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Exact Traveling Wave Solutions of the Generalized Fractional Differential mBBM Equation

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Abstract

By using the fractional complex transform and the bifurcation theory to the generalized fractional differential mBBM equation, we first transform this fractional equation into a plane dynamic system, and then find its equilibrium points and first integral. Based on this, the phase portraits of the corresponding plane dynamic system are given. According to the phase diagram characteristics of the dynamic system, the periodic solution corresponds to the limit cycle or periodic closed orbit. Therefore, according to the phase portraits and the properties of elliptic functions, we obtain exact explicit parametric expressions of smooth periodic wave solutions. This method can also be applied to other fractional equations.

Keywords

A Generalized Fractional Differential mBBM Equation, Traveling Wave Solution, Phase Portrait

1. Introduction

BBM (Benjamin-Bona-Mahoney) type equations are widely used in fluid mechanics. In this paper, we consider the generalized fractional differential mBBM equation

$$D_t^{\alpha} u(t,x) + aD_x^{\alpha} u(t,x) + bu^2(t,x) D_x^{\alpha} u(t,x) + cD_x^{3\alpha} u(t,x) = 0, \tag{1}$$

where a, b and c are non-zero constants, and $0<\alpha<1$, $t>0$. When $a=c=1$ and $0<\alpha<1$, Alzaidy [1] constructed the analytical solutions of Equation (1) by the fractional sub-equation method. Guo and Sirendaoerji [2] obtained the exact solutions of Equation (1) by using the auxiliary equation method. Feng [3] introduced a new approach for seeking exact solutions of the space-time fractional BBM equation. When $\alpha = 1$, Equation (1) becomes the generalized differential mBBM equation. Many different methods were used to investigate the BBM equation and mBBM equation (see, e.g., [4] [5] [6] [7]). These methods can only obtain partial solutions of the BBM type equations, and cannot explain the dynamic behavior of various traveling wave solutions. The limitations of these methods make it impossible for us to have a comprehensive and systematic understanding of the equations. Therefore, we will study the Equation (1) by using the bifurcation theory of plane dynamic system (see [8] [9] [10]).

Fractional differential equations have been widely used to describe complex problems in science and engineering. For example, Wang, Long and Liu [11] studied the oscillatory theory for two classes of fractional neutral differential equations by using fractional calculus and the Laplace transform. The investigation of exact solutions of nonlinear evolution equations plays an important role in nonlinear mathematical physics. In recent years, many authors have applied the theory of plane dynamic systems to solve the travelling wave solutions of nonlinear wave equations [12] [13]. The main goal of this paper is to show that the generalized fractional differential mBBM equation has some traveling wave solutions by using the bifurcation theory of planar dynamical systems.

This paper is organized as follows. In Section 2, we discuss the phase portraits of Equation (1). In Section 3, we obtain all the explicit exact expressions of smooth periodic traveling waves.

2. Phase Portraits of Equation (1)

In this paper, we consider the common fractional derivatives introduced by Khalil *et al.* [14]. The common fractional derivatives of order α is defined as

$$D_t^\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon} \tag{2}$$

for all $0 < \alpha < 1$, $t > 0$. In order to find the traveling wave solutions, and inspired by [15], we make the following transformation

$$u(x,t) = \phi(\xi), \quad \xi = \frac{k}{\alpha} x^\alpha + \frac{l}{\alpha} t^\alpha, \tag{3}$$

where k and l are non-zero constants, and $0 < \alpha < 1$. By (3) and Theorem 2.2 of [14], it infers

$$D_t^\alpha u = t^{1-\alpha} \frac{\mathrm{d}\phi(\xi)}{\mathrm{d}\xi} \cdot \frac{\mathrm{d}\xi}{\mathrm{d}t} = l \frac{\mathrm{d}\phi(\xi)}{\mathrm{d}\xi} = l\phi'. \tag{4}$$

Similarly, it can be obtained

$$D_x^\alpha u = k\phi', D_x^{3\alpha} u = k^3 \phi'''. \tag{5}$$

Substituting (4) and (5) into Equation (1), it obtains

$$ck^3\phi''' + bk\phi^2\phi' + (ak + l)\phi' = 0. \tag{6}$$

Then, integrating (6) and ignoring the integral constant, we find

$$\phi'' = -\frac{b}{3ck^2}\phi^3 - \frac{ak+l}{ck^3}\phi. \tag{7}$$

Denote $A = -\frac{b}{3ck^2}$, $B = -\frac{ak+l}{ck^3}$ and let $\frac{\mathrm{d}\phi}{\mathrm{d}\xi} = y$. Then Equation (7) is equivalent to the following planar Hamiltonian system

$$\begin{cases} \frac{\mathrm{d}\phi}{\mathrm{d}\xi} = y, \\ \frac{\mathrm{d}y}{\mathrm{d}\xi} = A\phi^3 + B\phi \end{cases} \tag{8}$$

with the first integral

$$H(\phi, y) = \frac{1}{2}y^2 - \frac{A}{4}\phi^4 - \frac{B}{2}\phi^2 = h. \tag{9}$$

