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An Introduction to the Theory of Matrix Near-Rings

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Abstract

Matrix rings are prominent in abstract algebra. In this paper we give an overview of the theory of matrix near-rings. A near-ring differs from a ring in that it does not need to be abelian and one of the distributive laws does not hold in general. We introduce two ways in which matrix near-rings can be defined and discuss the structure of each. One is as given by Beildeman and the other is as defined by Meldrum. Beildeman defined his matrix near-rings as normal arrays under the operation of matrix multiplication and addition. He showed that we have a matrix near-ring over a near-ring if, and only if, it is a ring. In this case it is not possible to obtain a matrix near-ring from a proper near-ring. Later, in 1986, Meldrum and van der Walt defined matrix near-rings over a near-ring as mappings from the direct sum of n copies of the additive group of the near-ring to itself. In this case it can be shown that a proper near-ring is obtained. We prove several properties, introduce some special matrices and show that a matrix notation can be introduced to make calculations easier, provided that n is small.

Keywords

Near-Rings, First Near-Ring Isomorphism, Zero Symmetric Near-Ring, Near-Ring Module and Matrix Near-Rings

1. Introduction

Near-rings were first discovered as nearfields by Leonard E. Dickson in 1905 when he constructed an algebraic structure that had all the properties of a field except with one distributive law missing.

Near-rings are generalisation of rings but having only one distributive law and addition is not necessarily commutative in general. Near-rings have found many applications including in areas like cryptography. In [1], the author tried to define matrix near-rings over an arbitrary near-ring N as normal arrays with the usual matrix addition and matrix multiplication. From the results obtained it was concluded that we can define matrix near-rings over N if, and only if, N is a ring. Later, [2] defined matrix near-rings over a near-ring N as mappings from N^n to itself, where N^n is the direct sum of n copies of the additive group of N. Matrix near-rings were defined over near-rings with identity and near-rings without identity. In this paper we only consider matrix near-rings over near-rings with an identity element. A natural question would be, can matrix near-rings be defined over an arbitrary near-ring N with the usual matrix addition and matrix multiplication? The answer turns up to be yes and that matrix near-rings over a near-ring N can only be defined if, and only if, N is a ring.

This paper consists of 4 chapters. In Chapter 2 we give a brief introduction to near-rings and the background material that will help us understand matrix near-rings. In Chapter 3, the main chapter of this paper, we introduce matrix near-rings. We first define them as given by [1] and then later on as mappings as seen in [2]. We also work out some examples. In Chapter 4 we give a conclusion.

2. Preliminary Material

Before we introduce the concept of matrix near-rings we will give a brief background of near-rings. Near-rings are similar to rings, with one distributive law missing and unlike rings, the additive groups of near-rings need not be abelian. Near-rings are a generalisation of rings, so every ring is a near-ring. In this paper we will include the proofs of only some selected results from [2]-[6].

We begin by defining what a near-ring is.

Definition 2.1. [7] A near-ring is a triple $\langle N,+,\cdot\rangle$ where N is a non empty set, in which the following holds,

1) $\langle N,+\rangle$ is a group, not necessarily abelian;
2) $\langle N,\cdot\rangle$ is a semigroup;
3) $n_1\cdot(n_2+n_3)=n_1\cdot n_2+n_1\cdot n_3$, for all n_1,n_2 and $n_3\in N$.

This defines a left near-ring since the left distributive law holds. The definition of a right near-ring follows from the above where we have the right distributive law instead of the left one in Definition 2.1 part (3). In this project when we refer to a near-ring we mean a left near-ring. Just as in ring theory, near-rings have a unique identity element.

If in addition $N\setminus\{0\}$ is a group under multiplication, then we have that N is a nearfield.

We have a wide range of examples of near-rings. We will list some of them in the next example.

Example 2.2. [3] Let G be a group (not necessarily abelian) with identity 0.

Then the following are examples of near-rings.

1) $M(G)=\{f\mid f:G\to G\}$, the set of all mappings from G to itself.
2) $M_0(G)=\{f\in M(G)\mid(0)f=0\}$ the set of all mappings that map the identity element of G to itself.
3) $M_c(G)=\{f\in M(G)\mid f \text{ is constant}\}$.

We verify that (1) is a left near-ring below.

Let $M(G)$ be the set of all mappings from G to itself, where $\langle G,+\rangle$ is a group. We can verify that $M(G)$ is a left near-ring with pointwise addition and composition of functions.

We define λ to be the zero map, that is for all $x\in G$, $(x)\lambda=0$. We have that $\lambda\in M(G)$, so $M(G)$ is non-empty.

For $f,g\in M(G)$, $(x)(f\circ g)=((x)f)g$, for all $x\in G$, so $f\circ g\in M(G)$ and function composition is a binary operation on $M(G)$.

Now, for all $f,g,h\in M(G)$ and $x\in G$ we have,

$$\begin{aligned}(x)(f\circ(g\circ h))&=((x)f)(g\circ h)\\&=(((x)f)g)h\\&=((x)(f\circ g))h\\&=(x)(f\circ g)\circ h,\end{aligned}$$

so $\langle M(G),\circ\rangle$ is a semigroup.

Using pointwise addition we have that for all $x\in G$,

$$(x)(f+g)=(x)f+(x)g\in M(G),$$

so function addition is a binary operation on $M(G)$.

Then for $f,g,h\in M(G)$ we have,

$$\begin{aligned}(x)(f+(g+h))&=(x)f+(x)(g+h)\\&=(x)f+(x)g+(x)h\\&=((x)f+(x)g)+(x)h\\&=(x)(f+g)+(x)h\\&=(x)((f+g)+h),\end{aligned}$$

that is, addition is associative.

For all $x\in G$, $f\in M(G)$ we have that,

$$(x)(\lambda+f)=(x)\lambda+(x)f=0+(x)f=(x)f$$
$$(x)(f+\lambda)=(x)f+(x)\lambda=(x)f+0=(x)f.$$

We let $(x)(-f)=-((x)f)$, then for all $x\in G$,

$$(x)((-f)+f)=(x)(-f)+(x)f=-((x)f)+(x)f=0=(x)\lambda,$$
$$(x)(f+(-f))=(x)f+(x)(-f)=(x)f-(x)f=0=(x)\lambda.$$

Thus, we have that λ is the additive identity element and each element has an additive inverse. Therefore, $M(G)$ is a group under addition.

The left distributive law holds, that is function composition distributes over point wise addition from the left, since we have that,

$$\begin{aligned}(x)(f\circ(g+h))&=((x)(f))(g+h)\\&=((x)f)g+((x)f)h\\&=(x)(f\circ g)+(x)(f\circ h)\\&=(x)((f\circ g)+(f\circ h)).\end{aligned}$$

Thus, $M(G)$ is a left near-ring since the left distributive law holds.

The right distributive law fails to hold if G contains more than one element. To check this, let $a, b \in G$. We define functions f_a and f_b by $(x) f_a = a$, $(x) f_b = b$, for all $x \in G$. Then for any $g \in M(G)$,

$$(x)\left[(f_a + f_b) \circ g\right] = \left((x)(f_a + f_b)\right) g = \left((x) f_a + (x) f_b\right) g = (a+b) g$$

while,

$$\begin{aligned}(x)\left[f_a \circ g + f_b \circ g\right] &= (x)(f_a \circ g) + (x)(f_b \circ g) \\ &= \left((x) f_a\right) g + \left((x) f_b\right) g \\ &= (a) g + (b) g.\end{aligned}$$

Therefore, the right distributive law can only hold when $(a+b) g = (a) g + (b) g$ for all $a, b \in G$. We conclude that g needs to be an endomorphism for the right distributive law to hold. But when G contains more than one element, not all the mappings of $M(G)$ are endomorphisms, (for example $(x) f_a$ for $a \neq 0$).

Just as in ring theory we have the notion of sub-near-rings. We give the formal definition below.

Definition 2.3. [8] A non-empty subset A of a near-ring N is said to be a sub-near-ring of N if A satisfies all the properties in Definition 2.1.

As for rings it can be shown that a subset A of N, $\langle A, +, \cdot \rangle$ is a sub-near-ring of $\langle N, +, \cdot \rangle$ if A is non empty and for every $a, a' \in A$ we have that $a - a' \in A$ and $a \cdot a' \in A$. This is the sub-near-ring test.

Now we show some properties of near-rings.

Lemma 2.4. [8] *Let* $\langle N, +, \cdot \rangle$ *be a left near-ring. Then,*

1) $n \cdot 0 = 0$,

2) $n \cdot (-m) = -n \cdot m$,

for all $n, m \in N$.

