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Pythagorician Divisors and Applications to Some Diophantine Equations

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Abstract

We consider the Pythagoras equation $X^2+Y^2=Z^2$, and for any solution of the type $(a,b=2^s b_1\neq 0,c)\in\mathbb{N}^{*3}$, $s\geq 2$, b_1 odd, $(a,b,c)\equiv(\pm 1,0,1)(\bmod 4)$, $c>a$, $c>b$, and $gcd(a,b,c)=1$, we then prove the Pythagorician divisors Theorem, which results in the following: i) $a^2+b^2=c^2$ $\Leftrightarrow$ ii) $\begin{cases}c-b=d^2\\ c+b=d''^2\end{cases}$ $\Leftrightarrow$ iii) $\begin{cases}c-a=\frac{e^2}{2}\\ c+a=2e''^2\end{cases}$, where (d,d'') (resp. (e,e'')) are unique particular divisors of *a* and *b*, such that $a=dd''$ (resp. $b=ee''$), these divisors are called: Pythagorician divisors from *a*, (resp. from *b*). Let's put $\lambda\in\{0,1\}$, defined by: $\frac{c-a}{2}\equiv\lambda(\bmod 2)$ and $S=s-\lambda(s-1)$. Then $\exists\overline{e}\in 2\mathbb{N}+1$ such that $(e,e'')=\left(2^S\overline{e},2^{s-S}\frac{b_1}{\overline{e}}\right)$. Moreover the map $\pi:(a,b,c)\mapsto(d,\overline{e},S)$ is a bijection. We apply this new tool to obtain a new classification of the primitive, positive and non-trivial solutions of the Pythagoras equations: $a^2+b^2=c^2$ via the Pythagorician parameters $(d,\overline{e},S)$. We obtain for $(d,\overline{e})$ fixed, the equivalence class of any Pythagorician solution (a,b,c), checking $\pi(a,b,c)=(d,\overline{e},S)$, namely: $\overline{(a,b,c)}=\pi^{-1}(\{d\}\times\{\overline{e}\}\times\mathbb{N}^*)$. We also update the solutions of some Diophantine equations of degree 2, already known, but very important for the resolution of other equations. With this tool of Pythagorean divisors, we have obtained (in another paper) new recurrent methods to solve Fermat's equation: $a^4+b^4=c^4$, other than usual infinite descent method; and to solve congruent numbers problem. We believe that this tool can bring new arguments, for Diophantine resolution, of the general equations of Fermat: $a^{2p}+b^{2p}=c^{2p}$ and $a^p+b^p=c^p$.

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Keywords

Pythagoras Equation, Pythagorician Triplets, Diophantine Equations of Degree 2, Factorisation-Gcd-Fermat's Equations

1. Introduction

A question that many mathematicians have asked themselves since the 17th century is the following: "Is there a Diophantine method, allowing to solve Fermat's equation: $a^p+b^p=c^p$, or $a^{2p}+b^{2p}=c^{2p}$, where p is prime". Very interesting results were obtained by Fermat himself, in the case: $a^4+b^4=c^4$, with his famous method of the "infinite descent", by Sophie Germain via his famous prime numbers, Ernst Kummer and by Guy Terjanian via Diophantine methods cf. [1] p. 110-123, p. 360, p. 209 and [2], and this, in particular with the assumption that p/abc or not.

But in the end, the proof of Wiles for the general case (1995) (cf. [3]) which is the culmination of the new methods developed in the 20th century, in particular those counting the number of integer points of the elliptic curves, has in a way closed the problem of solving , by non-diophantine means.

The goal of this paper is to prove Theorem 1.1. It's why, we therefore return naturally to the basis of this problem, by focusing on the well-known problem of solving the Pythagoras equation: $a^2+b^2=c^2$. This gives us, a new parametrization for the expression of Pythagorean triples (which is a new tool that we call: "Pythagorician divisors").

This new parametrization is very important because it allows to solve many Diophantine equations: Thus, the results found can be directly applied, to the case: $a^4+b^4=c^4$ (cf. [1] [4]) producing a new demonstration, and in some extent, to the equation $a^{2p}+b^{2p}=c^{2p}$, where p is any odd prime number.

From a historical point of view, note that the Pythagorean triples have been known since ancient Egypt, long before Pythagoras, as evidenced by the problems of the Berlin papyrus 6619, found in Thebes in 1858 and dated to 1680-1620BC, cf. Clagett M., in [5]. There are also Pythagorean triples in the Kahun papyrus (1800BC): $6^2+8^2=10^2$; $12^2+16^2=20^2$, and more surprisingly, rational Pythagorean triples, which are: $\left(1+\frac{1}{2}\right)^2+2^2=\left(2+\frac{1}{2}\right)^2$; $\left(\frac{3}{4}\right)^2+1^2=\left(1+\frac{1}{4}\right)^2$.

We will follow the following plan for the article: We first make some reminders, followed by the definitions of the Pythagorean divisors, as well as the associated theorems. We express the Pythagorean triples with these new parameters in different tables, and we explain an equivalence relation allowing a new classification of these triples. Finally we apply these methods to the resolution of certain important equations of degree 2, before concluding.

1.1. Notations-Reminders

Let's make some reminders:

Reminders 1.1 $\forall a,c\in\mathbb{N}^*$, $\forall n\in\mathbb{N}$, $n\ge 2$, *we have*:

1) $c^n-a^n=(c-a)\times T_n(c,a)$, *where* $T_n(c,a)=\sum_{k=0}^{n-1}c^{n-1-k}a^k$.

2) $gcd(c-a,T_n(c,a))=gcd(n,c-a)$

- *Therefore if* $n=p$ *is prime, then* $gcd(c-a,T_p(c,a))\in\{1,p\}$.
- *In particular* $gcd(c-a,T_2(c,a))\in\{1,2\}$.

3) *In addition, if* $gcd(a,c)=1$, *then*: $gcd(c,T_n(c,a))=1=gcd(a,T_n(c,a))$.

Convention 1.1 *Let* $(a,b,c)\ne(0,0,0)$ *be a solution of the Pythagoras equation*: $a^2+b^2=c^2$. *We agree for the following, unless otherwise stated, that* $(a,b,c)\equiv(\pm1,0,1)(\mathrm{mod}\,4)$.

This in no way restricts the expression of the generality of the solutions of said equation, because (b,a,c) is also a solution called "associated with (a,b,c)", such that $(b,a,c)\equiv(0,\pm1,1)(\mathrm{mod}\,4)$.

Reminders 1.2 *Let* $(a,b,c)\ne(0,0,0)$ *be a solution of* $a^2+b^2=c^2$. *So*:

- (a,b,c) *is said to be a positive solution if* $a>0$, $b\ge0$, $c>0$.
- $(a,b,c)=(1,0,1)$ *is said to be the positive trivial solution.*
- $(a>0,b=2^s b_1\ne0,c>0)$, *is said to be a positive non-trivial solution* (*then* $s\ge2$, b_1 *odd*).
- (a,b,c) *is called a primitive solution if and only if* $gcd(a,b,c)=1$.
- *Let* (a,b,c) *be a positive primitive solution, then* $(\pm ta,\pm tb,\pm tc)$ $t\in\mathbb{N}$, *form the set of solutions known as generated by* (a,b,c).

Proposition 1.1 *The set of all the solutions of the Pythagoras equation* $a^2+b^2=c^2$, *is formed from the solutions generated by all the positive primitive solutions, and their associates* (*notice that the null solution belong to*).

It is well known cf. [1], [6] or [7] (see also **Table 3**, at the end of paragraph 1.3), that:

Proposition 1.2 *The set of positive primitive solutions of* $a^2+b^2=c^2$, *is given by the following set*:

$$\left\{(u^2-v^2,2uv,u^2+v^2);u,v\in\mathbb{N},u>v,u+v\equiv1(\mathrm{mod}\,2)\text{ and } gcd(u,v)=1\right\}.$$

Note that in this article, we obtain in Corollary 1.2, our new parametrization, (in [8] there is another parametrization).

1.2. Pythagorician Divisors

We will describe and set the parameters we will need.