Thus, the coefficient matrix of the linearized system of (8) is

$$M(\phi_i, y_i) = \begin{pmatrix} 0 & 1 \\ 3A\phi_i^2 + B & 0 \end{pmatrix}. \tag{10}$$

And the determinant of $M(\phi_i, y_i)$ has the form

$$J(\phi_i, y_i) = -\left(3A\phi_i^2 + B\right). \tag{11}$$

By the theory of planar dynamical systems [16], we know that for an equilibrium point of a planar integrable system, if $J < 0$, then the equilibrium point is a saddle point; if $J > 0$ and $\mathrm{Trace}\left(M(\phi_i, y_i)\right) = 0$, then it is a center point. Therefore, according to this theory and (10)-(11), we obtained the following propositions.

Proposition 1. Suppose that $AB > 0$. The system (8) has only one equilibrium point $E_0(0,0)$.

1) When $A > 0$ and $B > 0$, $E_0(0,0)$ is a saddle point.

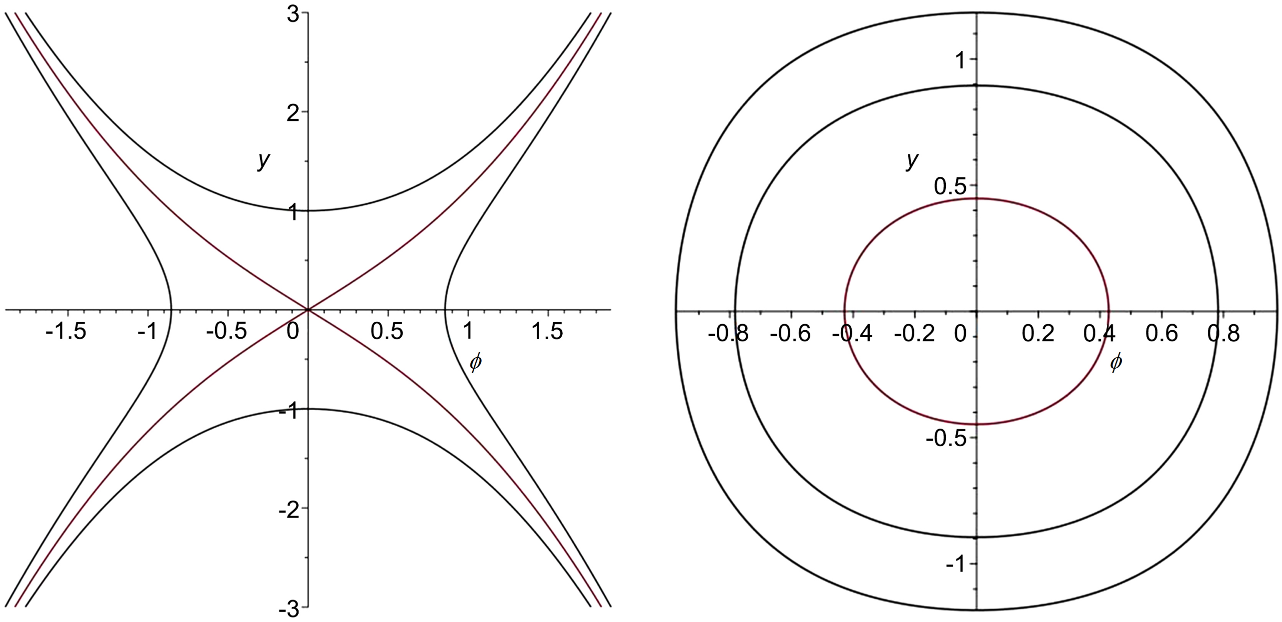
2) When $A < 0$ and $B < 0$, $E_0(0,0)$ is a center point.

Proposition 2. Suppose that $AB < 0$. The system (8) has three equilibrium points $E_0(0,0)$, $E_1\left(-\sqrt{-\frac{B}{A}}, 0\right)$, $E_2\left(\sqrt{-\frac{B}{A}}, 0\right)$.

