Estimating Sums of Convergent Series via Rational Polynomials

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Abstract

Sums of convergent series for any desired number of terms, which may be infinite, are estimated very accurately by establishing definite rational polynomials. For infinite number of terms the sum infinite is obtained by taking the asymptotic limit of the rational polynomial. A rational function with second-degree polynomials both in the numerator and denominator is found to produce excellent results. Sums of series with different characteristics such as alternating signs are considered for testing the performance of the proposed approach.

Keywords

Sums of Series, Rational Polynomials, Extrapolation to Limit, Asymptotic Value

1. Introduction

In his well-known book *Methodus Differentialis* [1] Stirling (1692-1770) made substantial contributions to summation of infinite series, interpolation, and quadrature calculations. By a novel approach, a given series is re-arranged to yield the remaining sum after a definite number of terms. More importantly, the transformed series converges much more rapidly and if used for calculation of the remaining sum after the first ten or twenty terms the convergence becomes even faster. *Methodus Differentialis* treats quite a number of special series and in the same vein presents the derivation of now famous *Stirling formula* for estimating $n!$ for large values of n.

Christiaan Huygens (1629-1695) was probably the first to use the idea of *extrapolation to the limit* for estimating π. Much later, in the early 20th century, Richardson [2] introduced a similar concept for accelerating the convergence rate in finite-difference solution of definite problems involving differential equa-

tions. Richardson and Gaunt [3] put the method on firmer mathematical grounds and termed it *the deferred approach to the limit*. In essence, the method assumes that numerically computed derivatives with different step-sizes at a point may be expressed as an analytical function and then obtains the most accurate value corresponding to zero step-size by a limiting process. This approach requires the selection of a definite function, typically a polynomial. Bulirsch and Stoer [4] used this idea together with rational polynomials to establish a numerical integrator which gives accurate results for relatively large step sizes for the solution of ordinary differential equations.

Nyengeri *et al.* [5] produced *economized* series truncated at relatively lower orders by the use of power series together with Chebyshev polynomials and then applied this approach to the solution of ordinary differential equations via Frobenius and Taylor methods. Abrarov *et al.* [6] introduced a method to approximate the Fourier transform of a function as rational polynomials.

The present work employs rational polynomials for predicting sums of series in a novel way by establishing a function to be used at discreet integer values which correspond to the number of terms. Accordingly, an appropriate rational polynomial is made to satisfy exactly the sums of a given series for a few selected number of terms and then this function is used to estimate the sums for any desired number of terms, which may tend to infinity. Presently, only two different rational polynomials are employed to test the reliability of estimates for a variety of series. The rational polynomial composed of second-degree polynomials both in the numerator and denominator is found to produce estimates typically accurate to the five or six decimal places.

2. Rational Polynomials

A rational polynomial is a function constructed by diving a polynomial to another polynomial:

$$R_m^n(x) = \frac{P_n(x)}{Q_m(x)} = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n}{b_0 + b_1 x + b_2 x^2 + \cdots + b_m x^m}, \tag{1}$$

where $a_0, a_1, \cdots, a_n$ and $b_0, b_1, \cdots, b_m$ are coefficients of the polynomials $P_n(x)$ and $Q_m(x)$, respectively. Rational polynomials typically have *vertical* and *horizontal* asymptotes. Wherever the denominator $Q_m(x)$ is zero, there is a vertical asymptote and the rational polynomial $R_m^n(x)$ approaches either $+\infty$ or $-\infty$. A horizontal asymptote is a horizontal line that the rational polynomial $R_m^n(x)$ approaches as x tends to $+\infty$ or $-\infty$. For the purpose of the present work it is essential that the rational polynomial used has a nonzero finite positive horizontal asymptote as $x \to +\infty$. A necessary but not sufficient condition to achieve this is to equate the highest powers of polynomials in the numerator and denominator. Accordingly, the rational polynomials used here is of the form

$$R_n^n(x) = \frac{P_n(x)}{Q_n(x)} = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n}{b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n}, \tag{2}$$

so that

$$\lim_{x \to +\infty} R_n^n(x) = \frac{a_0/x^n + a_1/x^{n-1} + a_2/x^{n-2} + \cdots + a_n}{b_0/x^n + b_1/x^{n-1} + b_2/x^{n-2} + \cdots + b_n} \to \frac{a_n}{b_n}. \tag{3}$$