Definition 1.1 *Let* $(a,b=2^s b_1\ne0,c)\in\mathbb{N}^{*3}$, $s\ge2$, b_1 *odd*,

$(a,b,c)\equiv(\pm 1,0,1)(\bmod 4)$, $c>a>0$, $c>b>0$, *and* a,b,c *pairwise relativly prime.*

1) *We call divisors of such a triplet, coming from a* (*resp. from b*) *the unique couple* (d,d'') (*resp.* (e,e'')) *defined by:*

$$\begin{cases}(d,d'')=\left(gcd(a,c-b),\dfrac{a}{d}\right), \text{note that } a=dd'';\\ (e,e'')=\left(gcd(b,c-a),\dfrac{b}{e}\right), \text{note that } b=ee''.\end{cases}$$

2) *When in addition, we suppose that* (a,b,c) *is a non-trivial, primitive and positive Pythagorician triplet, that is to say that:* $a^2+b^2=c^2$, *then those just defined divisors are call Pythagorician divisors of such a triplet, coming from a* (*resp. From b*) *and are the unique couples* (d,d'') (*resp.* (e,e'')):

Definition 1.2 *Let, as above, be a triplet:* $(a,b=2^s b_1,c)\in\mathbb{N}^{*3}$, $s\geq 2$, b_1 *odd. So:*

1) $\lambda\in\{0,1\}$ *is defined by:* $\dfrac{c-a}{2}\equiv\lambda(\bmod 2)$.

2) $S\in\mathbb{N}^*$, *is defined by:* $S=s-\lambda(s-1)=\begin{cases}s, \text{if } \lambda=0;\\ 1, \text{otherwise.}\end{cases}$

And note that: $s-S=\lambda(s-1)=\begin{cases}0, \text{if } \lambda=0;\\ s-1, \text{otherwise.}\end{cases}$

Proposition 1.3 *Let as above, a triplet:* $(a,b=2^s b_1,c)$. *In addition, we suppose that* (a,b,c) *is a non-trivial, primitive and positive Pythagorician triplet, that is to say that:* $a^2+b^2=c^2$. *Then:*

1) d,d'' *are odd and* $gcd(d,d'')=1$.

2) *e is always even and* $gcd(e,e'')=2^{\lambda}=\begin{cases}1, \text{if } \lambda=0;\\ 2, \text{otherwise.}\end{cases}$

Specifically, $\exists!$ $\overline{e}\equiv 1(\bmod 2)$ *and* $pgcd\left(\overline{e},\dfrac{b_1}{\overline{e}}\right)=1$, *such that:*

$$\begin{cases}e=2^S\overline{e}=\begin{cases}2^s\overline{e}, \text{ if } \lambda=0;\\ 2\overline{e}, \text{otherwise;}\end{cases}\\ e''=2^{s-S}\dfrac{b_1}{\overline{e}}=\begin{cases}\dfrac{b_1}{\overline{e}}, \text{if } \lambda=0;\\ 2^{s-1}\dfrac{b_1}{\overline{e}}, \text{otherwise;}\end{cases}\end{cases}\quad \text{Implies that:}$$

$$gcd(e,e'')=2^{\lambda}=\begin{cases}gcd\left(2^s\overline{e},\dfrac{b_1}{\overline{e}}\right)=1, \text{if } \lambda=0;\\ gcd\left(2\overline{e},2^{s-1}\dfrac{b_1}{\overline{e}}\right)=2 \text{ otherwise.}\end{cases}$$

Note that: $b_1=\begin{cases}\dfrac{e}{2^s}\times e''=\overline{e}e'', \text{if } \lambda=0;\\ \dfrac{e}{2}\times\dfrac{e''}{2^{s-1}}=\overline{e}\times\dfrac{e''}{2^{s-1}}, \text{otherwise.}\end{cases}$

And that the factors are coprime, given the fact that $gcd(e,e'')=2^{\lambda}$.

For the proof of this proposition, see the proof of the first case and second case, of the the following Theorem.

Theorem 1.1 (***of Pythagorician divisors***). *Let* $(a, b = 2^s b_1, c) \in \mathbb{N}^{*3}$, $s \geq 2$, b_1 *odd and* λ, S *as defined in definition* 1.1 & 1.2. *and proposition* 1.3. *Denote by* (d, d'') *and* $(e, e'') = \left(2^S \overline{e}, 2^{s-S} \frac{b_1}{\overline{e}}\right)$ *the Pythagorician divisors of* (a, b, c).

So, there is an equivalence between the following propositions:

$$\text{(i)} \quad a^2 + b^2 = c^2; \text{ (ii) } \begin{cases} c - b = d^2 \\ c + b = d''^2 \end{cases}; \text{ (iii) } \begin{cases} c - a = \frac{e^2}{2} = \frac{\left(2^S \overline{e}\right)^2}{2}; \\ c + a = 2e''^2 = 2\left(2^{s-S} \frac{b_1}{\overline{e}}\right)^2. \end{cases}$$

Proof 1 *Let us show that:* (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

- *It is clear that each system* (ii) *and* (iii) *leads to the equality* (i) $a^2 + b^2 = c^2$.
- *Conversely, let us show that:*
  $a^2 + b^2 = c^2$ *realized* $\Rightarrow$ *each one of the systems* (ii) *and* (iii) *is solvable.*
  Note that: $a^2 + b^2 = c^2 \Rightarrow a^2 = (c-b)(c+b)$ *and* $b^2 = (c-a)(c+a)$.
  Let us put: $\Delta_a = gcd(c-b, c+b)$ *and* $\Delta_b = gcd(c-a, c+a)$.
  Consequently (*cf. Reminders* 1.1.):
  $\Delta_a = gcd(c-b, c+b) = gcd(c-b, T_2(c,b)) = gcd(2, c-b) = 1$
  (*because* $c-b$ *is odd*),
  While:
  $\Delta_b = gcd(c-a, c+a) = gcd(c-a, T_2(c,a)) = gcd(2, c-a) = 2$
  (*because* $c-a$ *is even*).
- 1st *case:* $\Delta_a = 1 \Rightarrow c - b \equiv c + b \equiv 1 \pmod 2$.
  But $\begin{cases} a^2 = (c-b)(c+b) \text{ and} \\ \Delta_a = gcd(c-b, c+b) = 1 \end{cases} \Rightarrow \exists!\, \delta, \delta'' \in 2\mathbb{N}+1 / gcd(\delta, \delta'') = 1$,

  Checking: $\begin{cases} c - b = \delta^2 \\ c + b = \delta''^2 \end{cases} \Rightarrow a^2 = (\delta\delta'')^2 \Rightarrow a = \delta\delta''$.

  Let's take a look at the quantities d and d'':

  $$gcd(a, c-b) = gcd(\delta\delta'', \delta^2) = \delta = d \Rightarrow \delta'' = \frac{a}{d} = d'' \Rightarrow \begin{cases} c - b = d^2; \\ c + b = d_2''^2. \end{cases}$$

  Hence as announced in the system (ii).

  In conclusion the point ii) *is checked by* $(d, d'') = \left(gcd(a, c-b), \frac{a}{d}\right)$, *the Pythagorician divisors coming from a, as defined in definition* 1.1.; *moreover, as stated in proposition* 1.3., *we have* $gcd(d, d'') = gcd(\delta, \delta'') = 1$, *as noted in the proof.*
- 2nd *case:* $\Delta_b = pgcd(c-a, c+a) = 2$.
  Then: $c - a \equiv c + a \equiv 0 \pmod 2$.

  Thus: $b^2 = (c-a)(c+a) = 4\left(\frac{c-a}{2}\right)\left(\frac{c+a}{2}\right)$ *with* $gcd\left(\frac{c-a}{2}, \frac{c+a}{2}\right) = 1$ (*cf. Reminders* 1.1.2)). *So we have:*

$$\left(\frac{b}{2}\right)^2=\left(\frac{c-a}{2}\right)\left(\frac{c+a}{2}\right). \qquad (1.1)$$

We have 2 *sub-cases*:

α) 1^{st} *sub-case*: $\frac{c-a}{2}\equiv 0\pmod 2 \Leftrightarrow \frac{c+a}{2}\equiv 1\pmod 2$ *i.e.* $\lambda=0$, *so in this case*: $S=s$.

Then $\exists!\, E,\varepsilon''\in\mathbb{N}^*$ *such that*: $(E,\varepsilon'')\equiv(0,1)\pmod 2$ *and* $gcd(E,\varepsilon'')=1$,

Checking: $\begin{cases}\frac{c-a}{2}=E^2;\\ \frac{c+a}{2}=\varepsilon''^2.\end{cases}$

Hence $(1.1)\Rightarrow\left(\frac{b}{2}\right)^2=(E\varepsilon'')^2\Rightarrow\frac{b}{2}=E\varepsilon''\Rightarrow b=2E\varepsilon''$.

Consider then the Pythagorician divisors coming from $b: e \underset{def}{=} gcd(b,c-a)$.

Then: $e=gcd(2E\varepsilon'',2E^2)=2gcd(E\varepsilon'',E^2)=2E$;

From where: $\begin{cases}e=2E\\ b=2E\varepsilon''=e\varepsilon'',\end{cases}$ *Thus, according to definition* 1.1: $e''=\frac{b}{e}=\varepsilon''$, *and is indeed the second Pythagorician divisors coming from b and associated with e*.

