Scientific Research Publishing

Advances in Pure Mathematics, 2023, 13, 207-210
https://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

A Discussion on the Establishment That a Fibre Metric on the Positive Definite Real Inner-Product of a Properly Embedded Smooth Submanifold Be Always Extended to a Riemannian Metric on the Positive Definite Real Inner-Product

Zhiwei Yan

Department of Mathmetics, Jimei University, Xiamen, China
Email: 2070975437@qq.com

How to cite this paper: Yan, Z.W. (2023) A Discussion on the Establishment That a Fibre Metric on the Positive Definite Real Inner-Product of a Properly Embedded Smooth Submanifold Be Always Extended to a Riemannian Metric on the Positive Definite Real Inner-Product. *Advances in Pure Mathematics*, **13**, 207-210.
https://doi.org/10.4236/apm.2023.135014

Received: March 28, 2023
Accepted: May 8, 2023
Published: May 11, 2023

Abstract

Let M be a smooth manifold and $S \subseteq M$ a properly embedded smooth submanifold. Suppose that we have a fibre metric on $TM|_s$ *i.e.* a positive definite real inner-product on T_pM for all $p \in S$, which depends smoothly on $p \in S$. The purpose of this article is to figure out that the fibre metric on $TM|_s$ can always be extended to a Riemannian metric on TM from a special perspective.

Keywords

Embedded Smooth Manifold, Superplane, Riemannian Metric

1. Introduction

The initial idea of the construction that the paper is going to discuss comes from the inspiration of the study among the property maps between superplane $R^{0|2}$ and a smooth manifold M in Danie, Berwick-Evan's work [1], which included a new proof using $0|\delta$ Euclidean field theories to get Chern-Gauss-Bonet formula, which is constructed by Chern in [2]. In fact, their methods and techniques are similar to Rastogi's classical work [3]. In their paper, such relationships and structures have good properties, and specific calculations can be made in the exchange relationship between them to obtain classical properties in certain new situations. This inspired me to use similar results to cover the ideological characteristics of many properties in classical situations so that we can obtain extensions and generalizations of differential structures under certain general conditions. So, this is about putting forward assumptions and what I will consider and discuss next. More specifically, I will give a proof that a positive definite real inner-product embedded of a smooth submanifold can be extended to a Riemannian metric on the manifold.

The paper next is mainly divided into two parts, the first part introduces the specific content of inspiration, extending some known results to the tangent space of each point on a general smooth manifold. The second part is devoted to discussing how to prove the problem from the condition itself and the discussion in the first part.

2. Related Background and Introduction

Let M be an ordinary m dimensional smooth manifold and over it is a sheaf of supercommutative algebras C^{∞} on M, where C^{∞} can be thought of as the sheaf of functions on the supermanifold, which is locally isomorphic to $C^{\infty}\otimes\Lambda^{*}R^{n}$. The space underlying $R^{m|n}$ is R^{m} and the sheaf of function is $U\mapsto C^{\infty}\otimes\Lambda^{*}R^{2}\cong\left(*,\Lambda^{*}R^{2}\right)$. People can define a map between supermanifolds $\left(M,C_{M}^{\infty}\right)\to\left(N,C_{N}^{\infty}\right)$ as a smooth map $f:M\to N$ and a morphism $f^{*}:C_{N}^{\infty}\to f_{*}C_{M}^{\infty}$, where $f_{*}C_{M}^{\infty}$ denotes the push forward sheaf of C_{M}^{∞} with respect to f, that is $f_{*}C_{M}^{\infty}\left(U\right)=C_{M}^{\infty}\left(f^{-1}\left(U\right)\right)$. One can see it clearly in [1] [2].

As defined, a map $f:R^{0|2}\to M$ between the superplane $R^{0|2}$ and a smooth manifold M consists of a map $:*\to M$ and a morphism of sheaves of superalgebras $C_{M}^{\infty}\to f_{*}\Lambda^{*}R^{2}$ where $f_{*}\Lambda^{*}R^{2}$ if $f\left(*\right)\notin U$ and $f_{*}\Lambda^{*}R^{2}\left(U\right)\to\Lambda^{*}R^{2}$. For those above, we have the following lemma:

Lemma (2.1): Given a connection on a closed Riemannian manifold X, there exists an isomorphism of supermanifolds:

$$Hom\left(R^{0|2},X\right)\cong p^{*}\left(\pi\left(TX\oplus TX\right)\right)$$

where $p:TX\to X$ is the usual projection. Hense, after a choice of connection, a point in $Hom\left(R^{0|2},X\right)$ is a point of X, two odd tagent vector vectors, and one even tagent vector.

Theorem (2.2): There exists a natural mapping from $\pi\left(TX\oplus TX\right)$ to TX, which maintains a continuous structure from $Hom\left(R^{0|2},X\right)$ to TX, we note it $i^{*}\left(X\right):Hom\left(R^{0|2},X\right)\to TX$, which is an isomorphism.

For the finite dimension situation about the initial part I have mentioned, combining with the related Δ-cohomologus results in the $0|\delta$ EFTS, they illustrate the fact that structures on certain types of supermanifolds have good analogies in suitable general differential structures, and that they can be transformed into each other under suitable conditions, establishing profound relationships, this is why in the relevant literature we can overcome certain technical details by using them to obtain certain classical invariants, such as the Chen-Gauss-Bonnet formular, which also implies that such structures may have some rigidity.