Before proceeding further an important detail must be clarified. Obviously, both the numerator and denominator of a rational polynomial may be divided by any nonzero quantity without incurring any change on the functional values of the rational polynomial. This indicates that not all coefficients expressing the rational polynomial can be selected as desired. Instead, out of $2n+2$ coefficients we can freely determine $2n+1$ coefficients. To implement this condition we divide the numerator and denominator of (2) by b_n:

$$R_n^n(x) = \frac{P_n(x)}{Q_n(x)} = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n}{b_0 + b_1 x + b_2 x^2 + \cdots + x^n}, \tag{4}$$

where the divided coefficients a_k/b_n for $k = 0, \cdots, n$ and b_k/b_n for $k = 0, \cdots, n-1$ are all denoted as a_k and b_k for the sake of convenience. Note that the asymptotic value of (4) for $x \to +\infty$ is simply a_n. Equation (4) is the general form of the rational polynomials to be used in this work and once the coefficient a_n is determined the summation of the series for infinite number of terms is obtained.

3. Rational Polynomial Formulations for Sums of Series

Rational polynomials are now used as functions of discrete integers N instead of a continuously changing argument x. For a given number of terms of a series, say N, a corresponding definite sum S is assigned. Thus, for the argument N, which is an integer, the rational polynomial is expected to yield the sum S. For the present purposes two different rational polynomial formulations with highest powers of one and two; namely, $R_1^1(N)$ and $R_2^2(N)$, are considered.

3.1. Rational Polynomial $R_1^1(N)$

Following the general expression in Equation (4) we propose a rational polynomial of the simplest form

$$R_1^1(N) = \frac{a_0 + a_1 N}{b_0 + N} \tag{5}$$

to represent the sum of a given series for a given number of terms, N. As the approach is ultimately a curve-fitting process there are several choices to determine the coefficients a_0, a_1, and b_0 by the use of given data. Cooper [7] presents an excellent work on rational polynomial fitting and determination of coefficients by a number of different algorithms. Here, for demonstration purposes only we follow the simplest possible method and determine the coefficients by exactly satisfying three different data pairs. This approach may also be called the *collocation method*, where definite points are satisfied exactly as in the approximate solution of differential equations.

Supposing that we are to satisfy the data pairs (N_1, S_1), (N_2, S_2), and (N_3, S_3), which represent the number of terms taken in the series under consideration and the corresponding summation, we then have

$$\frac{a_0 + a_1 N_1}{b_0 + N_1} = S_1, \quad \frac{a_0 + a_1 N_2}{b_0 + N_2} = S_2, \quad \frac{a_0 + a_1 N_3}{b_0 + N_3} = S_3, \tag{6}$$

for the *collocation points* (N_1, S_1), (N_2, S_2), and (N_3, S_3). Solving the above set of equations yields

$$a_0 = \frac{N_1 N_2 S_3 (S_1 - S_2) + N_1 N_3 S_2 (S_3 - S_1) + N_2 N_3 S_1 (S_2 - S_3)}{(N_1 - N_2)(S_2 - S_3) - (N_2 - N_3)(S_1 - S_2)},$$

$$a_1 = \frac{(N_1 S_1 - N_2 S_2)(S_2 - S_3) - (N_2 S_2 - N_3 S_3)(S_1 - S_2)}{(N_1 - N_2)(S_2 - S_3) - (N_2 - N_3)(S_1 - S_2)}, \tag{7}$$

$$b_0 = \frac{(N_1 - N_2)(N_3 S_3 - N_2 S_2) - (N_2 - N_3)(N_2 S_2 - N_1 S_1)}{(N_1 - N_2)(S_2 - S_3) - (N_2 - N_3)(S_1 - S_2)}.$$