But: $ee''=2^sb_1=b\Rightarrow\exists!\,\overline{e}\equiv 1\pmod 2$ and $\overline{e}=\frac{b_1}{e''}$, such that:

$$e=2E=2^s\overline{e}=2^{s-\lambda(s-1)}\overline{e}=2^S\overline{e}\text{, since we are in the case where } \lambda=0. \qquad (1.2)$$

Note that $E=2^{S-1}\overline{e}$.

But then: $\begin{cases}c-a=\frac{e^2}{2};\\ c+a=2e''^2.\end{cases}$ *Hence exactly point* (iii).

In addition:

As: $e''\equiv 1\pmod 2$ *and* $b=ee''=2^sb_1$, *we deduce that all the points of Proposition* 1.3. *are proved, since*:

$gcd(E,e'')=1\Leftrightarrow pgcd\left(\frac{e}{2},e''\right)=1\Rightarrow gcd(e,e'')=1=2^0=2^\lambda$:

Let's note that $gcd(\overline{e},e'')=1$ (*because* $gcd(e,e'')=1$).

This demonstrates point 2) *of proposition* 1.3. *in the case* $\lambda=0$.

β) 2^{nd} *sub-case*: $\frac{c-a}{2}\equiv 1\pmod 2\Leftrightarrow\frac{c+a}{2}\equiv 0\pmod 2$ *i.e.* $\lambda=1$, *in this case*: $S=1$.

This time in (1.1): $\exists!(\overline{\varepsilon},\varepsilon'')\equiv(1,0)\pmod 2$ *with* $gcd(\overline{\varepsilon},\varepsilon'')=1$, *verifying*:

$$\begin{cases}\frac{c-a}{2}=\overline{\varepsilon}^2;\\ \frac{c+a}{2}=\varepsilon''^2.\end{cases}\Rightarrow\left(\frac{b}{2}\right)^2=(\overline{\varepsilon}\varepsilon'')^2.$$

$$\Rightarrow\frac{b}{2}=\overline{\varepsilon}\varepsilon''\Rightarrow b=2^sb_1=2\overline{\varepsilon}\varepsilon''.$$

We have: $\varepsilon'' = 2^{s-1}\frac{b_1}{\overline{\varepsilon}} = 2^{s-S}\frac{b_1}{\overline{\varepsilon}}$ (*remember that* $s \geq 2$). *Note that the points of proposition* 1.3. *are thus demonstrated, if* $\overline{\varepsilon} = \overline{e}$ *and* $\varepsilon'' = e''$.

- *Let us calculate the Pythagorician divisors coming from b:*

$$e \underset{def}{=} gcd\left(b, c-a\right) = gcd\left(2\overline{\varepsilon}\varepsilon'', 2\overline{\varepsilon}^2\right) = 2gcd\left(\overline{\varepsilon}\varepsilon'', \overline{\varepsilon}^2\right) = 2\overline{\varepsilon} = 2^S\,\overline{\varepsilon} \underset{def}{=} 2^S\,\overline{e} \quad (1.3)$$

Then: $\overline{e} = \overline{\varepsilon}$, *and from* $e\varepsilon'' = 2\overline{\varepsilon}\varepsilon'' = b \Rightarrow \varepsilon'' = e''$ (*cf. definition* 1.1).

So, we have: $\begin{cases} c-a = \dfrac{e^2}{2}; \\ c+a = 2e''^2. \end{cases}$ *hence point* (iii), *which completes the proof of the theorem.*

Note that $\frac{e}{2} = \overline{e}$ *is odd and* e'' *is even. However* $gcd\left(\overline{e}, e''\right) = 1$, *hence:*

$$gcd\left(e, e''\right) = gcd\left(2\overline{e}, e''\right) = 2gcd\left(\overline{e}, \frac{e''}{2}\right) = 2 = 2^{\lambda}.$$

Which completes the proof of Proposition 1.3.

■

From which we deduce, the notations remaining the same, the following corollary, and which shows that the usual well-known parameters u, v (cf. Proposition 1.2.) are in fact intimately linked to the Pythagorician divisors, which is a remarkable fact in itself, and that the Pythagorician divisors are interdependent.

Corollary 1.1 *Suppose that:* $a^2 + b^2 = c^2$ *is realized, with* $\left(a,b,c\right)$ *taken as a non-trivial, primitive and positive solution. So: the usual well-known parameters* $u > v \in \mathbb{N}$ (*cf. proposition* 1.2), *which express the solution* $\left(a,b,c\right) = \left(u^2 - v^2, 2uv, u^2 + v^2\right)$, *are unique and are related to the Pythagorician divisors as follows:*

1) $\begin{cases} e = 2v; \\ e'' = u \end{cases}$; *and* $\begin{cases} d = u - v; \\ d'' = u + v. \end{cases}$; *in particular:* $\overline{e} = 2^{1-S}v$

2) *From where we deduce:*

$$\begin{cases} u = \dfrac{d''+d}{2} = e'' = \dfrac{e}{2} + d; \\ v = \dfrac{d''-d}{2} = \dfrac{e}{2} = e'' - d. \end{cases} \quad (1.4)$$

Then:

$$\begin{cases} d = -\dfrac{e}{2} + e'' = d'' - e = -d'' + 2e''; \\ d'' = \dfrac{e}{2} + e'' = d + e = -d + 2e''. \end{cases} \quad \textit{And} \quad \begin{cases} e = 2\left(d'' - e''\right); \\ e'' = \dfrac{e}{2} + d. \end{cases} \quad (1.5)$$

3) $\lambda = 1 \Leftrightarrow S = 1 \Leftrightarrow u \equiv 0\left(\bmod 2\right) \Leftrightarrow v = \overline{e} \equiv 1\left(\bmod 2\right) \Leftrightarrow a \equiv -1\left(\bmod 4\right)$.

4) $\lambda = 0 \Leftrightarrow S = s \geq 2 \Leftrightarrow u \equiv 1\left(\bmod 2\right) \Leftrightarrow v = 2^{S-1}\overline{e} \equiv 0\left(\bmod 2\right) \Leftrightarrow a \equiv 1\left(\bmod 4\right)$.

We have the following corollary, still under the same assumptions and notations as above:

Corollary 1.2 *There is equivalencies between the following propositions*:

(j) $a^2+b^2=c^2$; (jj) $\begin{cases} a=dd''=\left(\frac{d''+d}{2}\right)^2-\left(\frac{d''-d}{2}\right)^2; \\ b=2\left(\frac{d''+d}{2}\right)\left(\frac{d''-d}{2}\right); \\ c=\left(\frac{d''+d}{2}\right)^2+\left(\frac{d''-d}{2}\right)^2. \end{cases}$

(jjj) $\begin{cases} a=e''^2-\left(\frac{e}{2}\right)^2; \\ b=ee''=2e''\left(\frac{e}{2}\right); \\ c=e''^2+\left(\frac{e}{2}\right)^2. \end{cases}$ (jv) $\begin{cases} a=d^2+\left(2^S\overline{e}\right)d; \\ b=\frac{\left(2^S\overline{e}\right)^2}{2}+\left(2^S\overline{e}\right)d; \\ c=\frac{\left(2^S\overline{e}\right)^2}{2}+\left(2^S\overline{e}\right)d+d^2. \end{cases}$

Proof 2 *As soon as Theorem* 1.1. *is verified, then Corollary* 1.1. *is trivially verified, and the same is true for Corollary* 1.2., *indeed*:

- *We have* (j) $\Leftrightarrow$ (jj) $\Leftrightarrow$ (jjj) *as consequencies of corollary* 1.1.
- *Concerning the result*: (j) $\Leftrightarrow$ (vj) *We can compare this result to the one found in* [1] p. 7, *but ours being an update and more precise, which additionally includes the correct identification of the odd parameters, and the value of s, this because we are using Pythagorician divisors and the parameter* $\lambda \in \{0,1\}$, *defined by* $\frac{c-a}{2} \equiv \lambda \pmod 2$.

Given that: $a=dd''$ *and* $b=ee''$, *this result is easily demonstrated from the value of e in proposition* 1.3. 2) *and the formulas* (1.4) *of Corollary* 1.1. *Then, for that of c, we use Theorem* 1.1. (ii) *or* (iii). *So from the hereafter formulas*:

$$\begin{cases} a=d(d+e)=d^2+ed; \\ b=e\left(\frac{e}{2}+d\right)=\frac{e^2}{2}+ed; \\ c=a+\frac{e^2}{2}=b+d^2=\frac{e^2}{2}+ed+d^2. \end{cases}$$

It suffices then to replace $e=2^S\overline{e}$.