1) When $A > 0$ and $B < 0$, $E_0(0,0)$ is a center point, $E_1\left(-\sqrt{-\frac{B}{A}}, 0\right)$ and $E_2\left(\sqrt{-\frac{B}{A}}, 0\right)$ are two saddle points.

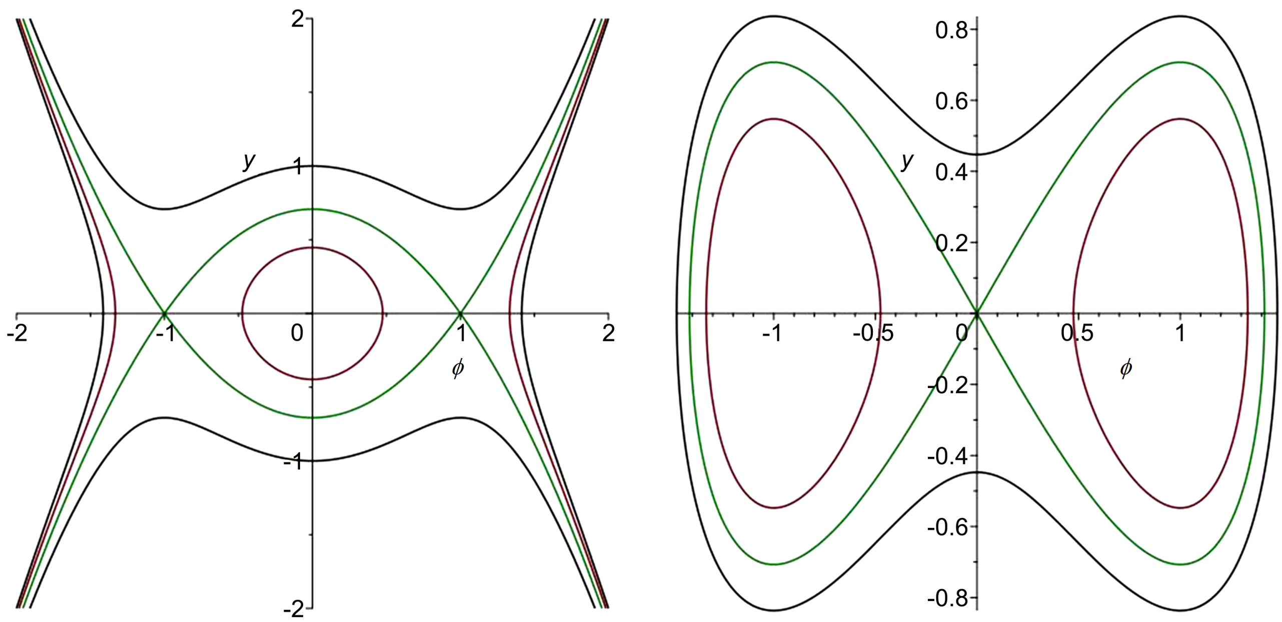
2) When $A < 0$ and $B > 0$, $E_0(0,0)$ is a saddle point, $E_1\left(-\sqrt{-\frac{B}{A}}, 0\right)$ and $E_2\left(\sqrt{-\frac{B}{A}}, 0\right)$ are two center points.

By Proposition 1 and Proposition 2, we obtain the following phase portraits of System (8), see **Figure 1** and **Figure 2**.



(a) $A > 0$, $B > 0$ (b) $A < 0$, $B < 0$

Figure 1. The phase portraits of the system (8) for $AB > 0$.



(a) $A > 0$, $B < 0$ (b) $A < 0$, $B > 0$

Figure 2. The phase portraits of the system (8) for $AB < 0$.

3. Explicit Parametric Expressions of the Solutions of Equation (1)

In this section, according to **Figure 1** and **Figure 2**, and by applying the elliptic integral theory [17] and the direct integration method, all possible explicit parametric representations of the traveling wave solutions of Equation (1) will be given.

3.1. Consider Proposition 1 in Section 2 (See Figure 1)

Suppose that $A<0$, $B<0$. In this case, we have the phase portraits of the system (8) shown in **Figure 1(b)**. Equation (1) has a family of smooth periodic wave solutions defined by $H(\phi,y)=h$, $h\in(0,+\infty)$. Denote $\Delta=B^2-2Ah$. Then $\sqrt{\Delta}>-B>0$. By (9), we obtain the expressions of the closed orbits

$$y^2=-\frac{A}{2}\left(\frac{B-\sqrt{\Delta}}{A}+\phi^2\right)\left(\frac{-B-\sqrt{\Delta}}{A}-\phi^2\right). \tag{12}$$