3.2. Rational Polynomial $R_2^2(N)$

A much better approximation can be achieved by taking a second-order polynomial both in the numerator and denominator:

$$R_2^2(N) = \frac{a_0 + a_1 N + a_2 N^2}{b_0 + b_1 N + N^2}. \tag{8}$$

It is now possible to satisfy five different collocation points (N_1, S_1), (N_2, S_2), (N_1, S_1), (N_2, S_2), (N_3, S_3) and the coefficients are expressed as

$$b_0 = \Delta_{b_0}/\Delta, \quad b_1 = \Delta_{b_1}/\Delta, \quad a_2 = (D_{123} b_0 + E_{123} b_1 + F_{123})/(N_1 - N_3),$$

$$a_1 = -(N_1 + N_2) a_2 + \big[(S_1 - S_2) b_0 + (N_1 S_1 - N_2 S_2) b_1$$

$$+ (N_1^2 S_1 - N_2^2 S_2)\big]\big/(N_1 - N_2), \tag{9}$$

$$a_0 = -N_1 a_1 - N_1^2 a_2 + S_1 (b_0 + N_1 b_1 + N_1^2).$$

where

$$D_{123} = \frac{S_1 - S_2}{N_1 - N_2} - \frac{S_2 - S_3}{N_2 - N_3}, \quad D_{234} = \frac{S_2 - S_3}{N_2 - N_3} - \frac{S_3 - S_4}{N_3 - N_4},$$

$$D_{345} = \frac{S_3 - S_4}{N_3 - N_4} - \frac{S_4 - S_5}{N_4 - N_5}, \quad E_{123} = \frac{N_1 S_1 - N_2 S_2}{N_1 - N_2} - \frac{N_2 S_2 - N_3 S_3}{N_2 - N_3},$$

$$E_{234} = \frac{N_2 S_2 - N_3 S_3}{N_2 - N_3} - \frac{N_3 S_3 - N_4 S_4}{N_3 - N_4}, \quad E_{345} = \frac{N_3 S_3 - N_4 S_4}{N_3 - N_4} - \frac{N_4 S_4 - N_5 S_5}{N_4 - N_5},$$

$$F_{123} = \frac{N_1^2 S_1 - N_2^2 S_2}{N_1 - N_2} - \frac{N_2^2 S_2 - N_3^2 S_3}{N_2 - N_3}, \quad F_{234} = \frac{N_2^2 S_2 - N_3^2 S_3}{N_2 - N_3} - \frac{N_3^2 S_3 - N_4^2 S_4}{N_3 - N_4},$$

$$F_{345} = \frac{N_3^2 S_3 - N_4^2 S_4}{N_3 - N_4} - \frac{N_4^2 S_4 - N_5^2 S_5}{N_4 - N_5}, \quad D_{1234} = \frac{D_{123}}{N_1 - N_3} - \frac{D_{234}}{N_2 - N_4},$$

$$E_{1234} = \frac{E_{123}}{N_1 - N_3} - \frac{E_{234}}{N_2 - N_4}, \quad F_{1234} = \frac{F_{123}}{N_1 - N_3} - \frac{F_{234}}{N_2 - N_4},$$

$$D_{2345} = \frac{D_{234}}{N_2 - N_4} - \frac{D_{345}}{N_3 - N_5}, \quad E_{2345} = \frac{E_{234}}{N_2 - N_4} - \frac{E_{345}}{N_3 - N_5},$$

$$F_{2345} = \frac{F_{234}}{N_2 - N_4} - \frac{F_{345}}{N_3 - N_5}, \quad \Delta_{b_0} = E_{1234}F_{2345} - E_{2345}F_{1234}, \quad (10)$$

$$\Delta_{b_1} = D_{2345}F_{1234} - D_{1234}F_{2345}, \quad \Delta = D_{1234}E_{2345} - D_{2345}E_{1234}.$$

It is also possible to try *rational functions* such as
$R_e^e(N) = \left(a_0 + a_1 e^{\alpha N}\right) \big/ \left(b_0 + b_1 e^{\beta N}\right)$ but the results are not as good as those obtained from *rational polynomials*. Therefore, only the rational polynomials are considered.

4. Applications to Special Series

The rational polynomial approximations, $R_1^1(N)$ and $R_2^2(N)$, to estimate series summations are now applied to four different special series taken from the classic studies.