Definition 1.3 *We now define the following sets*:

1) $\overline{T^+}$: *The set of non-trivial, primitive and positive Pythagoras solutions of the type* $\left(a,b=2^s b_1,c\right)\equiv(\pm1,0,1)\pmod 4$.

2) *The set*: $2\mathbb{N}+1 \underset{cop}{\times} 2\mathbb{N}+1=\left\{(x,y)\in(2\mathbb{N}+1)^2 / gcd(x,y)=1\right\}$.

It is clear that $Card\left(2\mathbb{N}+1 \underset{cop}{\times} 2\mathbb{N}+1\right)=+\infty$, *because there is an infinity of pairs of distinct odd prime numbers i.e.* (p,p'), *and* $p\neq p'$.

Corollary 1.3 $\overline{T^+}$ *is in bijection with*: $\left(2\mathbb{N}+1 \underset{cop}{\times} 2\mathbb{N}+1\right)\times\mathbb{N}^*$, *as follows*:

$$\pi:\overline{T^+}\to\left(2\mathbb{N}+1 \underset{cop}{\times} 2\mathbb{N}+1\right)\times\mathbb{N}^*$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}\mapsto\left(d,\overline{e},S\right)_{(a,b,c)}=\begin{pmatrix} d=gcd(a,c-b) \\ \overline{e}=\frac{gcd(b,c-a)}{2^S} \\ S \end{pmatrix}$$

where $S = s - \lambda(s-1)$, *with* $s = v_2(b)$ *and* λ *defined in Definition* 1.2;

Whose reciprocal bijection is:

$$\pi^{-1} : \left(2\mathbb{N}+1 \underset{cop}{\times} 2\mathbb{N}+1\right) \times \mathbb{N}^* \to \overline{T^+}$$

$$\begin{pmatrix} d \\ \overline{e} \\ S \end{pmatrix} \mapsto (a,b,c)_{(d,\overline{e},S)} = \begin{pmatrix} a = d^2 + \left(2^S \overline{e}\right) d \\ b = \dfrac{\left(2^S \overline{e}\right)^2}{2} + \left(2^S \overline{e}\right) d \\ c = \dfrac{\left(2^S \overline{e}\right)^2}{2} + \left(2^S \overline{e}\right) d + d^2 \end{pmatrix}$$

Remark 1.1 *When* $(a,b,c) = \left(u^2 - v^2, 2uv, u^2 + v^2\right) \in \overline{T^+}$, *then* $\frac{c-a}{2} = v^2 \equiv \lambda(\text{mod}\,2)$, *thus* λ *is none other than, the parity indicator of v. Recall that*: $2uv = 2^s b_1$ *with* $b_1 \equiv 1(\text{mod}\,2)$, *but then* $S = s - \lambda(s-1)$, $d = u - v$ *and* $\overline{e} = 2^{1-S} v$ (*cf. corollary* 1.1. 1). *As a result, we also have*:

$$\pi : \overline{T^+} \to \left(2\mathbb{N}+1 \underset{cop}{\times} 2\mathbb{N}+1\right) \times \mathbb{N}^*$$

$$\begin{pmatrix} u^2 - v^2 \\ 2uv \\ u^2 + v^2 \end{pmatrix} \mapsto (d,\overline{e},S)_{(a,b,c)} = \begin{pmatrix} d = u - v \\ \overline{e} = 2^{1-S} v \\ S = s - \lambda(s-1) \end{pmatrix}$$

Proof 3 1) *Firstly*, π *is well defined, because*:

$gcd(d,\overline{e}) = 1$, *and* $S \in \mathbb{N}^*$. *Let's remind that* (*cf. corollary* 1.1.):

$S = s - \lambda(s-1) = 1 \Leftrightarrow \lambda = 1$ *and* $S = s - \lambda(s-1) = s \ge 2 \Leftrightarrow \lambda = 0$.

2) π *is indeed surjective, because*: $\forall (d,\overline{e},S) \in \left(2\mathbb{N}+1 \underset{cop}{\times} 2\mathbb{N}+1\right) \times \mathbb{N}^*$, *its antecedent by* π *is*:

$$\left(a = d^2 + \left(2^S \overline{e}\right) d, b = \frac{\left(2^S \overline{e}\right)^2}{2} + \left(2^S \overline{e}\right) d, c = \frac{\left(2^S \overline{e}\right)^2}{2} + \left(2^S \overline{e}\right) d + d^2\right)$$

which indeed belongs to $\overline{T^+}$.

Since:

- $a^2 + b^2 = c^2$ *and* $(a,b,c) \ne (1,0,1)$ *is primitive and positive, therefore* $(a,b,c) \in \overline{T^+}$.

We distinguish the cases: $S = 1$ *and* $S = s \ge 2$. *That is to say* $(d,\overline{e},1)$ *and* $(d,\overline{e},S = s \ge 2)$. *Denote by* $v_2(x)$ *the* 2-*adic valuation of the integer x. Then*:

- *For case* $S = 1$:

$$\begin{cases} \dfrac{c-a}{2} = \overline{e}^2 \equiv 1(\text{mod}\,2) \Rightarrow \lambda = 1; \\ b = 2^{v_2\left(2\left(\overline{e}^2 + \overline{e}d\right)\right)} \left(\dfrac{b}{2^{v_2\left(2\left(\overline{e}^2 + \overline{e}d\right)\right)}}\right) \Rightarrow s = v_2\left(2\left(\overline{e}^2 + \overline{e}d\right)\right); b_1 = \dfrac{b}{2^{v_2\left(2\left(\overline{e}^2 + \overline{e}d\right)\right)}} \equiv 1(\text{mod}\,2); \\ gcd(a, c-b) = gcd\left(d(d + 2\overline{e}), d^2\right) = d; \\ \dfrac{gcd(b, c-a)}{2^S} = \dfrac{gcd\left(2\overline{e}(\overline{e} + d), 2\overline{e} \times \overline{e}\right)}{2} = \overline{e}. \end{cases}$$

- *And in the second case* $S = s \geq 2$:

$$\begin{cases} \frac{c-a}{2} = 2^{2(S-1)}\overline{e}^2 \equiv 0 \pmod 2 \Rightarrow \lambda = 0; \\ b = 2^S \overline{e}\left(2^{S-1}\overline{e} + d\right) \Rightarrow s = S \text{ and } b_1 = \frac{b}{2^S \overline{e}} \equiv 1 \pmod 2; \\ gcd\left(a, c-b\right) = gcd\left(d\left(d + 2\overline{e}\right), d^2\right) = d; \\ \frac{gcd\left(b, c-a\right)}{2^S} = \frac{gcd\left(2^S \overline{e} \times \left(2^{S-1}\overline{e} + d\right), 2^S \overline{e} \times \left(2^{S-1}\overline{e}\right)\right)}{2^S} = \overline{e}. \end{cases}$$

So in all cases: $\pi\left(a, b = 2^s b_1, c\right) = \left(d, \overline{e}, S\right) \Rightarrow \pi$ *is surjective.*

3) *Elsewhere,* π *is also injective, because if:* $\left(a_1, b_1, c_1\right)$, $\left(a_2, b_2, c_2\right) \in \overline{T^+}$ *and that* $\pi\left(a_1, b_1, c_1\right) = \pi\left(a_2, b_2, c_2\right)$, *then:*

- $d_1 = gcd\left(a_1, c_1 - b_1\right) = gcd\left(a_2, c_2 - b_2\right) = d_2 \Rightarrow d_1 = d_2$.
- $S_1 = s_1 - \lambda_1\left(s_1 - 1\right) = s_2 - \lambda_2\left(s_2 - 1\right) = S_2$ *and*

$$\overline{e}_1 = \frac{gcd\left(b_1, c_1 - a_1\right)}{2^{S_1}} = \frac{gcd\left(b_2, c_2 - a_2\right)}{2^{S_2}} = \overline{e}_2 \Rightarrow e_1 = 2^{S_1}\,\overline{e}_1 = 2^{S_2}\,\overline{e}_2 = e_2$$

But then:

$$\left(d_1 = d_2 \text{ and } e_1 = e_2\right) \Rightarrow \begin{cases} a_1 = d_1\left(d_1 + e_1\right) = d_2\left(d_2 + e_2\right) = a_2; \\ b_1 = e_1\left(\frac{e_1}{2} + d_1\right) = e_2\left(\frac{e_2}{2} + d_2\right) = b_2; \\ c_1 = b_1 + d_1^2 = b_2 + d_2^2 = c_2. \end{cases}$$ (*cf. corollary* 1.2. *jv*)).