By using the first equation of (8), (12) and [17], we obtain

$$\int_{\phi}^{\frac{-B-\sqrt{\Delta}}{A}}\left(\frac{1}{\sqrt{\left(\frac{B-\sqrt{\Delta}}{A}+\phi^2\right)\left(\frac{-B-\sqrt{\Delta}}{A}-\phi^2\right)}}\right)\mathrm{d}\phi=\sqrt{-\frac{A}{2}}|\xi|.$$

Therefore, the parametric expression of the periodic solutions as follow

$$\phi=\sqrt{\frac{-B-\sqrt{\Delta}}{A}}Cn\left(\sqrt[4]{\Delta}|\xi|,\sqrt{\frac{B+\sqrt{\Delta}}{2\sqrt{\Delta}}}\right).$$

3.2. Consider Proposition 2 in Section 2 (See Figure 2)

Suppose that $A>0$, $B<0$. In this case, we have the phase portraits of the system (8) shown in **Figure 2(a)**. Equation (1) has a family of smooth periodic wave solutions defined by $H(\phi,y)=h$, $h\in\left(0,\frac{B^2}{4A}\right)$. Denote $\Delta=B^2-2Ah$. By (9), we obtain the expressions of the closed orbits

$$y^2=\frac{A}{2}\left(\phi+\sqrt{\frac{\sqrt{\Delta}-B}{A}}\right)\left(\phi+\sqrt{\frac{-B-\sqrt{\Delta}}{A}}\right)\left(\phi-\sqrt{\frac{-B-\sqrt{\Delta}}{A}}\right)\left(\phi-\sqrt{\frac{\sqrt{\Delta}-B}{A}}\right). \tag{13}$$

By using the first equation of (8), (13) and [17], it infers the following parametric expression of the periodic solutions

$$\phi=\frac{w\sqrt{\frac{-B-\sqrt{\Delta}}{A}}sn^2\left(4\sqrt{h}|\xi|,\frac{-B+\sqrt{2Ah}}{2\sqrt{2Ah}}\right)-\sqrt{\frac{\sqrt{\Delta}-B}{A}}}{wsn^2\left(4\sqrt{h}|\xi|,\frac{-B+\sqrt{2Ah}}{2\sqrt{2Ah}}\right)-1},$$

where $w=\frac{\sqrt{\sqrt{\Delta}-B}+\sqrt{-B-\sqrt{\Delta}}}{2\sqrt{-B-\sqrt{\Delta}}}$.

Suppose that $A<0$, $B>0$. In this case, we have the phase portraits of the system (8) shown in **Figure 2(b)**. Equation (1) has two families of smooth periodic wave solutions defined by $H(\phi,y)=h$, $h\in\left(\frac{B^2}{4A},0\right)$. Denote $\Delta=B^2-2Ah$. By (9), we obtain the expressions of the closed orbits

$$y^2 = -\frac{A}{2}\left(\sqrt{\frac{-B-\sqrt{\Delta}}{A}}-\phi\right)\left(\phi-\sqrt{\frac{\sqrt{\Delta}-B}{A}}\right)\left(\phi+\sqrt{\frac{\sqrt{\Delta}-B}{A}}\right)\left(\phi+\sqrt{\frac{-B-\sqrt{\Delta}}{A}}\right). \quad (14)$$

By using the first equation of (8), (14) and [17], it infers

$$\int_{\phi}^{\sqrt{\frac{-B-\sqrt{\Delta}}{A}}} \frac{\mathrm{d}\phi}{\sqrt{\left(\sqrt{\frac{-B-\sqrt{\Delta}}{A}}-\phi\right)\left(\phi-\sqrt{\frac{\sqrt{\Delta}-B}{A}}\right)\left(\phi+\sqrt{\frac{\sqrt{\Delta}-B}{A}}\right)\left(\phi+\sqrt{\frac{-B-\sqrt{\Delta}}{A}}\right)}} = \sqrt{-\frac{A}{2}}|\xi|.$$

Thus,

$$\phi = \sqrt{\frac{-B-\sqrt{\Delta}}{A}} \frac{1-\alpha sn^2\left(\sqrt{B+2Ah}|\xi|, \sqrt{\frac{B-\sqrt{2Ah}}{B+\sqrt{2Ah}}}\right)}{1+\alpha sn^2\left(\sqrt{B+2Ah}|\xi|, \sqrt{\frac{B-\sqrt{2Ah}}{B+\sqrt{2Ah}}}\right)},$$

where $\alpha = \frac{\sqrt{B+\sqrt{\Delta}}+\sqrt{B-\sqrt{\Delta}}}{\sqrt{B+\sqrt{\Delta}}-\sqrt{B-\sqrt{\Delta}}}$.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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