4.1. Zeta Function

The first application is to the well-known series introduced by Euler (1707-1783) ([8], p. 139). This series was extended to imaginary domain by Riemann and now known as Riemann's zeta function:

$$\zeta_p(N) = \sum_{n=1}^{N} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{N^p}, \quad (11)$$

where p is an integer. For even powers $p = 2, 4, 6, \cdots$, Euler obtained the sums in closed forms:

$$\zeta_2(\infty) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{2^0}{1 \cdot 2 \cdot 3} \cdot \frac{1}{1} \pi^2$$

$$\zeta_4(\infty) = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \cdots = \frac{2^2}{1 \cdot 2 \cdots 5} \cdot \frac{1}{3} \pi^4 \quad (12)$$

$$\zeta_6(\infty) = 1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \cdots = \frac{2^4}{1 \cdot 2 \cdots 7} \cdot \frac{1}{3} \pi^6$$

$$\vdots$$

We now use $R_1^1(N)$ and $R_2^2(N)$ to obtain estimates for $\zeta_2(N)$. To establish the approximate function $R_1^1(N)$ it is necessary to select three collocation points, which are taken as $N_1 = 1$, $N_2 = 7$, and $N_3 = 15$ with corresponding summations $S_1 = 1$, $S_2 = 1 + 1/2^2 + \cdots + 1/7^2 = 1.511797$, and $S_3 = 1 + 1/2^2 + \cdots + 1/15^2 = 1.580440$. Satisfying these three pairs exactly yields $a_0 = -0.07957289$, $a_1 = 1.64538$, and $b_0 = 0.5657849$ as can be obtained from equation (7). The rational polynomial is then

$$R_1^1(N) = \frac{-0.07957289 + 1.645358N}{0.56567849 + N} \quad (13)$$

which gives $R_1^1(1) = 1$, $R_1^1(7) = 1.511797$, and $R_1^1(15) = 1.580440$ as imposed. The selection of collocation points is arbitrary, in particular the first point need not be the first term in series but using it fixes the rational polynomial to the ex-

act starting value. Making the selection with the first 15 - 20 terms evenly as is done here is usually enough for obtaining quite good estimates for sums. However, depending on the characteristics of the series considered, use of the same arbitrarily selected points would not give equally good results, especially for $R_1^1(N)$, which satisfies only three points exactly. A separate work on optimal determination of coefficients by employing different approaches such as the least-squares should be useful.

Figure 1 shows the numerically computed exact sums, the sum of *infinite* number of terms, which is termed here as the *sum infinite*, and the estimated sums from $R_1^1(N)$. The asymptotic limit of $R_1^1(\infty) \to a_1 = 1.645358$, which is the estimate for the sum infinity, compares well with Euler's exact result $\zeta_2(\infty) = \pi^2/6 = 1.644934$ given in equation (12). Rational polynomial $R_1^1(N)$, the simplest possible function, shows and good agreement with exact sums over the entire range of terms from $N = 1$ to $N \to \infty$. Since small differences cannot be distinguished from **Figure 1**, **Table 1** lists the exact sums up to a definite number of terms $\zeta_2(N)$ as obtained numerically and the corresponding estimates predicted by $R_1^1(N)$ for nine different N values including $N \to \infty$. Relative error percentage is computed as $100 \cdot \left[\zeta_2(N) - R_1^1(N) \right] / \zeta_2(N)$ and collocation points, for which the errors are zero, are shown in red.
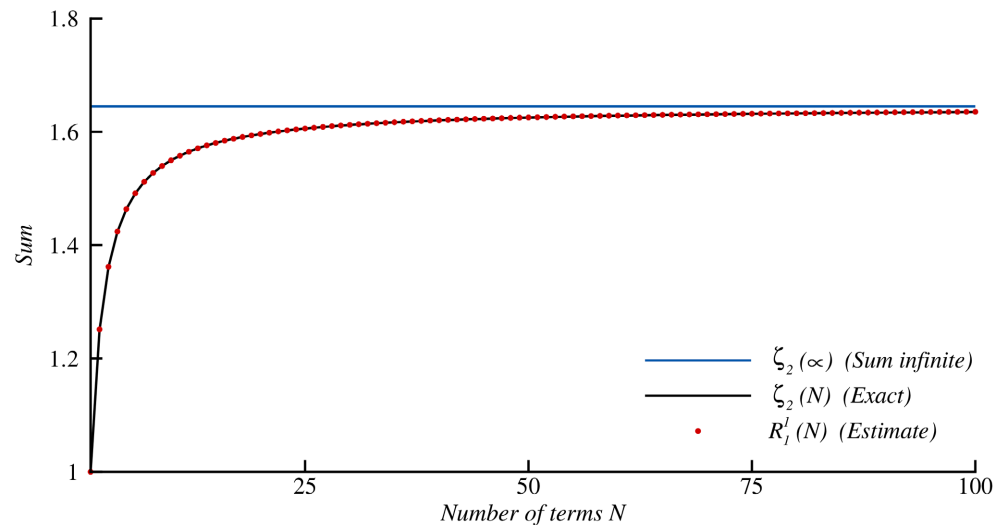
Figure 1. Sum infinite $\zeta_2(\infty)$ (blue line), computed exact sums $\zeta_2(N)$ (black line), and estimated sums $R_1^1(N)$ (red circles).