Hence $\left(a_1, b_1, c_1\right) = \left(a_2, b_2, c_2\right) \Rightarrow \pi$ *is injective.*

Ultimately π *is bijective as stated.*

We will retain the following proposition, (which is the counterpart of proposition 1.2.), the notations being unchanged:

Proposition 1.4

$$\overline{T^+} = \left\{\left(d^2 + \left(2^S\overline{e}\right)d, \frac{\left(2^S\overline{e}\right)^2}{2} + \left(2^S\overline{e}\right)d, \frac{\left(2^S\overline{e}\right)^2}{2} + \left(2^S\overline{e}\right)d + d^2\right) / \left(d, \overline{e}\right) \in 2\mathbb{N}+1 \underset{cop}{\times} 2\mathbb{N}+1 \text{ and } S \in \mathbb{N}^*\right\}.$$

The proof comes directly from the bijectivity of π^{-1}.

1.3. Pythagorician Classes of Pythagoras Solutions and Tables

Let us keep the notations of the previous corollaries and propositions, then:

$$\overline{T^+} = \left\{\left(a, b, c\right)_{\left(d, \overline{e}, S\right)}, \text{ such that } \left(d, \overline{e}\right) \in 2\mathbb{N}+1 \underset{cop}{\times} 2\mathbb{N}+1 \text{ and } S \in \mathbb{N}^*\right\}.$$

This makes it possible to define the following concepts:

Definition 1.4 1) *Let* $\left(a, b, c\right) \in \overline{T^+}$, *we call Pythagorician parameters associated with* $\left(a, b, c\right)$, *the elements forming the unique triplet* $\left(d, \overline{e}, S\right) \in \left(2\mathbb{N}+1 \underset{cop}{\times} 2\mathbb{N}+1\right) \times \mathbb{N}^*$, *such that:*

$$d = gcd(a, c-b),\ \overline{e} = \frac{gcd(b, c-a)}{2^S} \quad where \quad S = s - \lambda(s-1).$$

It is clear that the Pythagorician parameters of $(a,b,c)_{(d,\overline{e},S)}$ *are worth* $(d,\overline{e},S)$.

2) *When* $(d,\overline{e}) \in 2\mathbb{N}+1 \underset{cop}{\times} 2\mathbb{N}+1$ *is fixed, we call Pythagorician class crossing through the solution* $(a,b,c)_{(d,\overline{e},1)}$, *the set noted*:

$\overline{(a,b,c)_{(d,\overline{e},1)}} = \left\{(a,b,c)_{(d,\overline{e},S)} / S \in \mathbb{N}^*\right\}$.

It is clear that classes of 2 *different solutions are pairwise disjoint, hence*:

$\overline{(a,b,c)_{(d_1,\overline{e}_1,1)}} \cap \overline{(a,b,c)_{(d_2,\overline{e}_2,1)}} = \varnothing$, $\forall (d_1,\overline{e}_1) \neq (d_2,\overline{e}_2) \in 2\mathbb{N}+1 \underset{cop}{\times} 2\mathbb{N}+1$; *and that*:

$\overline{T^+} = \bigcup_{(a,b,c)_{(d,\overline{e},1)} \in \mathcal{C}_1} \overline{(a,b,c)_{(d,\overline{e},1)}}$, *where*

$\mathcal{C}_1 = \left\{(a,b,c)_{(d,\overline{e},1)} / (d,\overline{e}) \in 2\mathbb{N}+1 \underset{cop}{\times} 2\mathbb{N}+1\right\}$.

3) *We define an equivalence relation on* $\overline{T^+}$, *noted* $\underset{\mathcal{P}}{\cong}$, *as follows*:

We say that $(a_1, b_1 = 2^{s_1} b_{1,1}, c_1)$, $(a_2, b_2 = 2^{s_2} b_{1,2}, c_2) \in \overline{T^+}$ *are* $\mathcal{P}$*-equivalent in* $\overline{T^+}$, *and we note*:

$(a_1,b_1,c_1) \underset{\mathcal{P}}{\cong} (a_2,b_2,c_2) \Leftrightarrow \exists (d,\overline{e}) \in 2\mathbb{N}+1 \underset{cop}{\times} 2\mathbb{N}+1 / (a_1,b_1,c_1)$ *and* $(a_2,b_2,c_2) \in \overline{(a,b,c)_{(d,\overline{e},1)}}$ $\Leftrightarrow$

$\left(gcd(a_1, c_1-b_1), \frac{gcd(a_1, c_1-b_1)}{2^{S_1}}\right) = \left(gcd(a_2, c_2-b_2), \frac{gcd(a_2, c_2-b_2)}{2^{S_2}}\right)$, *where*:

$\forall i \in \{1,2\}, \lambda_i \in \{0,1\} / \frac{c_i - a_i}{2} \equiv \lambda_i \pmod 2$, *and* $S_i = s_i - \lambda_i(s_i - 1)$ *where* $s_i = v_2(b_i)$.

- *The equivalence class* $\overline{(a,b,c)}$ *containing the element* $(a, b = 2^s b_1, c) \in \overline{T^+}$, *is called the Pythagorician class of the solution* (a,b,c), *and if* $(a,b,c) \in \overline{(a,b,c)_{(d,\overline{e},1)}}$, *we get*:

$$\overline{(a,b,c)} = \overline{(a,b,c)_{(d,\overline{e},1)}} = \pi^{-1}\left(\{d\} \times \{\overline{e}\} \times \mathbb{N}^*\right)$$
$$= \pi^{-1}\left(\{gcd(a,c-b)\} \times \left\{\frac{gcd(b,c-a)}{2^S}\right\} \times \mathbb{N}^*\right).$$

This unique element: $(a,b,c)_{(d,\overline{e},1)}$, *is called the canonical exceptional representative of the class* $\overline{(a,b,c)}$.

3) *We note*: $\overline{(a,b,c)_{(d,\overline{e},1)}}_{\leq \ell} = \left\{(a,b,c)_{(d,\overline{e},S)} / 1 \leq S \leq \ell\right\}$. *This is the class of* $(a,b,c)_{(d,\overline{e},1)}$ *truncated in* ℓ.

1.3.1. Equivalent Expression, for the Equivalence Relation $\underset{\mathcal{P}}{\cong}$ with Respect to the Parameterization $(u^2 - v^2, 2uv, u^2 + v^2)$

Let $(u^2 - v^2, 2uv, u^2 + v^2) \in \overline{T^+}$. We then ask ourselves the question of knowing under what conditions a Pythagorician triplet $(u_1^2 - v_1^2, 2u_1v_1, u_1^2 + v_1^2)$ is $\mathcal{P}$-

equivalent (in $\overline{T^+}$), to the chosen one?

Recall that if $(d,\overline{e},S),(d_1,\overline{e}_1,S_1)\in\left(2\mathbb{N}+1\underset{cop}{\times}2\mathbb{N}+1\right)\times\mathbb{N}^*$ are respectively, the Pythagorician parameters associated with the Pythagorician triplets $\left(u^2-v^2,2uv,u^2+v^2\right)$ and $\left(u_1^2-v_1^2,2u_1v_1,u_1^2+v_1^2\right)$, then:

$\begin{cases} d=u-v; \\ \overline{e}=2^{1-S}v. \end{cases}$ and $\begin{cases} d_1=u_1-v_1; \\ \overline{e}_1=2^{1-S_1}v_1. \end{cases}$, and thus we get:

Proposition 1.5 *Let* $\left(u^2-v^2,2uv,u^2+v^2\right),\left(u_1^2-v_1^2,2u_1v_1,u_1^2+v_1^2\right)\in\overline{T^+}$, *two Pythagorician triplets, and* $\underset{\mathcal{P}}{\cong}$ *be the relation of* $\mathcal{P}$*-equivalence in* $\overline{T^+}$. *Then, there is an equivalence between the following* 2 *propositions*:

1) $\left(u_1^2-v_1^2,2u_1v_1,u_1^2+v_1^2\right)\underset{\mathcal{P}}{\cong}\left(u^2-v^2,2uv,u^2+v^2\right)$.

2) $\begin{cases} u_1=u+\left(2^{S_1-S}-1\right)v; \\ v_1=2^{S_1-S}v. \end{cases}\Leftrightarrow\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}=\begin{pmatrix} 1 & 2^{S_1-S}-1 \\ 0 & 2^{S_1-S} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$

Remark 1.2 1) *Note that necessarily when the equivalence takes place, then simultaneously*: $v_1=2^{S_1-S}v$ *and* $v=2^{S-S_1}v_1\in\mathbb{N}^*$.

2) *Let's remark that we can replace in an equivalent way, the first proposition of* (ii) *by*:

$\exists k\in\mathbb{Z}$ *such that* $\begin{cases} u_1=u+\left(2^k-1\right)v; \\ v_1=2^k v. \end{cases}$

Note that for such a k, we have $2^k v$ *and* $2^{-k}v_1\in\mathbb{N}^*$, *and in case that v is odd, then necessarily* $k=0$.