Proof. 1) For all $n \in N$, we have, $n \cdot 0 = n \cdot (0+0) = n \cdot 0 + n \cdot 0$, so that $n \cdot 0 = 0$.

2) Also, for all $n, m, \in N$, we have that $0 = n \cdot 0 = n \cdot \left(m + (-m)\right) = n \cdot m + n \cdot (-m)$ so that $n \cdot (-m) = -n \cdot m$. □

In our near-ring N, we have that $n \cdot 0 = 0$ for all $n \in N$, but $0 \cdot n = 0$ for all $n \in N$ is not generally true, this brings us to the following parts of a near-ring.

Definition 2.5. [8] Let N be a near-ring.

1) $N_0 = \{n \in N \mid 0 \cdot n = 0\}$ is the zero symmetric part of N.

2) $N_c = \{n \in N \mid 0 \cdot n = n\}$ is the constant part of N.

Both N_0 and N_c are sub-near-rings and $\langle N_0, + \rangle$ is a normal subgroup of $\langle N, + \rangle$. But we will not show that in this paper.

A near-ring N is called a **zero symmetric** near-ring if $N = N_0$. Since most researchers in this field require that this be an extra property, we will only consider zero symmetric near-rings in this paper.

We now give another part of near-rings, the distributive part.

Definition 2.6. [8] An element $d \in N$ is distributive if for every $m, m' \in N$,

$$(m+m')\cdot d = m\cdot d + m'\cdot d.$$

We also define,

$$N_d = \{d \in N \mid d \text{ is distributive}\}.$$

A subset S of a group G is said to be a generating set of G if every element of G can be expressed as a combination (under the group binary operation) of finitely many elements of S. In other words G is the intersection of all subgroups containing elements of S.

Now we give the following definition of distributively generated near-ring.

Definition 2.7. [1] A near-ring N is said to be distributively generated if, and only if, N contains a multiplicative group B of distributive elements that generate the additive group of N.

If we have that N_d generates $\langle N,+\rangle$, then N is said to be distributively generated or *d.g* for short. It can easily be shown that $N_d \subseteq N_0$.

We now give the following theorem which tells us about the decomposition of a near-ring into a zero symmetric part and a constant part.

Theorem 2.8. [8] Let N be a near-ring. Then for every N, we have that $N_0 \cap N_c = \{0\}$ and $N = N_0 + N_c$.

Proof. Let $x \in N_0 \cap N_c$. Then $x \in N_0$ and $x \in N_c$, so we have that,

$$0\cdot x = 0, \text{ since } x \in N_0.$$

Also,

$$0\cdot x = x, \text{ since } x \in N_c.$$

Therefore, $x = 0$ and this implies that,

$$N_0 \cap N_c = \{0\}.$$

Now, for any $n \in N$ and suppose that $(n-0\cdot n) \in N_0$, we have,

$$\begin{aligned} 0\cdot(n-0\cdot n) &= 0\cdot n - 0\cdot(0\cdot n) \\ &= 0\cdot n - 0\cdot n \\ &= 0, \end{aligned}$$

which shows that $(n-0\cdot n) \in N_0$.

Similarly, suppose that $0\cdot n \in N_c$,

$$\begin{aligned} 0\cdot(0\cdot n) &= 0\cdot(n-(n-0\cdot n)), \text{since } 0\cdot n = n-(n-0\cdot n) \\ &= 0\cdot n - 0\cdot(n-0\cdot n) \\ &= 0\cdot n - 0 \\ &= 0\cdot n. \end{aligned}$$

which shows that $0\cdot n \in N_c$.

Finally if $n \in N$ is defined by,

$$n = (n-0\cdot n) + 0\cdot n,$$

Therefore, since $0\cdot n \in N_c, (n-0\cdot n) \in N_0$ we have that,

$$N = N_0 + N_c.$$

□

As in ring theory we have modules of near-rings. Since we are working with left near-rings we will define a right module below.

From now on we will not write $\cdot$ for multiplication of elements of the near-ring N, but use juxtaposition instead.

Definition 2.9. [1] A right near-ring module M over a near-ring N is an additive group M, together with a near-ring N and a mapping

$$\gamma : M \times N \to M$$

defined by

$$(m,n)\gamma = m \cdot n \text{ where } m \in M, n \in N,$$

such that for any $m \in M$ and $n_1, n_2 \in N$ we have the following axioms,

1) $m \cdot (n_1 + n_2) = m \cdot n_1 + m \cdot n_2$,

2) $m \cdot (n_1 n_2) = (m \cdot n_1) \cdot n_2$.

Let N have an identity element, 1. If we have the extra axiom,

3) $m \cdot 1 = m$, for all $m \in M$,

then M is said to be a **unitary** module.

We denote a near-ring module M over N by M_N and it is called an N-module.

Below we give some examples of modules.

Example 2.10. [8] Let H be a group and $n \in \mathbb{Z}$, the set of integers. We define for all $h \in H$ and $n \in \mathbb{Z}$,

$$h \cdot 0 = 0_h, \text{ where } 0_h \text{ is the identity of } H,$$

$$\begin{aligned} h \cdot n &= h + h + \cdots + h, (n \text{ elements}) \text{ if } n > 0, \\ &= -h - h - \cdots - h, (-n \text{ elements}) \text{ if } n < 0. \end{aligned}$$

Then H is a near-ring $\mathbb{Z}$-module.

We will show that the axioms of a module are satisfied.

For any $h \in H$ and $n, m \in \mathbb{Z}$ we have

$$\begin{aligned} h \cdot (n+m) &= \underbrace{h + h + \cdots + h}_{n+m \text{ times}}, \\ &= \underbrace{h + h + \cdots + h}_{n \text{ times}} + \underbrace{h + h + \cdots + h}_{m \text{ times}}, \\ &= h \cdot n + h \cdot m. \end{aligned}$$

Also,

$$\begin{aligned} h \cdot (nm) &= \underbrace{h + h + \cdots + h}_{n \cdot m \text{ times}} \\ &= \left(\underbrace{h + h + \cdots + h}_{n \text{ times}} \right) \cdot m. \\ &= \underbrace{h \cdot n + h \cdot n + \cdots + h \cdot n}_{m \text{ times}} \\ &= (h \cdot n) \cdot m. \end{aligned}$$

And since $1 \in \mathbb{Z}$, we have that $h \cdot 1 = h$ for every $h \in H$. Thus, H is a **unitary** $\mathbb{Z}$-module.

Example 2.11. Let N be a near-ring. Then, the set N^n for n an integer whose elements are of the form $(n_1, n_2, \cdots, n_n) \in N^n$ for every $n_i \in N$, and

$i \in \{1, 2, \cdots, n\}$ with coordinate-wise addition defined for every $(n_1, n_2, \cdots, n_n), (m_1, m_2, \cdots, m_n) \in N^n$ by,

$$(n_1, n_2, \cdots, n_n) \oplus (m_1, m_2, \cdots, m_n) = (n_1 + m_1, n_2 + m_2, \cdots, n_n + m_n),$$

and scalar multiplication defined for all $(m_1, m_2, \cdots, m_n) \in N^n$ and $m \in N$ by,

$$(m_1, m_2, \cdots, m_n) \cdot m = (m_1 m, m_2 m, \cdots, m_n m),$$

is an N-module.

For any $(m_1, m_2, \cdots, m_n) \in N^n, m, n \in N$, with 1 the identity of N, we have,

$$\begin{aligned}(m_1, m_2, \cdots, m_n) \cdot (n+m) &= (m_1(n+m), m_2(n+m), \cdots, m_n(n+m)) \\ &= (m_1 n + m_1 m, m_2 n + m_2 m, \cdots, m_n n + m_n m) \\ &= (m_1 n, m_2 n, \cdots, m_n n) + (m_1 m, m_2 m, \cdots, m_n m) \\ &= (m_1, m_2, \cdots, m_n) \cdot n + (m_1, m_2, \cdots, m_n) \cdot m.\end{aligned}$$

Also,

$$\begin{aligned}(m_1, m_2, \cdots, m_n) \cdot (nm) &= (m_1(nm), m_2(nm), \cdots, m_2(nm)) \\ &= ((m_1 n) m, (m_2 n) m, \cdots, (m_n n) m) \\ &= (m_1 n, m_2 n, \cdots, m_n n) \cdot m \\ &= ((m_1, m_2, \cdots, m_n) \cdot n) \cdot m.\end{aligned}$$

To check the identity axiom,

$$(m_1, m_2, \cdots, m_n) \cdot 1 = (m_1 1, m_2 1, \cdots, m_n 1) = (m_1, m_2, \cdots, m_n).$$

Therefore, N^n is a **unitary** near-ring module over N.

