(V)\right) \in m_{ij}pC(X)$ as a result,

$$\begin{aligned} & m_{ij}(X)pCl\left(F^{+}(V)\right) \\ & = \bigcap\left\{F^{+}(K) : F^{+}(V) \subseteq F^{+}(K) \text{ and } X \setminus F^{+}(K) \in m_{ij}pO(X)\right\} \\ & \subseteq F^{+}\left(\bigcap\left\{K : V \subseteq K \text{ and } Y \setminus K \in m_{ij}pO(Y)\right\}\right) \\ & = F^{+}\left(m_{ij}(Y)pCl(V)\right) \end{aligned}$$

Consequently $m_{ij}(X)pCl(F^+(V))$ is a subset of $F^+(m_{ij}(Y)pCl(V))$.

(4) $\Rightarrow$ (5): For any subset $U \neq \varnothing$ of *X*, set $V = F(U)$, whence $U \subseteq F^+(V)$. Supposed (iv) holds, then $U \subseteq F^+(m_{ij}(Y)pCl(F(U)))$. By Lemma 2.19, $m_{ij}(Y)pCl(F(U)) \in m_{ij}pC(Y)$ and our hypothesis, $F^+(m_{ij}(Y)pCl(F(U))) \in m_{ij}pC(X)$. Hence,

$$
\begin{aligned}
&F^+(m_{ij}(Y)pCl(F(U)))\\
&= F^+\left(\bigcap\{K : F(U) \subseteq K \text{ and } Y \setminus K \in m_{ij}pO(Y)\}\right)\\
&= F^+\left(\bigcap\{K : V \subseteq K \text{ and } Y \setminus K \in m_{ij}pO(Y)\}\right)\\
&\supseteq \bigcap\{F^+(K) : F^+(V) \subseteq F^+(K) \text{ and } X \setminus F^+(K) \in m_{ij}pO(X)\}\\
&= m_{ij}(X)pCl(F^+(V)) \supseteq m_{ij}(X)pCl(U).
\end{aligned}
$$

Clearly, $m_{ij}(Y)pCl(F(U)) \supseteq F(m_{ij}(X)pCl(U))$.

(5) $\Rightarrow$ (6): If (5) holds, then we have by Lemma 2.15 and from Definition 2.23 for any arbitrary subset *W* of *Y* that,

$$
\begin{aligned}
F^-(m_{ij}(Y)pInt(W)) &= F^-(Y \setminus m_{ij}(Y)pCl(Y \setminus W))\\
&= X \setminus F^+(m_{ij}(Y)pCl(Y \setminus W))\\
&\subseteq X \setminus m_{ij}(X)pCl(F^+(Y \setminus W))\\
&= X \setminus m_{ij}(X)pCl(X \setminus F^-(W))\\
&= m_{ij}(X)pInt(F^-(W)).
\end{aligned}
$$

And the result follows.

(6) $\Rightarrow$ (1): Suppose $F(x_o) \cap G \neq \varnothing$ for any arbitrary $m_{ij}(Y)$-preopen set *G* and point x_o in *X*. Then, Lemma 2.19 and Lemma 2.20 implies $G = m_{ij}(Y)pInt(G)$. Assume (6) holds, then $x_o \in F^-(G) = F^-(m_{ij}(Y)pInt(G)) \subseteq m_{ij}(X)pInt(F^-(G))$. Thus, there exists an $m_{ij}(X)$-preopen neighborhood *O* of x_o such that $F(x) \cap G \neq \varnothing$ for every $x \in O$. Hence, $F^-(G) \in m_{ij}pO(X)$ as a results, the multifunction *F* is a lower m_{ij}-asymmetric preirresolute at x_o in *X*. □

Theorem 3.12. Let $F : (X, m_{ij}(X)) \to (Y, m_{ij}(Y))$, $i, j = 1, 2$; $i \neq j$ with *Y* satisfying property $\mathscr{B}$ be a lower *M*-asymmetric preirresolute multifunction at a point x_o in *X*. Then, $F^-(G)$ is $m_{ij}(X)$-preopen if and only if for every $m_{ij}(Y)$-preopen set *G*, there exists an $m_{ij}(X)$-preopen set *O* such that $x_o \in O$ and $F(x) \cap G \neq \varnothing$ for all *x* in *O*.