Table 1. Computed exact sums $\zeta_2(N)$ and estimates $R_1^1(N)$ for a range of N values between 1 and ∞ with collocation points colored red.

N	1	4	7	10	15	30	60	100	∞
$\zeta_2(N)$	1.00000	1.42361	1.51180	1.54977	1.58044	1.61215	1.62841	1.63498	1.64493
$R_1^1(N)$	1.00000	1.42404	1.51180	1.54972	1.58044	1.61229	1.62867	1.63531	1.64536
Error%	0.00000	−0.03009	0.00000	0.00310	0.00000	−0.00917	−0.01645	−0.01991	−0.02578

Summation estimates for the same series are now done by employing $R_2^2(N)$. Besides the three collocation points already selected for $R_1^1(N)$ we select two more as $(2,1.25)$ and $(25,1.605724)$. Then, using these five points in equations (9), (10) and (10), the coefficients are obtained as $a_0 = 0.010590$, $a_1 = 1.115741$, $a_2 = 1.644933$, $b_0 = 0.485085$, $b_1 = 1.286180$, which yield the following second-order rational polynomial

$$R_2^2(N) = \frac{0.010590 + 1.115741N + 1.644933N^2}{0.485085 + 1.286180N + N^2}. \tag{14}$$

At once it is seen that the predicted sum infinite $a_2 = 1.644933$ agrees with Euler's exact value $\zeta_2(\infty) = \pi^2/6 = 1.644934$ to five decimal places. Since a graphical representation would be indistinguishable from **Figure 1** the results are listed in **Table 2** for clear comparisons. Again, the collocation points are shown in red.

Note that the errors are all positive indicating that estimations consistently remain slightly below the computed exact values and the maximum error occurs for the sum infinite and only 0.00006% due to the disagreement in the sixth decimal value, which cannot be seen in the table. Comparing these estimates with those of $R_1^1(N)$, the second-order approach $R_2^2(N)$ is absolutely superior and should be preferred.

4.2. Brouncker Series

In *Methodus Differentialis*, Proposition 2-Example 5, Stirling [1] considers a series which is attributed to Viscount Brouncker as a formulation for the quadrature of hyperbola. The series is given by

$$B(N) = \sum_{n=1}^{N} \frac{1}{4n(n-1/2)} = \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \frac{1}{7\cdot 8} + \cdots + \frac{1}{4N(N-1/2)}. \tag{15}$$

Using his method of rendering the convergence of a series much faster Stirling [1] gives the sum infinite as $B(\infty) = 0.693147180$, which is correct to nine decimal places, and further points out that this sum is hyperbolic logarithm two; briefly, $\ln 2$.

We now estimate the summations for various N values as well as the sum infinite by employing the second-degree rational polynomial approach. To save space, the relatively inferior first-degree approach $R_1^1(N)$ is not considered anymore. For the Brouncker series using the same collocation points selected in

Table 2. Computed exact sums $\zeta_2(N)$ and estimates $R_2^2(N)$ for a range of N values between 1 and ∞ with collocation points colored red.