A near-ring module has different properties, we will list them below and verify each one of them.

Lemma 2.12. [1] *Let M_N be an N-module with an identity element 0_M. Then we* have,

1) $0_M \cdot 0 = 0_M$,
2) $x \cdot 0 = 0_M$, for all $x \in M$,
3) $0_M \cdot y = 0_M$ for all $y \in N$,
4) $x \cdot (-y) = -x \cdot y$ for all $y \in N, x \in M$.

Proof.

1) $0_M \cdot 0 = 0_M \cdot (0+0) = 0_M \cdot 0 + 0_M \cdot 0$, so that $0_M \cdot 0 = 0_M$.

2) For all $x \in M$ we have, $x \cdot 0 = x \cdot (0+0) = x \cdot 0 + x \cdot 0$, so that $x \cdot 0 = 0_M$.

3) Let $y \in N$. Then, $0_M \cdot y = (0_M \cdot 0) \cdot y = 0_M \cdot (0 \cdot y) = 0_M \cdot 0 = 0_M$, so that $0_M \cdot y = 0_M$.

4) Let $x \in M$ and $y \in N$. Then,
$x \cdot y + (-x \cdot y) = 0_M = x \cdot 0 = x \cdot (y + (-y)) = x \cdot y + x \cdot (-y)$ so that $x \cdot (-y) = -x \cdot y$.

□

We now give the definition of a submodule below.

Definition 2.13. [1] A subset H of an N-module M_N is said to be submodule if, and only if,

1) H is a normal subgroup of $\langle M,+\rangle$,

2) $(m+h)\cdot n-m\cdot n\in H$, where $m\in M_N, h\in H$ and $n\in N$.

Just like in ring theory we have homomorphisms to help find structural properties between two near-rings.

Definition 2.14. [1] Let N and N' be near-rings. A mapping ϕ from N into N' is called a near-ring homomorphism if for all $n,n'\in N$,

$$(n+n')\phi=(n)\phi+(n')\phi \text{ and } (nn')\phi=(n)\phi(n')\phi.$$

Remark 2.15. [8]

1) An injective (one-to-one) homomorphism is called a monomorphism.

2) A surjective (onto) homomorphism is called an epimorphism.

3) A homomorphism that is both one-to-one and onto is known as an isomorphism.

The term embed is used to mean, "*map by means of a monomorphism*."

We will now provide an interesting theorem about the embedding of near-rings into other algebraic structures.

Theorem 2.16. [3] Let $\langle N,+,\cdot\rangle$ be a near-ring and $\langle G,+\rangle$ a group which properly contains an isomorphic copy of $\langle N,+\rangle$. Then it is possible to embed $\langle N,+,\cdot\rangle$ in $M(G)$.

Proof. We identify $\langle N,+\rangle$ with its isomorphic copy contained in some group *G*. Let $\psi_n:G\to G$, where for $n\in N$, ψ_n is defined by,

$$(g)\psi_n=\begin{cases} gn & \text{if } g\in N, \\ n & \text{otherwise.}\end{cases}$$

We define a map $\theta:N\to M(G)$ for $n\in N$ by,

$$(n)\theta=\psi_n.$$

We now show that θ is a monomorphism.

For any $m,m'\in N$ we have,

$$\begin{aligned}(m+m')\theta&=\psi_{m+m'}\\ &=\psi_m+\psi_{m'}\\ &=(m)\theta+(m')\theta.\end{aligned}$$

Also,

$$\begin{aligned}(mm')\theta&=\psi_{mm'}\\ &=\psi_m\circ\psi_{m'}\\ &=((m)\theta)((m')\theta).\end{aligned}$$

The homomorphism property holds.

Now, we show that θ is an injective map.

Suppose m,m' are both in *N*. Then,

$$(m)\theta=(m')\theta$$

implies that $\psi_m=\psi_{m'}$

so that $m=m'$

Thus, θ is a monomorphism and thus an embedding map. □

The above theorem tells us that every near-ring can be considered as a sub-near-ring of some $M(G)$.

Since $M(G)$ is a nearing with an identity element we can now derive the following corollary.

Corollary 2.17. [8] *Every near-ring can be embedded in a near-ring with identity.*

Isomorphism theorems that apply in other algebraic structures such as groups and rings also apply in near-rings. We will take a moment to give the First near-ring Isomorphism Theorem. Before stating the theorem we give some important definitions we will need.

Definition 2.18. [1] Let N and N' be near-rings. Let ϕ be a near-ring homomorphism from N to N'. Then we have,

1) The image of ϕ in N' is,

$$\operatorname{im}(\phi) = \{(n)\phi : n \in N\},$$

2) The kernel of ϕ denoted by $\ker(\phi)$ is given by,

$$\ker(\phi) = \{n \in N \mid (n)\phi = 0 \in N'\}.$$

Theorem 2.19. [8] (First near-ring Isomorphism Theorem) Let N and N' be near-rings. Let ϕ be a near-ring homomorphism from N to N'. Then,

$$N/\ker(\phi) \cong \operatorname{im}(\phi).$$

Having discussed the necessary background material we will now introduce the concept of matrix near-rings in the next chapter.

3. Matrix Near-Rings

In this section we look at two possible ways of defining matrix near-rings. We restrict the discussion to near-rings with identity. Results, definitions and theorems are similar to those in [1] [9] [10] [11].

3.1. Defining Matrix Near-Rings as Arrays

If we try defining matrix near-rings as normal arrays with the usual matrix addition and multiplication over a near-ring as seen in [1], we observe that the set of $n \times n$ matrices is not associative under multiplication because of the missing distributive law in our near-ring. We begin proving some results. We will need the following definition.

Definition 3.1. [1] Let N be a left near-ring with an identity element. A matrix over N is an $n \times n$ rectangular array of the form,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

with n rows and n columns and elements a_{ij} from the near-ring N.

Let $M(N)$ be the set of $n \times n$ matrices over *N*. Two matrices *A* and $B \in M(N)$ are said to be equal if the corresponding elements $a_{ij} = b_{ij}$ for every i, j.

We define addition in $M(N)$ by,

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$$
$$= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & \cdots & a_{nn}+b_{nn} \end{pmatrix} \in M(N),$$

i.e., we add corresponding elements of the two matrices.

Multiplication in $M(N)$ is defined as,

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} = \begin{pmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{pmatrix} \in M(N),$$

where,

$$d_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj}.$$

We now give a theorem which tells us that $\langle M(N), \cdot \rangle$ is a semigroup if, and only if, *N* is a ring.

Theorem 3.2. [1] Let *N* be a near-ring with identity and $\langle N, + \rangle$ an abelian group. $\langle M(N), \cdot \rangle$, for $n > 1$ is a semigroup if, and only if, *N* is a ring.

Proof. Suppose *N* is a ring, then for any $A, B, C \in M(N)$ we have that,

$$A(BC) = (AB)C,$$

because rings have the associative law. Therefore, $\langle M(N), \cdot \rangle$ is a semigroup.

Conversely, suppose $\langle M(N), \cdot \rangle$ is a semigroup. Since $\langle N, + \rangle$ is abelian, it suffices to show that $\langle N, +, \cdot \rangle$ satisfies the right distributive law. For any $n, n_1, n_2 \in N$, let $A, B, C \in M(N)$ be defined by

$$A = \begin{pmatrix} n_1 & 1 & 0\cdots & 0 \\ 0 & 1 & 0\cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & n_2 & 0\cdots & 0 \\ n_2 & 0 & 0\cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} n & 0 & 0\cdots & 0 \\ 1 & 0 & 0\cdots & 0 \\ 0 & 0 & 0\cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

Then

$$A(BC) = \begin{pmatrix} n_1 & 1 & 0\cdots & 0 \\ 0 & 1 & 0\cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \left[\begin{pmatrix} 1 & n_2 & 0\cdots & 0 \\ n_2 & 0 & 0\cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \begin{pmatrix} n & 0 & 0\cdots & 0 \\ 1 & 0 & 0\cdots & 0 \\ 0 & 0 & 0\cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \right]$$

$$= \begin{pmatrix} n_1 & 1 & 0\cdots & 0 \\ 0 & 1 & 0\cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \begin{pmatrix} n+n_2 & 0 & 0\cdots & 0 \\ n_2 n & 0 & 0\cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} n_1 n + n_1 n_2 + n_2 n & 0 & 0\cdots & 0 \\ n_2 n & 0 & 0\cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$

Also we have that,

$$(AB)C = \left[\begin{pmatrix} n_1 & 1 & 0\cdots & 0 \\ 0 & 1 & 0\cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \begin{pmatrix} 1 & n_2 & 0\cdots & 0 \\ n_2 & 0 & 0\cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}\right] \begin{pmatrix} n & 0 & 0\cdots & 0 \\ 1 & 0 & 0\cdots & 0 \\ 0 & 0 & 0\cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} n_1+n_2 & n_1 n_2 & 0\cdots & 0 \\ n_2 & 0 & 0\cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \begin{pmatrix} n & 0 & 0\cdots & 0 \\ 1 & 0 & 0\cdots & 0 \\ 0 & 0 & 0\cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} (n_1+n_2)n + n_1 n_2 & 0 & 0\cdots & 0 \\ n_2 n & 0 & 0\cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

Since corresponding entries of the matrices are equal, we have that,

$$(n_1+n_2)n + n_1 n_2 = n_1 n + n_1 n_2 + n_2 n$$

$(n_1+n_2)n + n_1 n_2 = n_1 n + n_2 n + n_1 n_2$, since $\langle N,+\rangle$ is abelian,
so that $(n_1+n_2)n = n_1 n + n_2 n$.