Proof. Supposed that $F(x_o) \cap G$ whence $x_o \in F^-(G)$ for a point x_o in *X* and an $m_{ij}(Y)$-preopen set *G*. By our hypothesis, there is some $m_{ij}(X)$-preopen neighborhood *O* of x_o such that for any other $x \in O$, $F(x) \cap G \neq \varnothing$. Put $\bigcup_{x \in F^-(G)} O = F^-(G)$, then $F^-(G) \in m_{ij}pO(X)$ by Lemma 2.8.

On the others hand, let us assume that $F(x_o) \cap G \neq \varnothing$, whence $x_o \in F^-(G) \in m_{ij}pO(X)$ for every $m_{ij}(Y)$-preopen set *G* and point x_o in *X*. Set $F^-(G) = O$ then, $x_o \in O$. Hence, by our hypothesis $F(x) \cap G \neq \varnothing$ for any other $x \in O$, giving $F(O) \subseteq G$. □

As a consequence to Lemma 2.22 and Theorem 3.11, we have:

Theorem 3.13. Let $F:((X,\mathscr{T}_i,\mathscr{T}_j),m_{ij}(X)) \to ((X,\mathscr{Q}_i,\mathscr{Q}_j),m_{ij}(Y))$, $i,j=1,2$; $i \neq j$ with $((X,\mathscr{Q}_i,\mathscr{Q}_j),m_{ij}(Y))$ satisfying property $\mathscr{B}$ be a multifunction. Then, the statements that follows are equivalent:

1) F is lower M-asymmetric preirresolute;

2) The inclusion $F^-(G) \subseteq Int_{m_i}\left(Cl_{m_j}\left(F^-(G)\right)\right)$ holds for any $m_{ij}(Y)$-preopen set G;

3) The set inclusion $Cl_{m_i}\left(Int_{m_j}\left(F^+(K)\right)\right) \subseteq F^+(K)$ holds for any given $m_{ij}(Y)$-preclosed set K;

4) $F\left(Cl_{m_i}\left(Int_{m_j}(U)\right)\right) \subseteq m_{ij}(Y)pCl(F(U))$ for any given subset U of X;

5) $Cl_{m_i}\left(Int_{m_j}\left(F^+(V)\right)\right) \subseteq F^+\left(m_{ij}(Y)pCl(V)\right)$ given a subset V of Y;

6) $F^-\left(m_{ij}(Y)pInt(W)\right) \subseteq Int_{m_i}\left(Cl_{m_j}\left(F^-(W)\right)\right)$ for any given subset W of Y.

Proof.

(1) $\Rightarrow$ (2): Suppose (1) holds. Then, for some $m_{ij}(X)$-preopen neighborhood O of an arbitrary point x_o and for any $m_{ij}(Y)$-preopen set G, we have by Theorem 3.11 and Lemma 2.22 that,

$$\begin{aligned} x_o \in m_{ij}(X)pInt(O) &\subseteq m_{ij}(X)pInt\left(F^-(G)\right) \\ &\subseteq Int_{m_j}\left(Cl_{m_i}\left(m_{ij}(X)pInt\left(F^-(G)\right)\right)\right) \\ &\subseteq Int_{m_i}\left(Cl_{m_j}\left(F^-(G)\right)\right) \end{aligned}$$

giving $F^-(G) \subseteq Int_{m_i}\left(Cl_{m_j}\left(F^-(G)\right)\right)$.

(2) $\Rightarrow$ (3): Assume (2) holds. Then, given an $m_{ij}(Y)$-preclosed set $K \neq \varnothing$, we have from Lemma 2.22 that,

$$\begin{aligned} X \setminus F^+(K) = F^-(Y \setminus K) &\supseteq m_{ij}(X)pInt\left(F^-(Y \setminus K)\right) \\ &= m_{ij}(X)pInt\left(X \setminus F^+(K)\right) \\ &= X \setminus m_{ij}(X)pCl\left(F^+(K)\right) \\ &= X \setminus Cl_{m_j}\left(Int_{m_j}\left(F^+(K)\right)\right) \end{aligned}$$

As result, $Cl_{m_i}\left(Int_{m_j}\left(F^+(K)\right)\right) \subseteq F^+(K)$ by Theorem 3.11.

(3) $\Rightarrow$ (4): Let $U \neq \varnothing$ be any given subset of X. Suppose (3) holds. Since $Cl_{m_i}\left(Int_{m_j}(U)\right) \subseteq F^+\left(F\left(Cl_{m_i}\left(Int_{m_j}(U)\right)\right)\right)$, we obtain from Theorem 3.11 that,

$$\begin{aligned} F\left(Cl_{m_i}\left(Int_{m_j}(U)\right)\right) &\subseteq F\left(Cl_{m_i}\left(Int_{m_j}\left(m_{ij}(X)pCl(U)\right)\right)\right) \\ &\subseteq F\left(F^-\left(m_{ij}(Y)pCl(F(U))\right)\right) \\ &= m_{ij}(Y)pCl(F(U)) \end{aligned}$$

And the implication follows.