N	1	2	7	10	15	25	60	100	∞
$\zeta_2(N)$	1.00000	1.25000	1.51180	1.54977	1.58044	1.60572	1.62841	1.63498	1.64493
$R_2^2(N)$	1.00000	1.25000	1.51180	1.54977	1.58044	1.60572	1.62841	1.63498	1.64493
Error%	0.00000	0.00000	0.00000	0.00002	0.00000	0.00000	0.00002	0.00004	0.00006

§4.1, $R_2^2(N)$ is established as

$$R_2^2(N) = \frac{0.006566443 + 0.2957807N + 0.6931473N^2}{0.2036055 + 0.7873834N + N^2},\qquad(16)$$

which at once reveals the sum infinite as $a_2 = 0.6931473$ correct to the six decimal places. Let us remark that since the series converges relatively rapidly it is not necessary to extend the last collocation point to 25; but kept here as a part of the previous set of points.

Table 3 gives the computed exact values for the Brouncker series and the corresponding estimates from Equation (16) with collocation points shown in red. Computations reveals that except for $N = 3$ and $N = 4$ (correct to five decimals) all the estimates are correct to six decimal places within the range $N = 1$ to ∞.

4.3. Series with Alternating Signs

We now consider a series with alternating signs $A(N)$ from Stirling [1], Proposition 7-Example 1:

$$A(N) = \sum_{n=1}^{N} \frac{(-1)^{n-1}}{2n-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^{N-1}}{2N-1}.\qquad(17)$$

The sum infinite is given as $A(\infty) = \pi/4 = 0.78539816339$, which is the area of the circle whose diameter is unity. Since the series has alternating signs the sum values zigzag about the sum infinite $A(\infty)$, overshooting and undershooting while getting closer and closer to it. This oscillatory character of the series therefore requires a slightly different approach hence two rational polynomial representations are formed: one by the use of odd-numbered and another from even-numbered collocation points. Thus, the former can predict the sums corresponding to the odd number of terms $N = 1,3,5,\cdots$, while the latter those to the even number of terms $N = 2,4,6,\cdots$. Both of them however converge to the same sum infinite $A(\infty)$ from above and from below, respectively. Fig. 2 shows the sum infinite $A(\infty)$, the numerically computed exact sums $A(N)$, and the estimated sums from two different $R_2^2(N)$ functions which form upper and lower *envelopes* to the exact values. The asymptotic limit $R_2^2(\infty) \rightarrow a_1 = 0.785398$ of both functions agrees to the six decimal places with the exact result $\pi/4 = 0.78539816339$. Incidentally, the computational values could be obtained until six decimal places therefore it was not possible to make

Table 3. Computed Brouncker sums $B(N)$ and estimates $R_2^2(N)$ for a range of N values between 1 and ∞ with collocation points colored red.

N	1	2	7	10	15	25	60	100	∞
$B(N)$	0.50000	0.58333	0.65871	0.66877	0.67676	0.68325	0.68900	0.69065	0.69315
$R_2^2(N)$	0.50000	0.58333	0.65871	0.66877	0.67676	0.68325	0.68900	0.69065	0.69315
Error%	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
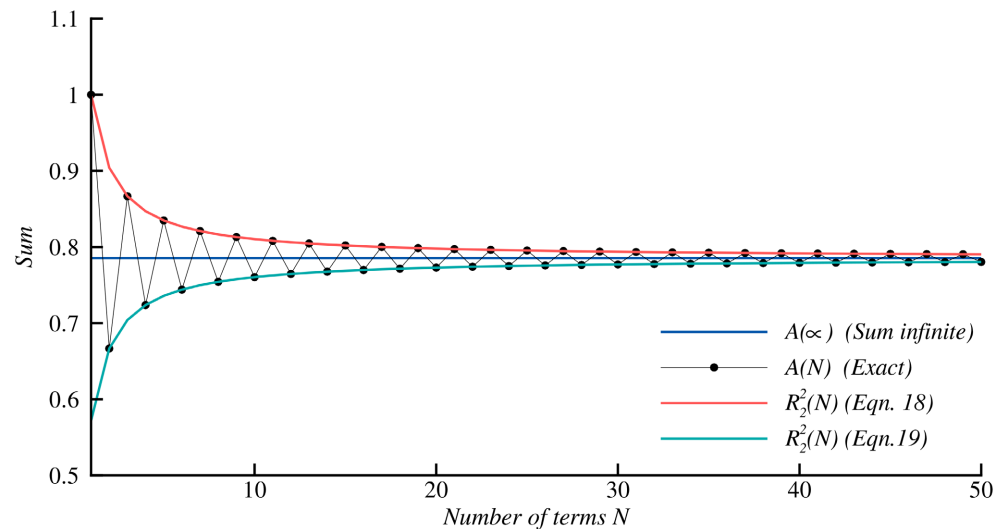
Figure 2. Sum infinite $A(\infty)$ (blue line), computed exact sums $A(N)$ (black dots), and estimates $R_2^2(N)$ (orange and cyan lines).