Therefore, the right distributive law holds. Thus, N is a ring. □

An immediate result is the following corollary which tells us that if the additive group of N is abelian, then $M(N)$ with multiplication forms a groupoid.

Corollary 3.3. [1] *Let N be a proper near-ring with identity and* $\langle N,+\rangle$ *an abelian group. Then* $\langle M(N),\cdot\rangle$, *for* $n>1$ *forms a groupoid and not a semigroup.*

Proof. Let $M(N)$ be the set of $n\times n$ matrices. Since $0\in N$, we have that the zero matrix given by

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in M(N),$$

thus $M(N)$ is non empty.

Now, for all $A,B \in M(N)$ when we multiply two matrices, we have that the product $AB \in M(N)$. Therefore $M(N)$ is closed under multiplication.

Also, since $1 \in N$, we have an identity element $I \in M(N)$ given by,

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix},$$

such that, for any $A \in M(N)$,

$$AI = A = IA.$$

Clearly, $\langle M(N),\cdot\rangle$ is a groupoid.

Now, we show that the associativity property does not hold in general.

Since N is a left near-ring we can choose $a,b,c \in N$ so that $(a+b)c \neq ac+bc$.

Now, let $A,B,C \in M(N)$ be defined as,

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix},\ C = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}.$$

Then,

$$\begin{aligned} A(BC) &= \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\left[\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}\right] \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} ac & 0 \\ bc & 0 \end{pmatrix} = \begin{pmatrix} ac+bc & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

while,

$$\begin{aligned} (AB)C &= \left[\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\right]\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} a+b & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (a+b)c & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Therefore, associativity fails to hold in general. Thus, $\langle M(N),\cdot\rangle$ is not a semigroup. □

The definition of a near-ring $\langle N,+,\cdot\rangle$ does not require that $\langle N,+\rangle$ is abelian. So now, we state a theorem that tells us that for matrix near-rings $\langle M(N),+,\cdot\rangle$, $\langle N,+\rangle$ needs to be an abelian group.

Theorem 3.4. [1] Let N be a near-ring with identity. $\langle M(N),+,\cdot\rangle$ has a left distributive law if, and only if, $\langle N,+\rangle$ is abelian.

Proof. Suppose $\langle M(N),+,\cdot\rangle$ satisfies the left distributive law. Then for any $n_1,n_2 \in N$, let $A,B,C \in M(N)$ be defined by,

$$A = \begin{pmatrix} n_1 & 1 & 0\cdots & 0 \\ 0 & 0 & 0\cdots & 0 \\ 0 & 0 & 0\cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & n_2 & 0\cdots & 0 \\ 0 & n_2 & 0\cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 1 & 0\cdots & 0 \\ 1 & 1 & 0\cdots & 0 \\ 0 & 0 & 0\cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

Then we have that,

$$A(B+C)=\begin{pmatrix} n_1 & 1 & 0\cdots & 0 \\ 0 & 0 & 0\cdots & 0 \\ 0 & 0 & 0\cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}\left[\begin{pmatrix} 1 & n_2 & 0\cdots & 0 \\ 0 & n_2 & 0\cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}+\begin{pmatrix} 1 & 1 & 0\cdots & 0 \\ 1 & 1 & 0\cdots & 0 \\ 0 & 0 & 0\cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}\right]$$

$$=\begin{pmatrix} n_1+n_1+1 & n_1n_2+n_1+n_2+1 & 0\cdots & 0 \\ 0 & 0 & 0\cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

Also we have that,

$$AB=\begin{pmatrix} n_1 & 1 & 0\cdots & 0 \\ 0 & 0 & 0\cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}\begin{pmatrix} 1 & n_2 & 0\cdots & 0 \\ 0 & n_2 & 0\cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}=\begin{pmatrix} n_1 & n_1n_2+n_2 & 0\cdots & 0 \\ 0 & 0 & 0\cdots & 0 \\ 0 & 0 & 0\cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix},$$

and

$$AC=\begin{pmatrix} n_1 & 1 & 0\cdots & 0 \\ 0 & 0 & 0\cdots & 0 \\ 0 & 0 & 0\cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0\cdots & 0 \\ 1 & 1 & 0\cdots & 0 \\ 0 & 0 & 0\cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}=\begin{pmatrix} n_1+1 & n_1+1 & 0\cdots & 0 \\ 0 & 0 & 0\cdots & 0 \\ 0 & 0 & 0\cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix},$$

so now,

$$AB+AC=\begin{pmatrix} n_1+n_1+1 & n_1n_2+n_2+n_1+1 & 0\cdots & 0 \\ 0 & 0 & 0\cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

From the above we have that,

$$n_1n_2+n_1+n_2+1=n_1n_2+n_2+n_1+1$$

so that, $n_1+n_2=n_2+n_1$,

so that the additive group of N is abelian.

Conversely, suppose $\langle N,+\rangle$ is abelian. Then for all $A,B,C\in M(N)$ we have that,

$$[A(B+C)]_{ij}=\sum_{k=1}^{n}a_{ik}\left(b_{kj}+c_{kj}\right)=\sum_{k=1}^{n}\left(a_{ik}b_{kj}+a_{ik}c_{kj}\right).$$

Since,

$$[AB]_{ij}=\sum_{k=1}^{n}a_{ik}b_{kj}\ \text{ and }\ [AC]_{ij}=\sum_{k=1}^{n}a_{ik}c_{kj},$$

we also have that,

$$\begin{aligned}[AB]_{ij}+[AC]_{ij}&=\sum_{k=1}^{n}a_{ik}b_{kj}+\sum_{k=1}^{n}a_{ik}c_{kj},\\&=\sum_{k=1}^{n}a_{ik}\left(b_{kj}+c_{kj}\right),\ \text{since } \langle N,+\rangle \text{ is abelian.}\end{aligned}$$

From the above result we have that

$$A(B+C)=AB+AC.$$

Therefore, $\langle M(N),+,\cdot\rangle$ satisfies the left distributive law. □

From Theorems 3.2 and 3.4 we can conclude the following about $\langle M(N),+,\cdot\rangle$.

Corollary 3.5. [1] *Let N be a near-ring with an identity element. Then* $\langle M(N),+,\cdot\rangle$ *is a near-ring if, and only if, N is a ring.*

The following result tells us that the additive group of $M(N)$ forms a module.

Proposition 3.6. [1] *Let N be a near-ring with an identity element. Then, the additive group of* $M(N)$ *can be considered a unitary N-module.*

Proof. We will show the axioms of a module.