(*resp. the second proposition of* (ii) *by*: $\exists k\in\mathbb{Z}$ *such that* $\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}=\begin{pmatrix} 1 & 2^k-1 \\ 0 & 2^k \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$).

3) *Denote by* $\mathcal{M}_P=\left\{\begin{pmatrix} 1 & 2^k-1 \\ 0 & 2^k \end{pmatrix},\ k\in\mathbb{Z}\right\}$, *then* $\left(\mathcal{M}_P,\times\right)$ *is a commutative group*.

This relation is indeed an equivalence relation because

$$\begin{pmatrix} 1 & 2^{S_1-S}-1 \\ 0 & 2^{S_1-S} \end{pmatrix}^{-1}=\begin{pmatrix} 1 & 2^{S-S_1}-1 \\ 0 & 2^{S-S_1} \end{pmatrix}, \text{ and}$$

$$\begin{pmatrix} 1 & 2^{S_2-S_1}-1 \\ 0 & 2^{S_2-S_1} \end{pmatrix}\begin{pmatrix} 1 & 2^{S_1-S}-1 \\ 0 & 2^{S_1-S} \end{pmatrix}=\begin{pmatrix} 1 & 2^{S_2-S}-1 \\ 0 & 2^{S_2-S} \end{pmatrix}.$$

➢ Suppose that $u\equiv 0\pmod 2$ then $S=1$ and $v\equiv 1\pmod 2$.

If $u_1\equiv 0\pmod 2\Rightarrow 2^{S_1-1}-1\equiv 0\pmod 2\Rightarrow S_1=1$; but then $(u_1,v_1)=(u,v)$.

So in an equivalence class of a Pythagorician triplet, which contains a Pythagorician triplet $\left(u^2-v^2,2uv,u^2+v^2\right)$, with u even then it's the only triplet to have this property, *i.e.* all $\left(u_1^2-v_1^2,2u_1v_1,u_1^2+v_1^2\right)$ which are equivalent to it, are such that $u_1\equiv 1\pmod 2$. We can therefore choose $\left(u^2-v^2,2uv,u^2+v^2\right)$ as an canonical exceptional representative element of its class.

➢ Now suppose that $u\equiv 1\pmod 2$ then $S\geq 2$ and $v\equiv 0\pmod 2$. We note that $v = 2^{S-1}\overline{e}$.

Consider now the element $\left(u_1^2 - v_1^2, 2u_1v_1, u_1^2 + v_1^2\right)$ belonging to the class of $\left(u^2 - v^2, 2uv, u^2 + v^2\right)$, and defined by: $\begin{cases} u_1 = u + \left(2^{1-S} - 1\right)v \\ v_1 = 2^{1-S}v \end{cases}$, then $v_1 = \overline{e} \equiv 1 \pmod 2$, and $\dfrac{\left(u_1^2 + v_1^2\right) - \left(u_1^2 - v_1^2\right)}{2} = \overline{e}^2 \equiv 1 \pmod 2 \Rightarrow \lambda_1 = 1$, hence $u_1 \equiv 0 \pmod 2$, and it is the only Pythagorician triplet having this property, in the class of $\left(u^2 - v^2, 2uv, u^2 + v^2\right)$, it's why, we can therefore choose it as the canonical exceptional representative for this class.

We have just shown the following proposition:

Proposition 1.6 *Let* $\left(u_1^2 - v_1^2, 2u_1v_1, u_1^2 + v_1^2\right) \in \overline{T^+}$, *then there exists a single* $\left(u^2 - v^2, 2uv, u^2 + v^2\right) \in \overline{T^+}$ *such that* $u \equiv 0 \pmod 2$, *and* $\left(u_1^2 - v_1^2, 2u_1v_1, u_1^2 + v_1^2\right) \in \overline{\left(u^2 - v^2, 2uv, u^2 + v^2\right)}$.

So then: $\begin{cases} u_1 = u + \left(2^{S_1-1} - 1\right)v; \\ v_1 = 2^{S_1-1}v. \end{cases}$

And as a result:

$$\overline{\left(u^2 - v^2, 2uv, u^2 + v^2\right)} = \left\{\left(u_1^2 - v_1^2, 2u_1v_1, u_1^2 + v_1^2\right) / \begin{cases} u_1 = u + \left(2^{S_1-1} - 1\right)v; \\ v_1 = 2^{S_1-1}v. \end{cases}, S_1 \in \mathbb{N}^*\right\},$$

and those classes form a partition of $\overline{T^+}$, *when* $u, v \in \mathbb{N}^*$, $u \equiv 0 \pmod 2$, $u + v \equiv 1 \pmod 2$, $gcd(u, v) = 1$.

Definition 1.5 1) *For simplicity one can also note*: $\overline{(u, v)} = \left(u^2 - v^2, 2uv, u^2 + v^2\right)$

2) *When* $u \equiv 0 \pmod 2$, *one notes*:

$$\overline{(u, v)}_{\leq \ell} = \left\{\left(u + \left(2^{S_1-1} - 1\right)v, 2^{S_1-1}v\right) \text{ where } S_1 \leq \ell\right\},$$

it is the class of $\left(u^2 - v^2, 2uv, u^2 + v^2\right)$ *truncated in* ℓ.

1.3.2. Paradoxical Rarefaction of Pythagorician Triplets $\left(u^2 - v^2, 2uv, u^2 + v^2\right)$ Such That $u \equiv 0 \pmod 2$ *i.e.* Whose Adjacent Side Is $\equiv -1 \pmod 4$.

We get the following proposals:

Proposition 1.7 *Let* $\left(a, b = 2^s b_1, c\right) \in \overline{T^+}$, *let's recall cf. corollary* 1.1. *that* $S = s - \lambda(s-1) = 1 \Leftrightarrow \lambda = 1$.

1) *The Complete set of representatives (called canonical complete set of representatives) of* $\overline{T^+} / \underset{\mathcal{P}}{\cong}$: *Namely* $\mathcal{C}_1 = \left\{(a, b, c) \in \overline{T^+} / \lambda = 1\right\}$, *is the set of non-trivial Pythagorician solutions, positive, primitive* "*said evens* (*with reference to u even*)", *which correspond to the Pythagorician solutions of* $\overline{T^+}$ *whose adjacent side is* $\equiv -1 \pmod 4$.

2) $Card\left(\mathcal{C}_1\right) = Card\left(2\mathbb{N} + 1 \underset{cop}{\times} 2\mathbb{N} + 1\right) = +\infty$, (*because there is an infinity of prime numbers*), *However for* $\left(d, \overline{e}\right)$ *fixed, the element* $(a, b, c)_{\left(d, \overline{e}, 1\right)}$ *is the*

unique element of $\mathcal{C}_1$ (*i.e. even*), *which is present in* $\overline{(a,b,c)_{(d,\overline{e},1)}}$. *Thus: although they are in infinite number, we deduce that there is a relative rarefaction of the solutions* $(a,b,c)\in\mathcal{C}_1$ (*i.e. even ones*), *compared to those* $\notin\mathcal{C}_1$, *that is to say the set of "odd" solutions* (a,b,c), *whose cardinality is equal to:*

$Card\left(\left(2\mathbb{N}+1\underset{cop}{\times}2\mathbb{N}+1\right)\times\left(\mathbb{N}^*\setminus\{1\}\right)\right)=Card(\mathcal{C}_1)\times Card\left(\mathbb{N}^*\setminus\{1\}\right)$, *which is much greater than* $Card(\mathcal{C}_1)$.

Remark 1.3 *So, in any class* $\overline{(a,b,c)_{(d,\overline{e},S)}}$, *there is exactly one element such as* $u\equiv 0\pmod 2$, *i.e. whose adjacent side is* $\equiv -1\pmod 4$. *This one is* $(a,b,c)_{(d,\overline{e},1)}=\left(d^2+2\overline{e}d,2\overline{e}^2+2\overline{e}d,2\overline{e}^2+2\overline{e}d+d^2\right)$, *and then*

$u=\overline{e}+d\equiv 0\pmod 2$, *where* $d=gcd(a,c-b)$, *and* $e=gcd(b,c-a)$.

And there are on the other hand in this same class: $Card\left(\mathbb{N}^*\setminus\{1\}\right)$ *other elements, all having* $u\equiv 1\pmod 2$, *they are:*

$$(a,b,c)_{(d,\overline{e},S)}=\left(d^2+\left(2^S\overline{e}\right)d,\frac{\left(2^S\overline{e}\right)^2}{2}+\left(2^S\overline{e}\right)d,\frac{\left(2^S\overline{e}\right)^2}{2}+\left(2^S\overline{e}\right)d+d^2\right)$$ *where* $S\geq 2$, *then:* $u=2^{S-1}\overline{e}+d\equiv 1\pmod 2$.