(4) $\Rightarrow$ (5): Lets us assume (4) holds and let $V \neq \varnothing$ be any subset of Y. Then, from Lemma 2.22,

$$
\begin{aligned}
&F^{+}\left(m_{ij}\left(Y\right)pCl\left(V\right)\right)\\
&=F^{+}\left(\bigcap\left\{K:V\subseteq K \text{ and } Y\setminus K\in m_{ij}pO\left(Y\right)\right\}\right)\\
&\supseteq\bigcap\left\{F^{+}\left(K\right):F^{+}\left(V\right)\subseteq F^{+}\left(K\right) \text{ and } X\setminus F^{+}\left(K\right)\in m_{ij}pO\left(X\right)\right\}\\
&\quad=\bigcap\left\{H:F^{+}\left(V\right)\subseteq H \text{ and } X\setminus H\in m_{ij}pO\left(X\right)\right\}\\
&\quad=m_{ij}\left(X\right)pCl\left(F^{+}\left(V\right)\right)=Cl_{m_i}\left(Int_{m_j}\left(F^{+}\left(V\right)\right)\right)
\end{aligned}
$$

Hence, the result follows.

(5) $\Rightarrow$ (6): Assume (5) holds, then given a subset $W\neq\varnothing$ of Y we obtain from Lemma 2.13 and Lemma 2.22 that

$$
\begin{aligned}
Int_{m_i}\left(Cl_{m_j}\left(F^{-}\left(W\right)\right)\right)&=m_{ij}\left(X\right)pInt\left(F^{-}\left(W\right)\right)\\
&=X\setminus m_{ij}\left(X\right)pCl\left(X\setminus F^{-}\left(W\right)\right)\\
&=X\setminus m_{ij}\left(X\right)pCl\left(F^{+}\left(Y\setminus W\right)\right)\\
&\supseteq X\setminus F^{+}\left(m_{ij}\left(Y\right)pCl\left(Y\setminus W\right)\right)\\
&=F^{-}\left(Y\setminus m_{ij}\left(Y\right)pCl\left(Y\setminus W\right)\right)\\
&=F^{-}\left(m_{ij}\left(Y\right)pInt\left(W\right)\right)
\end{aligned}
$$

Thus, the implication holds.

(6) $\Rightarrow$ (1): Assume (6) holds. Let $G\neq\varnothing$ be any $m_{ij}\left(Y\right)$-preopen set such that $F\left(x_o\right)\cap G\neq\varnothing$, whence $x_o\in F^{-}\left(G\right)$ for an arbitrary point x_o in *X*. Then $x_o\in Int_{m_i}\left(Cl_{m_j}\left(F^{-}\left(W\right)\right)\right)$, by Theorem 3.9. Hence, $F^{-}\left(G\right)\subseteq Int_{m_i}\left(Cl_{m_j}\left(F^{-}\left(G\right)\right)\right)$ and so, $F^{-}\left(G\right)\in m_{ij}pO\left(X\right)$. Therefore, the multifunction *F* is lower *M*-asymmetric preirresolute at x_o in *X*. □

Remark 3.14. Example 3.7 clarifies the concepts of Theorem 3.9. We note that, the multifunction *F* so defined is lower *M*-asymmetric preirresolute and so lower *M*-asymmetric precontinuous but, F^{*} is a lower *M*-asymmetric precontinuous but not lower *M*-asymmetric preirresolute.

4. Conclusion

We have introduced and investigated a new form of point-to-set mappings namely; lower and upper *M*-asymmetric preirresolute multifunctions defined on weak form of asymmetric sets satisfying certain minimal structural conditions. Some relations between lower and upper *M*-asymmetric preirresolute multifunctions and, lower and upper *M*-asymmetric precontinuous multifunctions were established.

Acknowledgements

The authors wish to acknowledge the refereed authors for their helpful work towards this paper and the anonymous reviewers for their valuable comments and suggestions towards the improvement of the original manuscript. Further, we wish to acknowledge the financial support rendered by Mulungushi University towards the publication of this manuscript.

Conflicts of Interest

Regarding the publication of this paper, the authors declare that, there is no conflict of interest.