For $A\in M(N)$ and $m\in N$, we define,

$$A\cdot m=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}\cdot m=\begin{pmatrix} a_{11}m & a_{12}m & \cdots & a_{1n}m \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}m & a_{n2}m & \cdots & a_{nn}m \end{pmatrix}\in M(N).$$

Now, for any $A\in M(N)$ and $m,n\in N$, we have,

$$\begin{aligned} A\cdot(m+n) &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}\cdot(m+n) \\ &= \begin{pmatrix} a_{11}(m+n) & a_{12}(m+n) & \cdots & a_{1n}(m+n) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(m+n) & a_{n2}(m+n) & \cdots & a_{nn}(m+n) \end{pmatrix} \\ &= \begin{pmatrix} a_{11}m+a_{11}n & a_{12}m+a_{12}n & \cdots & a_{1n}m+a_{1n}n \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}m+a_{n1}n & a_{n2}m+a_{n2}n & \cdots & a_{nn}m+a_{nn}n \end{pmatrix} \\ &= \begin{pmatrix} a_{11}m & a_{12}m & \cdots & a_{1n}m \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}m & a_{n2}m & \cdots & a_{nn}m \end{pmatrix}+\begin{pmatrix} a_{11}n & a_{12}n & \cdots & a_{1n}n \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}n & a_{n2}n & \cdots & a_{nn}n \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}\cdot m+\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}\cdot n. \end{aligned}$$

Also we have,

$$\begin{aligned} A\cdot(mn) &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}\cdot(mn) \\ &= \begin{pmatrix} a_{11}mn & a_{12}mn & \cdots & a_{1n}mn \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}mn & a_{n2}mn & \cdots & a_{nn}mn \end{pmatrix} \\ &= \begin{pmatrix} a_{11}m & a_{12}m & \cdots & a_{1n}m \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}m & a_{n2}m & \cdots & a_{nn}m \end{pmatrix}\cdot n. \end{aligned}$$

Also, since $1\in N$, we have that

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \cdot 1 = \begin{pmatrix} a_{11}1 & a_{12}1 & \cdots & a_{1n}1 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}1 & a_{n2}1 & \cdots & a_{nn}1 \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \in M(N).$$

Therefore, $\langle M(N), + \rangle$ becomes a unitary near-ring module over *N*. □

We now find an alternate way of defining a proper near-ring of matrices in the next section.

3.2. Defining Matrix Near-Rings as Functions

As long as we view matrices as arrays of entries with the usual matrix addition and multiplication, it will not make sense to define a proper near-ring of matrices over an arbitrary near-ring. We could consider $n \times n$ matrices over a ring N as functions of N^n to N^n where N^n is the direct sum of n copies of the additive group of *N*. Before we give a formal definition of matrix near-rings as originally defined by [2], we will first take note of some notations we will need.

Let n be a natural number and $\langle N, +, \cdot \rangle$ a near-ring with an identity element. Let l_j be the j^{th}-coordinate injection function and π_j the j^{th}-coordinate projection function. That is, for any $(m_1, m_2, \cdots, m_n) \in N^n$ and $m \in N$, we have that,

$(m_1, m_2, \cdots, m_n)\pi_j = m_j$ and

$(m)l_j = (0, \cdots, 0, m, 0, \cdots, 0)$, with m in the j^{th} position and zeros elsewhere.

For each $k \in N$ there corresponds a function f^k from N to itself, defined by,

$$(s) f^k = sk, \forall s \in N.$$

We define our matrices using this embedding of N into $M(N)$ as seen in Theorem 2.16 of Chapter 2.

We now introduce the function given by,

$$f_{ij}^k : N^n \to N^n,$$

where, $f_{ij}^k = \pi_j f^k l_i, 1 \le i, j \le n, k \in N$.

In rings, $n \times n$ matrices over a ring can be expressed as sums and products of elementary matrices kE_{ij} with k in the $(i, j)^{\text{th}}$ position and zeros elsewhere,

$$kE_{ij} = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & k & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}.$$

So we can consider f_{ij}^k to be elementary matrices.

When using the f_{ij}^k functions in calculations we will use the following notation.

For any $m_1,\cdots,m_n \in N^n$,

$$\begin{aligned}(m_1,\cdots,m_n) f_{ij}^k &= (m_1,\cdots,m_n)\pi_j f^k l_i = (m_j) f^k l_i = (m_j k) l_i \\ &= (0,\cdots,0,m_j k,0,\cdots,0), \quad \text{with } m_j k \text{ in the } i^{\text{th}} \text{ position.}\end{aligned}$$

We now formally define a matrix near-ring using the concept introduced earlier where we consider $n\times n$ matrices as mappings from N^n to itself.

In the definition below, by saying $M_n(N)$ is generated by the set $\left\{f_{ij}^k : k\in N, 1\le i,j\le n\right\}$, we mean that it is closed under the operations of addition, differences and products.

Definition 3.7. [11] The near-ring of $n\times n$ matrices over N denoted by $M_n(N)$ is a sub-near-ring of $M(N^n)$, the near-ring of all mappings from N^n to itself, generated by the set,

$$\left\{f_{ij}^k : k\in N, 1\le i,j\le n\right\}.$$

The elements of $M_n(N)$ will be referred to as $n\times n$ matrices over *N*.

An immediate result from the definition of a matrix near-ring is the following proposition.

Proposition 3.8. [11] $M_n(N)$ *is a left near-ring with identity element* $f_{11}^1 + f_{22}^1 + \cdots + f_{nn}^1 \in M_n(N)$. *Where 1 is the identity element of N.*

Proof. $M_n(N)$ being a near-ring follows from Definition 3.7. So we now verify that $f_{11}^1 + f_{22}^1 + \cdots + f_{nn}^1$ is the identity element.

Take any $(m_1,m_2,\cdots,m_n)\in N^n$, so we have,

$$\begin{aligned}&(m_1,m_2,\cdots,m_n)\left(f_{11}^1 + f_{22}^1 + \cdots + f_{nn}^1\right) \\ &= (m_1,m_2,\cdots,m_n) f_{11}^1 + (m_1,m_2,\cdots,m_n) f_{22}^1 + \cdots + (m_1,m_2,\cdots,m_n) f_{nn}^1 \\ &= (m_1 1) l_1 + (m_2 1) l_2 + \cdots + (m_n 1) l_n \\ &= (m_1) l_1 + (m_2) l_2 + \cdots + (m_n) l_n \\ &= (m_1,0,\cdots,0) + (0,m_2,0,\cdots,0) + \cdots + (0,0,\cdots,0,m_n) \\ &= (m_1,m_2,\cdots,m_n).\end{aligned}$$

□

Proposition 3.9. [11] *If N is a ring with an identity element, then* $M_n(N)$ *is isomorphic to the usual full ring of* $n\times n$ *matrices over N.*

Proposition 3.9 tells us that if we have that N is a ring, then both distributive laws hold and we can define matrix near-rings as arrays with the usual matrix addition and multiplication and have a matrix ring.

In the next section we give an alternative notation for matrices.

3.3. Alternative Notation for Matrices

Now, the question the reader may have is whether or not we have an alternative notation for matrices which makes actual calculations feasible. We will show that for small *n*, we have a notation close to the normal notation used in matrix ring theory.

We make use of the following conventions. Although the elements of N^n are considered as column vectors, we represent them as n-tuples, $\mathcal{D} = (m_1, m_2, \cdots, m_n) \in N^n$.

Recall that 1 is the identity element of N. The matrix units are of the form $f_{ij}^1, 1 \le i, j \le n$ and the identity matrix is given by $f_{11}^1 + f_{22}^1 + \cdots + f_{nn}^1$ as shown in Proposition 3.8.

If $i \in \{1, 2, \cdots, n\}$, the function $A\pi_i : N^n \to N$ is called the i^{th} row of the matrix A, it follows that $A = \sum_{i=1}^{n} (A\pi_i) l_i$. The i^{th} column of the matrix A is defined as the function $l_i A : N \to N^n$.

Scalar multiplication on the right of the matrix A by an element $k \in N$ is defined by,

$$Ak = \sum_{i=1}^{n} A\pi_i f^k l_i.$$

We show the result below based on notation from [11].

It follows that $f_{ij}^1 k = f_{ij}^k$, if, and only if, $k \in N_0$. We show this below.

Suppose, $k \in N_0$. Then, for any $(m_1, m_2, \cdots, m_n) \in N^n$, we have that:

$$\begin{aligned}(m_1, m_2, \cdots, m_n) f_{ij}^1 k &= (0k, \cdots, 0k, m_j k, 0k, \cdots, 0k) \\ &= (0, \cdots, 0, m_j k, 0, \cdots, 0), \text{ with } m_j k \text{ in the } i^{\text{th}} \text{ position of the vector.}\end{aligned}$$

Also, we have

$$(m_1, m_2, \cdots, m_n) f_{ij}^k = (0, \cdots, 0, m_j k, 0, \cdots, 0), \text{ by definition of } f_{ij}^k.$$

This implies that $f_{ij}^1 k = f_{ij}^k$ for any $k \in N_0$.

Conversely, if $f_{ij}^1 k = f_{ij}^k$, then we have that:

$$\begin{aligned}(1, 1, \cdots, 1) f_{ij}^1 k &= (0k, \cdots, 0k, k, 0k, \cdots, 0k) \\ = (1, 1, \cdots, 1) f_{ij}^k &= (0, \cdots, 0, k, 0, \cdots, 0) \text{ with } k \text{ in the } i^{\text{th}} \text{ position.}\end{aligned}$$

Therefore, $0k = 0$.

Scalar multiplication on the left of A is defined by $kA = (f_{11}^k + f_{22}^k + \cdots + f_{nn}^k) A$. In this case we have that $kf_{ij}^1 = f_{ij}^k$ for any $k \in N$.