Table 4. Computed exact sums $A(N)$ and estimates $R_2^2(N)$ for a range of odd N values between 1 and ∞ with collocation points colored red.

N	1	3	7	13	25	33	45	99	∞
$A(N)$	1.00000	0.86667	0.82094	0.80460	0.79539	0.79297	0.79095	0.78792	0.78540
$R_2^2(N)$	1.00000	0.86667	0.82093	0.80460	0.79539	0.79297	0.79095	0.78792	0.78540
$Error\%$	0.00000	0.00000	0.00025	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

comparisons with more decimals.

First, using the odd-numbered collocation points $N_1 = 1$, $N_2 = 3$, $N_3 = 13$, $N_4 = 25$, and $N_5 = 45$ we establish the following $R_2^2(N)$ from equations (8) (9) and (10):

$$R_2^2(N) = \frac{0.4427708 + 0.8865442N + 0.7853984N^2}{0.3041914 + 0.8105220N + N^2}. \tag{18}$$

Note that the predicted sum infinite $a_2 = 0.7853984$ is correct to the six decimal places when compared with the exact result $\pi/4 = 0.78539816339$. **Table 4** compares the performance of (18) against exact values for a range of odd term numbers.

Second rational polynomial representing the lower sum values is established from the even-numbered collocation points $N_1 = 2$, $N_2 = 4$, $N_3 = 14$, $N_4 = 26$, and $N_5 = 46$:

$$R_2^2(N) = \frac{0.0608380 + 0.2621761N + 0.7853977N^2}{0.2859991 + 0.6520864N + N^2}. \tag{19}$$

In this case $a_2 = 0.7853977$ differs only slightly from the $a_2 = 0.7853984$ obtained in (18) and they become identical when rounded to the six decimals. The comparisons made for (18) are now repeated for (19) but for even-numbered N

Table 5. Computed exact sums $A(N)$ and estimates $R_2^2(N)$ for a range of even N values between 2 and ∞ with collocation points colored red.

N	2	4	8	14	26	34	46	100	∞
$A(N)$	0.66667	0.72381	0.75427	0.76756	0.77579	0.77805	0.77996	0.78290	0.78540
$R_2^2(N)$	0.66667	0.72381	0.75427	0.76756	0.77579	0.77805	0.77996	0.78290	0.78540
$Error\%$	0.00000	0.00000	−0.00010	0.00000	0.00000	0.00000	0.00000	0.00002	0.00000

values in Table 5 as always the collocation points are shown in red.

A notable point concerning the two rational approximations is that in the former case the sums are descending towards the sum infinity with increasing N values while in the latter case the opposite occurs and the values ascend towards the ultimate infinite sum. This upper and lower convergence patterns, which form envelopes, is visually quite clear from Figure 2 as well.

An interesting series with alternating signs, attributed to Newton (1643-1727) by Stirling [1] in Proposition 3-Example 1 as a series that could be used for an accurate calculation of the circumference of a circle, is given by

$$\begin{aligned}
C(N) &= \sum_{n=1}^{N} \frac{1}{2(4n-3)-1} + \frac{1}{2(4n-2)-1} - \frac{1}{2(4n-1)-1} - \frac{1}{2(4n)-1} \\
&= 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \cdots + \frac{1}{2(4N-3)-1} \\
&\quad + \frac{1}{2(4N-2)-1} - \frac{1}{2(4N-1)-1} - \frac{1}{2(4N)-1}.
\end{aligned} \tag{20}$$