References

[1] Levine, N. (1963) Semi-Open Sets and Semi-Continuity in Topological Spaces. *The American Mathematical Monthly*, **70**, 36-41. https://doi.org/10.1080/00029890.1963.11990039

[2] Maheshwari, S.N. and Prasad, R. (1977) Semi-Open Sets and Semi-Continuous Function in Bitopological Spaces. *Mathematical Notes*, **26**, 29-37.

[3] Bose, S. (1981) Semi Open Sets, Semi Continuity and Semi Open Mappings in Bitopological Spaces. *Bulletin of the Calcutta Mathematical Society*, **73**, 237-246.

[4] Berge, C. (1963) Topological Spaces: Including a Treatment of Multi-Valued Functions, Vector Spaces and Convexity. Oliver and Boyd Ltd, Edinburgh and London.

[5] Popa, V. (1975) Multifunctions and Bitopological Spaces. *Bulletin mathématique de la Société des Sciences Mathématiques de la République Socialiste de Roumanie*, **19**, 147-152.

[6] Noiri, T. and Popa, V. (2000) On Upper and Lower m-Continuous Multifunctions. *Filomat*, **14**, 73-86.

[7] Popa, V. and Noiri, T. (2000) On m-Continuous Functions. *Annals of the University Dunarea de Jos of Galat*, **18**, 31-41.

[8] Matindih, L.K. and Moyo, E. (2021) On m-Asymmetric Semiopem and m-Asymmetric Semcontinuous Multifunctions in Bitopological Spaces. *Advances in Pure Mathematics*, **11**, 218-236. https://doi.org/10.4236/apm.2021.114016

[9] Mashhour, A.S., Abd El-Monsef, M. and El-Deeb, S. (1982) On Precontinuous and Weak Precontinuous Mappings. *Proceedings of the Mathematical and Physical Society of Egypt*, **53**, 47-53.

[10] Jelić, M. (1990) A Decomposition of Pairwise Continuity. *Journal of Institute of Mathematics and Computer Sciences. Mathematics Series*, **3**, 25-29.

[11] Khedr, F., Al-Areefi, S. and Noiri, T. (1992) Precontinuity and Semi-Precontinuity in Bitopological Spaces. *Indian Journal of Pure and Applied Mathematics*, **23**, 625-625.

[12] Min, W. and Kim, Y. (2009) m-Preopen Sets and m-Precontinuity on Spaces with Minimal Structures. *Advances in Fuzzy Sets and Systems*, **4**, 237-245.

[13] Boonpok, C., Viriyapong, C. and Thongmoon, M. (2018) On Upper and Lower (τ1, τ2)-Precontinuous Multifunctions. *Journal of Mathematics and Computer Science*, **18**, 282-293. https://doi.org/10.22436/jmcs.018.03.04

[14] Crossley, G.S. and Hildebrand, S. (1972) Semi-Topological Properties. *Fundamenta Mathematicae*, **74**, 233-255. https://doi.org/10.4064/fm-74-3-233-254

[15] Ewert, J. and Lipski, T. (1983) Quasi-Continuous Multivalued Mappings. *Mathematica Slovaca*, **33**, 69-74.

[16] Popa, V. (1988) On Characterizations of Irresolute Multifunctions. *Journal of the University of Kuwait* (*Science*), **15**, 21-26.

[17] Popa, V., Kucuk, K. and Noiri, T. (1997) On Upper and Lower Preirresolute Multifunctions. *International Journal of Pure and Applied Mathematical Sciences*, **46**, 5-16.

[18] Matindih, L.K., Moyo, E., Manyika, D.K. and Sinyangwe, T. (2021) Some Results of

Upper and Lower m-Asymmetric Irresolute Multifunctions in Bitopological Spaces. *Advances in Pure Mathematics*, **11**, 611-627. https://doi.org/10.4236/apm.2021.116040

[19] Maki, H., Rao, K.C. and Gani, A.N. (1999) On Generalizing Semi-Open Sets and Preopen Sets. *Pure and Applied Mathematika Sciences*, **49**, 17-30.

[20] Kelly, J.C. (1963) Bitopological Spaces. *Proceedings of the London Mathematical Society*, **3**, 71-89. https://doi.org/10.1112/plms/s3-13.1.71

[21] Kar, A. and Bhattacharyya, P. (1992) Bitopological Preopen Sets, Precontinuity and Preopen Mappings. *Indian Journal of Mathematics*, **34**, 295-309.