Suppose $k \in N$. Then, for any $(m_1, m_2, \cdots, m_n) \in N^n$, we have that,

$$\begin{aligned}(m_1, m_2, \cdots, m_n) k f_{ij}^1 &= (m_1 k, m_2 k, \cdots, m_j k, \cdots, m_n k) f_{ij}^1 \\ &= (0, \cdots, 0, m_j k, 0, \cdots, 0), \text{ with } m_j k \text{ in the } i^{\text{th}} \text{ position of the vector.}\end{aligned}$$

Also, we have

$$(m_1, m_2, \cdots, m_n) f_{ij}^k = (0, \cdots, 0, m_j k, 0, \cdots, 0), \text{ by definition of } f_{ij}^k.$$

This implies that $kf_{ij}^1 = f_{ij}^k$ for any $k \in N$.

Since we have restricted our study to zero symmetric near-rings, it is clear that scalar multiplication on the left and right is the same.

Our alternative notation for matrices will be column vectors whose entries are functions from N^n to N. Each function is the appropriate row of the matrix defined previously. The following rules provide this representation in a recursive manner , where T represents transposition.

1) The matrix f_{ij}^k is represented by the vector with $\pi_j f^k$ in the i^{th} position

and zeros elsewhere and is given by

$$\left[0,\cdots,0,\pi_j f^k,0,\cdots,0\right]^{\mathrm{T}}.$$

2) If the matrices A and B are represented by

$$A=\left[a_1,a_2,\cdots,a_n\right]^{\mathrm{T}} \text{ and } B=\left[b_1,b_2,\cdots,b_n\right]^{\mathrm{T}}.$$

Then we have that,

$$A+B=\left[a_1+b_1,a_2+b_2,\cdots,a_n+b_n\right]^{\mathrm{T}}.$$

While AB is represented by the vector obtained from $A=\left[a_1,a_2,\cdots,a_n\right]^{\mathrm{T}}$ by replacing in a_k every occurrence of π_j by b_j.

Since we have assumed every element of N^n to be a column vector in this representation, we will write them as column vectors in the next example.

Example 3.10. We consider the case of 2×2 matrices, so we have that for any two matrices A and B given by,

$$A = f_{12}^{a}\left(f_{21}^{b}+f_{22}^{c}\right)+f_{22}^{d}$$

$$B = f_{21}^{p}+f_{11}^{q}\left(f_{12}^{s}+f_{11}^{t}\right).$$

We can represent the matrices by

$$A=\begin{bmatrix}\left(\pi_1 f^b+\pi_2 f^c\right)f^a\\ \pi_2 f^d\end{bmatrix} \text{ and } B=\begin{bmatrix}\left(\pi_2 f^s+\pi_1 f^t\right)f^q\\ \pi_1 f^p\end{bmatrix}.$$

To simplify further we may substitute f^r and $\pi_1 f^r$ by r and substitute $\pi_2 f^r$ by $\overline{r}$. So that A and B becomes:

$$A=\begin{bmatrix}\left(b+\overline{c}\right)a\\ \overline{d}\end{bmatrix} \text{ and } B=\begin{bmatrix}\left(\overline{s}+t\right)q\\ p\end{bmatrix}.$$

To illustrate how A acts as a function from N^2 to N^2. Let $\left[m_1,m_2\right]^{\mathrm{T}}\in N^n$. So we have:

$$\begin{bmatrix}m_1\\ m_2\end{bmatrix}A=\begin{bmatrix}m_1\\ m_2\end{bmatrix}\begin{bmatrix}\left(b+\overline{c}\right)a\\ \overline{d}\end{bmatrix}=\begin{bmatrix}\left(m_1 b+m_2 c\right)a\\ m_2 d\end{bmatrix}.$$

Multiplication is illustrated by:

$$AB=\begin{bmatrix}\left(b+\overline{c}\right)a\\ \overline{d}\end{bmatrix}\begin{bmatrix}\left(\overline{s}+t\right)q\\ p\end{bmatrix}=\begin{bmatrix}\left(b\left(\overline{s}+t\right)q+cp\right)a\\ dp\end{bmatrix}$$

Clearly, this notation is only convenient for small n. However, this notation shows us that the rows of a matrix are much more distinguishable than columns.

Just as in ring theory we do have the concept of special matrices which we will define in the next section.

3.4. Special Kinds of Matrices

We now define some special kinds of matrices which we know from matrix ring theory.

Definition 3.11. [11] Let $m_1,m_2,\cdots,m_n\in N$. A matrix is said to be a **diagonal**

matrix if it is of the form $f_{11}^{m_1} + f_{22}^{m_2} + \cdots + f_{nn}^{m_n}$. If we have that $m_1 = m_2 = \cdots = m_n$, the matrix is called a **scalar** matrix.

We can also define lower triangular matrices in two ways, one is that a matrix A is said to be a lower triangular matrix if, and only if, there is an expression for A consisting only of f_{ij}^{r} with $i \geq j$, apart from operators and parenthesis. The other way, which is equivalent to the first way is given in Definition 3.12.

Definition 3.12. [11] A matrix B in $M_n(N)$ is said to be **lower triangular** if, for any $i \in \{1, 2, \cdots, n\}$, we have that

$$m\pi_i B = m'\pi_i B, \text{ for all } m, m' \in N^n,$$

with $m\pi_j = m'\pi_j$, $j = \{1, 2, \cdots, i\}$.

We can also define an upper triangular matrix in a similar manner below.

Definition 3.13. [11] A matrix B is said to be an **upper triangular** matrix if, for any $i \in \{1, 2, \cdots, n\}$, we have that

$$m\pi_i B = m'\pi_i B, \text{ for all } m, m' \in N^n,$$

with $m\pi_j = m'\pi_j$, $j = \{i, i+1, \cdots, n\}$.

Now, we will denote the set of all lower triangular matrices by $\mathcal{L}$ and the set of all upper triangular matrices by $\mathcal{U}$.

Next we introduce a lemma that tells us that $\mathcal{L}$ and $\mathcal{U}$ are sub-near-rings of $M_n(N)$. We will use the sub-near-ring test to prove the following results.

Since we restricted our study to zero symmetric near-rings, we use that $mA = Am$ for $m \in N$ and $A \in M_n(N)$.

Lemma 3.14. [11] The *set of lower triangular matrices* $\mathcal{L}$ *and the set of upper triangular matrices* $\mathcal{U}$ *each form a sub-near-ring of* $M_n(N)$.

Proof. We first prove for the set of lower triangular matrices $\mathcal{L}$.

a) Suppose $A, B \in \mathcal{L}$. Let $i \in \{1, 2, \cdots, n\}$. Choose any $m, m' \in N^n$ with $m\pi_j = m'\pi_j$, $j \in \{1, 2, \cdots, i\}$. Then we have,

$$m\pi_i(A - B) = m\pi_i A - m\pi_i B = m'\pi_i A - m'\pi_i B = m'\pi_i(A - B).$$

Thus, $(A - B) \in \mathcal{L}$, this means $\langle \mathcal{L}, + \rangle$ is a subgroup of $\langle M_n(N), + \rangle$.

Further, we have that $m\pi_j A = m'\pi_j A$, $j \in \{1, 2, \cdots, i\}$ since $A \in \mathcal{L}$ and $m\pi_j = m'\pi_j$, $j \in \{1, 2, \cdots, i\}$. Therefore, we have that,

$$m\pi_i(AB) = (m\pi_i A)B = (m'\pi_i A)B = m'\pi_i(AB),$$

since $B \in \mathcal{L}$. Consequently $AB \in \mathcal{L}$, and $\mathcal{L}$ is a sub-near-ring of $M_n(N)$.

b) Similarly, we show for the set of upper triangular matrices $\mathcal{U}$.

Suppose $A, B \in \mathcal{U}$. Let $i \in \{1, 2, \cdots, n\}$. Choose any $m, m' \in N^n$ with $m\pi_j = m'\pi_j$, $j \in \{i, i+1, \cdots, n\}$. Then we have,

$$m\pi_i(A - B) = m\pi_i A - m\pi_i B = m'\pi_i A - m'\pi_i B = m'\pi_i(A - B).$$

Thus, $(A - B) \in \mathcal{U}$, so that $\langle \mathcal{U}, + \rangle$ is a subgroup of $\langle M_n(N), + \rangle$.