For general smooth manifolds, we can convert the structures obtained by the tangent bundles on the closed Riemannian manifolds to the tangent spaces of some points of them by suitable operations. For general smooth manifolds, we can transform the structures obtained by the tangent bundles on the closed Riemannian manifolds to the tangent space of some of their points through appropriate operations, by defining appropriate connections, we can strengthen the relationship to the tangent bundle, or even more weakly, the tangent space of each point on a smooth manifold. That is the corollary below:

Corollary (2.3): For a smooth manifold M and a closed Riemannian manifold X, the above structure on TX can deduce to TM, where TM is the tangent bundle on M.

In the next section, I will come back to the initial assumption, using the results discussed in this section in conjunction with some perspectives on connections on fiber bundlesand, which is strictly defined in [4]. Then I will show how a fibre metric on the positive definite real inner-product of a properly embedded smooth submanifold be always extended to a Riemannian metric on the positive definite real inner-product. Some related definitions can be seen in [5] [6] [7].

3. Discussion and Proof

Pay attention that the Riemann metric is a real-inner product field on a manifold, if we can make good use of the results of previous discussions on this basis, the further steps will not be complicated, then the focus is only on the condition of properly smooth embedded submanifold. At the same time, the connection on fiber bundle of the manifold can make good use of this condition in the hypothesis and the premise related to the results in Part 1. To make it easier to see these aspects clearly now let me firstly review the concept of connections on fibre bundle. Let $P \to B$ be a fibre bundle with fibre F, in order to introduce connections, P, B, F must be differential manifolds. A natural projection mapping $\pi : P \to B$ gives a plex mapping $T\pi : TP \to VP$ on the tangent bundles, whose kernel is a vector bundle, we note it as VB *i.e.* vertical vector bundle. In this context, a connection is a projection on tangent bundle. $K : TP \to VP$. This geometry of this definition is, when given a decomposition of TP *i.e.* $TP \cong VP \oplus HP$, where we note $HP \cong \ker K$, we can learn from it that the projection $T\pi$ gives an isomorphism of a vector bundle *i.e.* $HP \to \overset{TM}{\cong} \to TB$. From this perspective, we can see that a connection actually gives an embedding from the bottom space B to the fibre bundle P. Other details, however, which have new approaches to recover the classic formulation and even generalize it, for example, are listed in [8] [9].

Since S to M is a properly embedding, then through the equivalent expression of connection, we can define a fibre bundle $M \to S$, and the natural projection π gives a plex mapping $T\pi : TM \to TM|_S$. Suppose a Riemannnian metric

$g_{ij}dx^i \otimes dx^j$ was induced from *TM*. On the other hand, we have a fibre metric on $TM|_S$. Then we use the lemma (2.1):

$$Hom\left(R^{0|2}, X\right) \cong p^*\left(\pi\left(TX \oplus TX\right)\right)$$

and the corollary (2.3). We have a map $f: TM \to TM|_S$, which is a plex mapping. Let U be a neighbourhood of point p on *TM*, since $f(*) \in U$, so the morphism $\left(TM|_S, \langle *,* \rangle\right) \to \left(TM|_{U(p)}, g_{ij}dx^i \otimes dx^j\right)$ can be pullback through the morphism $\left(TM, g_{ij}dx^i \otimes dx^j\right) \to \left(TM|_{U(p)}, g_{ij}dx^i \otimes dx^j\right)$ which completes the proof.

Acknowledgements

The author is immensely grateful to his colleagues for their sincere advice, support and encouragement.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Danie, B.-E. (2013) The Chern-Gauss-Bonnet Theorem via Super Symmetric Euclidean Field Theories. https://doi.org/10.48550/arXiv.1310.5383

[2] Chern, S.-S. (1944) A Simple Intrinsic Proof of the Gauss-Bonet Formula for Closed Riemannian Mannifold. *Annals of Mathmetics*, **45**, 747-752. https://doi.org/10.2307/1969302

[3] Rastogi, S.C. (1978) On Quarter-Symmetric Metric Connection. Comptes Rendus delacad Bulgar des Science Press, 811-814.

[4] Yau, S.T. and Scheon, R. (1988) Differential Geometry. Science Publishing Company of China, Beijing, p. 37.

[5] Klingenberg, W. (1982) Riemannian Geometry. De Grunter, Berlin.

[6] Dombrowski, P. (1962) On the Geometry of the Tagent Bundle. *Journal für die reine und angewandte Mathematik*, **210**, 73-78. https://doi.org/10.1515/crll.1962.210.73

[7] Ermolitski, A.A. (1998) Riemannian Manifolds with Geometric Structures (Monograph). BSPU, Minsk. (in Russian)

[8] Kobayashi, S. and Nomizu, K. (1963) Foundations of Differential Geometry. Wiley, New York.

[9] Duggal, K.L. and Bejancu, A. (1996) Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications. Kluwer Academic Publishers, Dordrecht. https://doi.org/10.1007/978-94-017-2089-2