- It is difficult to observe this phenomenon, because if we fix a bound for the hypothenuse, and we use the parameterization in (u,v) the even and odd u will alternate, and in the end, after counting, there will be no significant difference (thus, **Table 2** gives 121 quantities u which are even, and 118 quantities u which are odd). We get the same conclusion likewise if we use the parametrization $(d,\overline{e},S)$ of **Table 3**, because given the fixed bound, the exponentiation 2^S, will be quickly out of control, and it will appear very quickly only $S=1$, thus in **Table 3** (which is a double entry one) we find the same conclusion, that is:121 triplets for which $S=1$, and 118 other triplets for which $S\geq 2$.
- For draw the below curve (cf. **Figure 1**), we consider the average of the cardinalities of the classes, taken per block of 5 classes, that is to say that we make the average obtained by taking the ratio of the cardinality of the i-th block of consecutive 5 classes (of type $\overline{(u,v)}$) by 5, this corresponds to the application: $i\mapsto\frac{Card(i)}{5}$). The curve above reveals a hyperbolic behavior of the representatives of the classes. Where data were obtained from **Table 3**. The curve shows the average positioning of the classes, when $c\leq 1500$, most of the classes do not open, strictly speaking, but reveal their canonical complete set of representatives who is then isolated, and it's why the curve tends towards the value 1.

  As sample data:

  1st block $=\left\{\overline{(2,1)},\overline{(4,1)},\overline{(4,3)},\overline{(6,1)},\overline{(6,5)}\right\}$ and $\frac{Card(1)}{5}=\frac{22}{5}=4.4$;

  5th block $=\left\{\overline{(14,9)},\overline{(14,11)},\overline{(14,13)},\overline{(16,1)},\overline{(16,3)}\right\}$ and $\frac{Card(5)}{5}=\frac{14}{5}=2.8$;

22^{th} block $=\left\{\overline{(34,3)},\overline{(34,5)},\overline{(34,7)},\overline{(34,9)},\overline{(34,11)}\right\}$ and $\frac{Card(22)}{5}=\frac{6}{5}=1.2$.

(The classes $\overline{(u,v)}$ cf. Definition 1.5., are easy to get from Table 3. The values use to draw the curve (with Excel software) are:

i^{th} block	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\frac{Card(i)}{5}$	4.4	3.8	3	3	2.8	2	2.6	2.4	1.4	2.6	1	2	2	1.4	1

i^{th} block	16	17	18	19	20	21	22	23	24	25
$\frac{Card(i)}{5}$	2	1	1.6	1.6	1.2	1.4	1.2	1.2	1	1

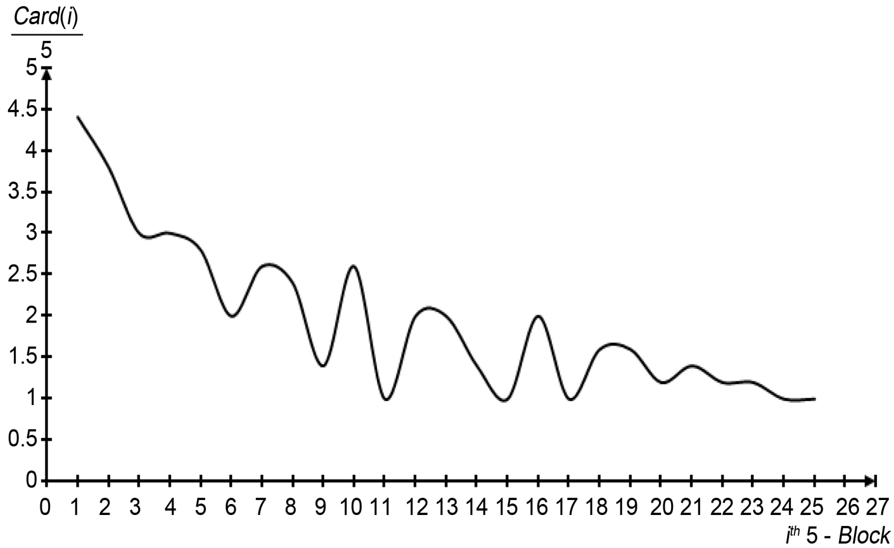

Figure 1. Rarefaction of certain Pythagorician triples.

Now, we get Tables 1-3 hereafter, by using the formal calculus software: Maple.

- In Table 1, we consider the truncated classes $\overline{(u,v)}_{\leq \ell}$, when $u,v\in\mathbb{N}^*$, $1\leq v<u\leq 10$, $u\equiv 0(\text{mod}\,2)$, $u+v\equiv 1(\text{mod}\,2)$, $gcd(u,v)=1$ and $\ell\leq 15$.

This classification shows as announced, the rarity of right-angled triangles, positive non-trivial primitives, with adjacent side congruent to $-1(\text{mod}\,4)$. In fact there are 13 pairs (u,v) canonical representatives, *i.e.* with even *u*, and 182 pairs (u,v) with odd *u*.

Table 1. Truncated classes $\overline{(u,v)}_{\leq \ell}, u,v\in\mathbb{N}^*, 0\leq v<u\leq 10$.

$\overline{(2,1)}$	$\overline{(4,1)}$	$\overline{(4,3)}$	$\overline{(6,1)}$	$\overline{(6,5)}$	$\overline{(8,1)}$	$\overline{(8,3)}$	$\overline{(8,5)}$	$\overline{(8,7)}$	$\overline{(10,1)}$	$\overline{(10,3)}$	$\overline{(10,7)}$	$\overline{(10,9)}$
(2, 1)	(4, 1)	(4, 3)	(6, 1)	(6, 5)	(8, 1)	(8, 3)	(8, 5)	(8, 7)	(10, 1)	(10, 3)	(10, 7)	(10, 9)
(3, 2)	(5, 2)	(7, 6)	(7, 2)	(11, 10)	(9, 2)	(11, 6)	(13, 10)	(15, 14)	(11, 2)	(13, 6)	(17, 14)	(19, 18)

Continued

(5, 4)	(7, 4)	(13, 12)	(9, 4)	(21, 20)	(11, 4)	(17, 12)	(23, 20)	(29, 28)	(13, 4)	(19, 12)	(31, 28)	(37, 36)
(9, 8)	(11, 8)	(25, 24)	(13, 8)	(41, 40)	(15, 8)	(29, 24)	(43, 40)	(57, 56)	(17, 8)	(31, 24)	(59, 56)	(73, 72)
(17, 16)	(19, 16)	(49, 48)	(21, 16)	(81, 80)	(23, 16)	(53, 48)	(83, 80)	(113, 112)	(25, 16)	(55, 48)	(115, 112)	(145, 144)
(33, 32)	(35, 32)	(97, 96)	(37, 32)	(161, 160)	(39, 32)	(101, 96)	(163, 160)	(225, 224)	(41, 32)	(103, 96)	(227, 224)	(289, 288)
(65, 64)	(67, 64)	(193, 192)	(69, 64)	(321, 320)	(71, 64)	(197, 192)	(323, 320)	(449, 448)	(73, 64)	(199, 192)	(451, 448)	(577, 576)
(129, 128)	(131, 128)	(385, 384)	(133, 128)	(641, 640)	(135, 128)	(389, 384)	(643, 640)	(897, 896)	(137, 128)	(391, 384)	(899, 896)	(1153, 1152)
(257, 256)	(259, 256)	(769, 768)	(261, 256)	(1281, 1280)	(263, 256)	(773, 768)	(1283, 1280)	(1793, 1792)	(265, 256)	(775, 768)	(1795, 1792)	(2305, 2304)
(513, 512)	(515, 512)	(1537, 1536)	(517, 512)	(2561, 2560)	(519, 512)	(1541, 1536)	(2563, 2560)	(3585, 3584)	(521, 512)	(1543, 1536)	(3587, 3584)	(4609, 4608)
(1025, 1024)	(1027, 1024)	(3073, 3072)	(1029, 1024)	(5121, 5120)	(1031, 1024)	(3077, 3072)	(5123, 5120)	(7169, 7168)	(1033, 1024)	(3079, 3072)	(7171, 7168)	(9217, 9216)
(2049, 2048)	(2051, 2048)	(6145, 6144)	(2053, 2048)	(10,241, 10,240)	(2055, 2048)	(6149, 6144)	(10,243, 10,240)	(14,337, 14,336)	(2057, 2048)	(6151, 6144)	(14,339, 14,336)	(18,433, 18,432)
(4097, 4096)	(4099, 4096)	(12,289, 12,288)	(4101, 4096)	(20,481, 20,480)	(4103, 4096)	(12,293, 12,288)	(20,483, 20,480)	(28,673, 28,672)	(4105, 4096)	(12,295, 12,288)	(28,675, 28,672)	(36,865, 36,864)
(8193, 8192)	(8195, 8192)	(24,577, 24,576)	(8197, 8192)	(40,961, 40,960)	(8199, 8192)	(24,581, 24,576)	(40,963, 40,960)	(57,345, 57,344)	(8201, 8192)	(24,583, 24,576)	(57,347, 57,344)	(73,729, 73,728)
(16,385, 16,384)	(16,387, 16,384)	(49,153, 49,152)	(16,389, 16,384)	(81,921, 81,920)	(16,391, 16,384)	(49,157, 49,152)	(81,923, 81,920)	(114,689, 114,688)	(16,393, 16,384)	(49,159, 49,152)	(114,691, 114,688)	(147,457, 147,456)