Further, we have that $m\pi_j A = m'\pi_j A$, $j \in \{i, i+1, \cdots, n\}$ since $A \in \mathcal{U}$ and $m\pi_j = m'\pi_j$, $j \in \{i, i+1, \cdots, n\}$. Therefore, we have that,

$$m\pi_i(AB) = (m\pi_i A)B = (m'\pi_i A)B = m'\pi_i(AB),$$

since $B \in \mathcal{U}$. In Consequence, $AB \in \mathcal{U}$, and $\mathcal{U}$ is a sub-near-ring of $M_n(N)$.

□

The binary operations on $M_n(N)$ are coordinate-wise addition and $\circ$ which indicates function composition.

We now define some rules for matrix calculations before we do some examples. We assume $\langle N,+\rangle$ is abelian.

Lemma 3.15. [11] *For all* $i,j,l,p=\{1,2,\cdots,n\}$ *and* $a,b,s,t \in N$ *we have,*

1) $f_{ij}^{a}+f_{ij}^{b}=f_{ij}^{a+b}$,

2) $f_{ij}^{a}+f_{pl}^{b}=f_{pl}^{b}+f_{ij}^{a}$, if $p \neq i$,

3) $f_{ij}^{a} \circ f_{pl}^{b}=\begin{cases} f_{pj}^{0b}, & \text{if } i \neq l \\ f_{pj}^{ab}, & \text{if } i=l, \end{cases}$

4) $-f_{ij}^{a}=f_{ij}^{-a}$.

5) a is zero symmetric in N if, and only if, f_{ij}^{a} is zero symmetric in $M_n(N)$.

6) a is constant in N if, and only if, f_{ij}^{a} is constant in $M_n(N)$.

7) a is distributive in N if, and only if, f_{ij}^{a} is distributive in $M_n(N)$.

8) If $a=s+t$ is the decomposition of a into the zero symmetric part s and the constant part t, then $f_{ij}^{a}=f_{ij}^{s}+f_{ij}^{t}$ is the corresponding decomposition of f_{ij}^{a} in $M_n(N)$.

Proof. For any $(m_1,m_2,\cdots,m_n) \in N^n$, we have,

1)

$$\begin{aligned}(m_1,m_2,\cdots,m_n)\left(f_{ij}^{a}+f_{ij}^{b}\right) &= (m_1,m_2,\cdots,m_n)\, f_{ij}^{a}+(m_1,m_2,\cdots,m_n)\, f_{ij}^{b} \\ &= (m_j a) l_i+(m_j b) l_i=(m_j a+m_j b) l_i \\ &= (m_j(a+b)) l_i=(m_1,m_2,\cdots,m_n)\, f_{ij}^{a+b}.\end{aligned}$$

Therefore, we have that $f_{ij}^{a}+f_{ij}^{b}=f_{ij}^{a+b}$.

2)

$$\begin{aligned}(m_1,m_2,\cdots,m_n)\, f_{ij}^{a}+f_{pl}^{b} &= (m_1,m_2,\cdots,m_n)\, f_{ij}^{a}+(m_1,m_2,\cdots,m_n)\, f_{pl}^{b} \\ &= (m_j a) l_i+(m_l b) l_p \\ &= \underbrace{(0,\cdots,0,m_j a,0,\cdots,0)}_{m_j a \text{ in the } i^{\text{th}} \text{ position}}+\underbrace{(0,\cdots,0,m_l b,0,\cdots,0)}_{m_l b \text{ in the } p^{\text{th}} \text{ position}} \\ &= (0,\cdots,0,m_j a,0,\cdots,0,m_l b,0,\cdots,0) \\ &= (m_l b) l_p+(m_j a) l_i \\ &= (m_1,m_2,\cdots,m_n)\, f_{pl}^{b}+f_{ij}^{a}.\end{aligned}$$

Therefore, we have that $f_{ij}^{a}+f_{pl}^{b}=f_{pl}^{b}+f_{ij}^{a}$ for $i \neq p$.

3)

$$\begin{aligned}(m_1,m_2,\cdots,m_n)\, f_{ij}^{a} \circ f_{pl}^{b} &= \left((m_j a) l_i\right) f_{pl}^{b}=\begin{cases}(0b) l_p & \text{if } i \neq l \\ (m_j ab) l_p & \text{if } i=l\end{cases} \\ &= \begin{cases} f_{pj}^{0b}, & \text{if } i \neq l \\ f_{pj}^{ab}, & \text{if } i=l.\end{cases}\end{aligned}$$

4)

$$\begin{aligned}(m_1,m_2,\cdots,m_n)\left(-f_{ij}^a\right) &= -(m_j a)l_i \\ &= \left(m_j(-a)\right)l_i \quad \text{since } m(-n) = -mn \\ &= (m_1,m_2,\cdots,m_n)\, f_{ij}^{-a}.\end{aligned}$$

Therefore we have that $-f_{ij}^a = f_{ij}^{-a}$.

5) Suppose a is zero symmetric in N, then we have,

$$\begin{aligned}(m_1,m_2,\cdots,m_n)0\, f_{ij}^a &= (0,0,\cdots,0)\, f_{ij}^a \\ &= (0,0,\cdots,0), \text{ since } 0a = 0 \\ &= (m_1,m_2,\cdots,m_n)0\end{aligned}$$

Thus, we have that $0 f_{ij}^a = 0$, so that f_{ij}^a is zero symmetric in $M_n(N)$.

Also, if f_{ij}^a is zero symmetric in $M_n(N)$, then,

$$(m_1,m_2,\cdots,m_n)0\, f_{ij}^a = (0,0,\cdots,0,0a,0,\cdots,0) = (0,0,\cdots,0).$$

This implies that $0a = 0$. Therefore a is zero symmetric in N.

6) Suppose a is constant in N, then we have,

$$\begin{aligned}(m_1,m_2,\cdots,m_n)0\, f_{ij}^a &= (0,0,\cdots,0)\, f_{ij}^a \\ &= (a)l_i, \text{ since } 0a = a, \\ &= (m_1,m_2,\cdots,m_n)\, f_{ij}^a.\end{aligned}$$

Thus, we have that $0 f_{ij}^a = f_{ij}^a$. So that f_{ij}^a is constant in $M_n(N)$.

Also, if f_{ij}^a is constant in $M_n(N)$, then,

$$(1,1,\cdots,1)0\, f_{ij}^a = (1,1,\cdots,1)\, f_{ij}^a,$$

$$(0,\cdots,0)\, f_{ij}^a = (1,1,\cdots,1)\, f_{ij}^a.$$

This implies that $0a = 1a = a$. Therefore a is constant in N.

7) Suppose a is distributive in N, choose matrices $A, B \in M_n(N)$ such that,

$$(m_1,m_2,\cdots,m_n)A = (s_1,s_2,\cdots,s_n), \text{ and } (m_1,m_2,\cdots,m_n)B = (t_1,t_2,\cdots,t_n).$$

Then,

$$\begin{aligned}&(m_1,m_2,\cdots,m_n)(A+B)\, f_{ij}^a \\ &= \left[(m_1,m_2,\cdots,m_n)A + (m_1,m_2,\cdots,m_n)B\right] f_{ij}^a \\ &= \left[(s_1,s_2,\cdots,s_n) + (t_1,t_2,\cdots,t_n)\right] f_{ij}^a \\ &= \left((s_j+t_j)a\right)l_i = (s_j a)l_i + (t_j a)l_i \text{ since } (m+m')a = ma + m'a \\ &= (s_1,s_2,\cdots,s_n)\, f_{ij}^a + (t_1,t_2,\cdots,t_n)\, f_{ij}^a \\ &= (m_1,m_2,\cdots,m_n)Af_{ij}^a + (m_1,m_2,\cdots,m_n)Bf_{ij}^a.\end{aligned}$$

Thus, we have that $(A+B)\, f_{ij}^a = Af_{ij}^a + Bf_{ij}^a$. So that f_{ij}^a is distributive in $M_n(N)$.

Also, if f_{ij}^a is distributive in $M_n(N)$, then, using our previous results we have that,

$$\left(f_{jl}^{s}+f_{jl}^{t}\right)f_{ij}^{a}=\left(f_{jl}^{s+t}\right)f_{ij}^{a}=f_{il}^{(s+t)a}=f_{il}^{sa}+f_{il}^{ta}=f_{il}^{sa+ta}.$$

Therefore, we have that $(1,1,\cdots,1)\,f_{il}^{(s+t)a}=(1,1,\cdots,1)\,f_{il}^{sa+ta}$, so that $(s+t)a=sa+ta$. Thus, a is distributive in N.

8) Suppose $a=s+t$, with s and t representing the zero symmetric part and constant part respectively. Then;

$$f_{ij}^{a}=f_{ij}^{s+t}=f_{ij}^{s}+f_{ij}^{t}.$$

where f_{ij}^{s} is the zero symmetric part, by Lemma 3.15 (5) and f_{ij}^{t} is the constant part by Lemma 3.15 (6).