This series can be treated very similar to $A(N)$ given in (17) only one must be careful in selecting the collocation points for establishing the rational polynomials for the upper and lower estimates. Accordingly, $N_1 = 2$, $N_2 = 6$, $N_3 = 10$, etc. can be used for the upper estimates which would form the envelope for the upper sums while $N_1 = 4$, $N_2 = 8$, $N_3 = 12$, etc. can be used for the lower estimates envelope. As always, these points need not be taken consecutively. For instance, computations performed by taking $N_1 = 2$, $N_2 = 6$, $N_3 = 14$, $N_4 = 30$, $N_5 = 46$ for the upper estimates and $N_1 = 4$, $N_2 = 8$, $N_3 = 16$, $N_4 = 32$, $N_5 = 48$ for the lower estimates yield a graph identical in appearance to Figure 2 and give the sum infinite estimates as $R_2^2(\infty) \to a_2 = 1.110723$ and $R_2^2(\infty) \to a_2 = 1.110720$, respectively for the upper and lower estimate functions. Direct computations by taking 10^6 terms result in $C(\infty) = 1.110722$ and increasing the number of terms does not change the result. In view of the agreement of the estimates with the direct computations the value obtained for the sum infinite, 1.11072, appears to be correct. Further, although not explicitly given in [1], this result can indeed be related to π, as implied in Stirling's claim that the series could be used for accurate calculation of the circumference of a circle. To a very good approximation we note that π can be computed as $\pi \approx 2\sqrt{2} \cdot C(\infty) = 2\sqrt{2} \cdot 1.11072 = 3.141590576$, which agrees

with the correct value 3.141592654 to five decimal places. Thus, we may conclude that the exact value of sum infinite is $C(\infty) = \pi/2\sqrt{2}$, which is 1.110720735 to nine decimal places.

5. Concluding Remarks

A functional approach based on rational polynomials is presented to estimate the sum of a series for any given number of terms as well the ultimate sum for infinite number of terms or the *sum infinite*, as termed here. A rational polynomial form composed of second-degree polynomials in the numerator and denominator is found to be quite satisfactory for obtaining estimates correct to no less than five decimal places and even in some cases six decimal places. When the powers of today's computational facilities are considered, the practical use of the approach introduced here is questionable; however, besides some theoretical considerations, there always arise possibilities to expand ideas into different areas where they may prove to be practically more useful.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Tweddle, I. (2003) James Stirling's Methodus Differentialis: An Annotated Translation of Stirling's Text. Springer-Verlag, London. https://doi.org/10.1007/978-1-4471-0021-8

[2] Richardson, L.F. (1910) The Approximate Arithmetical Solution by Finite Differences of Physical Problems Involving Differential Equations with an Application to the Stresses in a Masonry Dam. *Philosophical Transactions of the Royal Society London A*, **210**, 307-357. https://doi.org/10.1098/rsta.1911.0009

[3] Richardson, L.F. and Gaunt, J.A. (1927) The Deferred Approach to the Limit. *Philosophical Transactions of the Royal Society London A*, **226**, 299-361. https://doi.org/10.1098/rsta.1927.0008

[4] Bulirsch, R. and Stoer, J. (1966) Numerical Treatment of Ordinary Differential Equations by Extrapolation Methods. *Numerische Mathematik*, **8**, 1-13. https://doi.org/10.1007/BF02165234

[5] Nyengeri, H., Nizigiyimana, R., Mutankana, J.-P., Bayaga, H. and Bayubahe, F. (2021) Power and Chebyshev Series Transformation Formulas with Applications to Solving Ordinary Differential Equations via the Fröbenius and Taylor's Methods. *Open Access Library Journal*, **8**, e7142. https://doi.org/10.4236/oalib.1107142

[6] Abrarov, S.M., Siddiqui, R., Jagpal, R.K. and Quine, B.M. (2021) A Rational Approximation of the Fourier Transform by Integration with Exponential Decay Multiplier. *Applied Mathematics*, **12**, 947-962. https://doi.org/10.4236/am.2021.1211063

[7] Cooper, P. (2007) Rational Approximation of Discrete Data with Asymptomatic Behaviour. Doctoral Thesis, University of Huddersfield, Huddersfield. https://core.ac.uk/download/pdf/40039709.pdf

[8] Euler, L. (1988) Introduction to Analysis of the Infinite—Book I. In: *Introductio in Analysin Infinitorum*, Blanton, J.D., Trans., Springer-Verlag, New York.