□

Next we give some examples to practice working with the functions f_{ij}^{k} defined earlier.

Example 3.16. Let $\langle M_2(N),+,\cdot\rangle$ be a matrix near-ring. For any $r,s,t\in N$, we carry out some calculations.

<u>Addition</u>

a) $f_{11}^{r}+f_{12}^{s}$

$$\begin{aligned}(m_1,m_2)\left(f_{11}^{r}+f_{12}^{s}\right)&=(m_1,m_2)\,f_{11}^{r}+(m_1,m_2)\,f_{12}^{s}\\&=(m_1r,0)+(m_2s,0)=(m_1r+m_2s,0).\end{aligned}$$

b) $f_{22}^{r}+f_{12}^{s}$

$$\begin{aligned}(m_1,m_2)\left(f_{22}^{r}+f_{12}^{s}\right)&=(m_1,m_2)\,f_{22}^{r}+(m_1,m_2)\,f_{12}^{s}\\&=(0,m_2r)+(m_2s,0)=(m_2s,m_2r).\end{aligned}$$

<u>Function composition</u>

Function composition is operated from left to right as follows:

a) $f_{11}^{r}\circ f_{21}^{s}$

$$(m_1,m_2)\left(f_{11}^{r}\circ f_{21}^{s}\right)=\left((m_1,m_2)\,f_{11}^{r}\right)f_{21}^{s}=(m_1r,0)\,f_{21}^{s}=(0,m_1rs).$$

b) $f_{22}^{r}\circ f_{12}^{s}$

$$(m_1,m_2)\left(f_{22}^{r}\circ f_{12}^{s}\right)=\left((m_1,m_2)\,f_{22}^{r}\right)f_{12}^{s}=(0,m_2r)\,f_{12}^{s}=(m_2rs,0).$$

<u>Distribution of composition over addition</u>

We show the left distributive law:

a) $f_{12}^{r}\circ\left(f_{21}^{s}+f_{11}^{t}\right)$

$$\begin{aligned}(m_1,m_2)\left(f_{12}^{r}\circ\left(f_{21}^{s}+f_{11}^{t}\right)\right)&=(m_2r,0)\left(f_{21}^{s}+f_{11}^{t}\right)\\&=(m_2r,0)\,f_{21}^{s}+(m_2r,0)\,f_{11}^{t}\\&=(0,m_2rs)+(m_2rt,0)=(m_2rt,m_2rs).\end{aligned}$$

b) $f_{11}^{r}\circ\left(f_{21}^{s}+f_{12}^{t}\right)$

$$\begin{aligned}(m_1,m_2)\left(f_{11}^{r}\circ\left(f_{21}^{s}+f_{12}^{t}\right)\right)&=(m_1r,0)\left(f_{21}^{s}+f_{12}^{t}\right)\\&=(m_1r,0)\,f_{21}^{s}+(m_1r,0)\,f_{12}^{t}\\&=(0,m_1rs)+(0,0)=(0,m_1rs).\end{aligned}$$

Since our near-ring N is zero symmetric, we give a corollary that specifies when the near-ring $M_n(N)$ is zero symmetric. The following result follows from Lemma 3.15.

Corollary 3.17. [11] *N is zero symmetric if, and only if, $M_n(N)$ is zero symmetric.*

Proof. If $M_n(N)$ is zero symmetric, then each $f_{ij}^a \in M_n(N)$ is zero symmetric, this implies that $a \in N$ is zero symmetric by Lemma 3.15 part (5).

Conversely, if N is zero symmetric and $A \in M_n(N)$. Then,
$(m_1, m_2, \cdots, m_n)0A = (0,0,\cdots,0)A = (0,0,\cdots,0) = (m_1, m_2, \cdots, m_n)0$, since $0a = 0$, for all $a \in N$. Therefore, $0A = 0$. □

We now present a corollary that tells us about a sub-near-ring of $M_n(N)$ which is also isomorphic to our near-ring N, assuming $\langle M_n(N), + \rangle$ is abelian.

Corollary 3.18. [11] *If $\mathcal{A}$ is a non-empty subset of* $\{1, 2, \cdots, n\}$ then,

$$N_{\mathcal{A}} = \left\{ \sum_{i \in \mathcal{A}} f_{ii}^a : a \in N \right\},$$

is a sub-near-ring of $M(N)$ which is isomorphic to N.

Proof. From Lemma 3.15 part (1) and (2) and from the fact that

$$\left(\sum_{i \in \mathcal{A}} f_{ii}^a \right) \left(\sum_{i \in \mathcal{A}} f_{ii}^b \right) = \sum_{i \in \mathcal{A}} f_{ii}^{ab} \text{ for all } a, b \in N.$$

It follows that $N_{\mathcal{A}}$ is a sub-near-ring of $M_n(N)$.

We now show that the function

$$a \mapsto \sum_{i \in \mathcal{A}} f_{ii}^a, \forall a \in N,$$

is an isomorphism from N to $N_{\mathcal{A}}$.

For every $a, b \in N$, we have that,

$$\sum_{i \in \mathcal{A}} f_{ii}^a = \sum_{i \in \mathcal{A}} f_{ii}^b.$$

But since we have that for any $a, b, \in N$ and $(m_1, \cdots, m_n) \in N^n$, if $f_{ii}^a = f_{ii}^a$ then,

$$(m_1, \cdots, m_n) f_{ii}^a = (m_1, \cdots, m_n) f_{ii}^b$$

$$(m_i) f^a l_i = (m_i) f^b l_i$$

$$(m_i a) l_i = (m_i b) l_i$$

$$\underbrace{(0, \cdots, 0, m_i a, 0, \cdots, 0)}_{i^{\text{th}} \text{ position}} = \underbrace{(0, \cdots, 0, m_i b, 0, \cdots, 0)}_{i^{\text{th}} \text{ position}}$$

which is true if $m_i a = m_i b$, which implies that $a = b$. Thus, $a \mapsto \sum_{i \in \mathcal{A}} f_{ii}^a$ is well defined and clearly one-to-one.

Since we are taking the sum over every element in $\mathcal{A}$, then we have that for all $a \in N$, there exists an image in $N_{\mathcal{A}}$. Thus, $a \mapsto \sum_{i \in \mathcal{A}} f_{ii}^a$ is onto.

Now to check the homomorphism property,

$$a+b \mapsto \sum_{i\in\mathcal{A}} f_{ii}^{a+b} = \sum_{i\in\mathcal{A}} \left(f_{ii}^{a} + f_{ii}^{b} \right)$$
$$= \sum_{i\in\mathcal{A}} f_{ii}^{a} + \sum_{i\in\mathcal{A}} f_{ii}^{b} \text{ since } \langle M(N),+\rangle \text{ is abelian.}$$

□

As earlier stated, the near-ring N does not have to be abelian, so now we give a corollary that tells us that the near-ring of matrices $M_n(N)$ is abelian if, and only if, N is abelian.

Corollary 3.19. [11] *N is abelian if, and only if,* $\langle M_n(N),+\rangle$ *is* abelian.

Proof. Suppose N is abelian. Then, $\langle N^n,+\rangle$ is abelian. Take any $A,B \in M_n(N)$ and $\mathcal{D} \in N^n$, then we have

$$\mathcal{D}(A+B) = \mathcal{D}A + \mathcal{D}B$$
$$= \mathcal{D}B + \mathcal{D}A \text{ since } \langle N^n,+\rangle \text{ is abelian,}$$
$$= \mathcal{D}(B+A)$$

this implies that, $A+B=B+A$,

so that $M_n(N)$ is abelian.

Now, if $\langle M_n(N),+\rangle$ is abelian, then $N_{\mathcal{A}}$ with say, $\mathcal{A}=\{1\}$ is abelian. Consequently, since by Corollary 3.18 N is isomorphic to $N_{\mathcal{A}}$, N is therefore abelian. □

4. Conclusion

After understanding the background material on near-rings we went on to extend the idea to matrices. A natural question would be, can matrix near-rings be defined over an arbitrary near-ring N with the usual matrix addition and matrix multiplication? The answer was as seen in [1] who concluded that matrix near-rings over a near-ring N can only be defined if, and only if, N is a ring. Next we defined matrices as mappings from N^n to itself as seen in [2] and proved some results. In conclusion, proper near-rings of matrices can only be defined over an arbitrary near-ring if we consider all $n \times n$ matrices as elementary maps from N^n to itself.

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Conflicts of Interest

Regarding the publication of this paper, the authors declare that, there is no conflict of interest.

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