- In this second table, which is double entry: We give the positive non-trivial primitive Pythagorician triplets $(a,b,c)=(u^2-v^2,2uv,u^2+v^2)$, where $u>v$; $u+v\equiv 1(\bmod 2)$ and $gcd(u,v)=1$;

as well as the corresponding Pythagorician parameters $(d,\overline{e},S)\in\left(2\mathbb{N}+1 \underset{cop}{\times} 2\mathbb{N}+1\right)\times\mathbb{N}^*$.

Table 2. Table of positive primitive Pythagorician triplets of type (a,b,c) with $c<1500$.

N°	u	v	a	b	c	λ	s	d	$\overline{e}$	S	N°
1	2	1	3	4	5	1	2	1	1	1	1
2	3	2	5	12	13	0	2	1	1	2	2
3	4	1	15	8	17	1	3	3	1	1	27
4	4	3	7	24	25	1	3	1	3	1	6
5	5	2	21	20	29	0	2	3	1	2	28
6	5	4	9	40	41	0	3	1	1	3	3

Continued

7	6	1	35	12	37	1	2	5	1	1	44
8	6	5	11	60	61	1	2	1	5	1	10
9	7	2	45	28	53	0	2	5	1	2	45
10	7	4	33	56	65	0	3	3	1	3	29
11	7	6	13	84	85	0	2	1	3	2	7
12	8	1	63	16	65	1	4	7	1	1	64
13	8	3	55	48	73	1	4	5	3	1	49
14	8	5	39	80	89	1	4	3	5	1	32
15	8	7	15	112	113	1	4	1	7	1	13
16	9	2	77	36	85	0	2	7	1	2	65
17	9	4	65	72	97	0	3	5	1	3	46
18	9	8	17	144	145	0	4	1	1	4	4
19	10	1	99	20	101	1	2	9	1	1	84
20	10	3	91	60	109	1	2	7	3	1	69
21	10	7	51	140	149	1	2	3	7	1	35
22	10	9	19	180	181	1	2	1	9	1	15
23	11	2	117	44	125	0	2	9	1	2	85
24	11	4	105	88	137	0	3	7	1	3	66
25	11	6	85	132	157	0	2	5	3	2	50
26	11	8	57	176	185	0	4	3	1	4	30
27	11	10	21	220	221	0	2	1	5	2	11
28	12	1	143	24	145	1	3	11	1	1	99
29	12	5	119	120	169	1	3	7	5	1	72
30	12	7	95	168	193	1	3	5	7	1	53
31	12	11	23	264	265	1	3	1	11	1	17
32	13	2	165	52	173	0	2	11	1	2	100
33	13	4	153	104	185	0	3	9	1	3	86
34	13	6	133	156	205	0	2	7	3	2	70
35	13	8	105	208	233	0	4	5	1	4	47
36	13	10	69	260	269	0	2	3	5	2	33
37	13	12	25	312	313	0	3	1	3	3	8
38	14	1	195	28	197	1	2	13	1	1	119
39	14	3	187	84	205	1	2	11	3	1	104
40	14	5	171	140	221	1	2	9	5	1	89
41	14	9	115	252	277	1	2	5	9	1	55
42	14	11	75	308	317	1	2	3	11	1	37
43	14	13	27	364	365	1	2	1	13	1	19

Continued

44	15	2	221	60	229	0	2	13	1	2	120
45	15	4	209	120	241	0	3	11	1	3	101
46	15	8	161	240	289	0	4	7	1	4	67
47	15	14	29	420	421	0	2	1	7	2	14
48	16	1	255	32	257	1	5	15	1	1	138
49	16	3	247	96	265	1	5	13	3	1	124
50	16	5	231	160	281	1	5	11	5	1	107
51	16	7	207	224	305	1	5	9	7	1	92
52	16	9	175	288	337	1	5	7	9	1	75
53	16	11	135	352	377	1	5	5	11	1	57
54	16	13	87	416	425	1	5	3	13	1	39
55	16	15	31	480	481	1	5	1	15	1	21
56	17	2	285	68	293	0	2	15	1	2	139
57	17	4	273	136	305	0	3	13	1	3	121
58	17	6	253	204	325	0	2	11	3	2	105
59	17	8	225	272	353	0	4	9	1	4	87
60	17	10	189	340	389	0	2	7	5	2	73
61	17	12	145	408	433	0	3	5	3	3	51
62	17	14	93	476	485	0	2	3	7	2	36
63	17	16	33	544	545	0	5	1	1	5	5
64	18	1	323	36	325	1	2	17	1	1	148
65	18	5	299	180	349	1	2	13	5	1	127
66	18	7	275	252	373	1	2	11	7	1	110
67	18	11	203	396	445	1	2	7	11	1	77
68	18	13	155	468	493	1	2	5	13	1	59
69	18	17	35	612	613	1	2	1	17	1	22
70	19	2	357	76	365	0	2	17	1	2	149
71	19	4	345	152	377	0	3	15	1	3	140
72	19	6	325	228	397	0	2	13	3	2	125
73	19	8	297	304	425	0	4	11	1	4	102
74	19	10	261	380	461	0	2	9	5	2	90
75	19	12	217	456	505	0	3	7	3	3	71
76	19	14	165	532	557	0	2	5	7	2	54
77	19	16	105	608	617	0	5	3	1	5	31
78	19	18	37	684	685	0	2	1	9	2	16
79	20	1	399	40	401	1	3	19	1	1	164
80	20	3	391	120	409	1	3	17	3	1	153

Continued

81	20	7	351	280	449	1	3	13	7	1	130
82	20	9	319	360	481	1	3	11	9	1	112
83	20	11	279	440	521	1	3	9	11	1	94
84	20	13	231	520	569	1	3	7	13	1	79
85	20	17	111	680	689	1	3	3	17	1	40
86	20	19	39	760	761	1	3	1	19	1	23
87	21	2	437	84	445	0	2	19	1	2	165
88	21	4	425	168	457	0	3	17	1	3	150
89	21	8	377	336	505	0	4	13	1	4	122
90	21	10	341	420	541	0	2	11	5	2	108
91	21	16	185	672	697	0	5	5	1	5	48
92	21	20	41	840	841	0	3	1	5	3	12
93	22	1	483	44	485	1	2	21	1	1	180
94	22	3	475	132	493	1	2	19	3	1	169
95	22	5	459	220	509	1	2	17	5	1	156
96	22	7	435	308	533	1	2	15	7	1	143
97	22	9	403	396	565	1	2	13	9	1	132
98	22	13	315	572	653	1	2	9	13	1	96
99	22	15	259	660	709	1	2	7	15	1	80
100	22	17	195	748	773	1	2	5	17	1	60
101	22	19	123	836	845	1	2	3	19	1	41
102	22	21	43	924	925	1	2	1	21	1	24
103	23	2	525	92	533	0	2	21	1	2	181
104	23	4	513	184	545	0	3	19	1	3	166
105	23	6	493	276	565	0	2	17	3	2	154
106	23	8	465	368	593	0	4	15	1	4	141
107	23	10	429	460	629	0	2	13	5	2	128
108	23	12	385	552	673	0	3	11	3	3	106
109	23	14	333	644	725	0	2	9	7	2	93
110	23	16	273	736	785	0	5	7	1	5	68
111	23	18	205	828	853	0	2	5	9	2	56
112	23	20	129	920	929	0	3	3	5	3	34
113	23	22	45	1012	1013	0	2	1	11	2	18
114	24	1	575	48	577	1	4	23	1	1	188
115	24	5	551	240	601	1	4	19	5	1	172
116	24	7	527	336	625	1	4	17	7	1	158
117	24	11	455	528	697	1	4	13	11	1	134

Continued

118	24	13	407	624	745	1	4	11	13	1	114
119	24	17	287	816	865	1	4	7	17	1	81
120	24	19	215	912	937	1	4	5	19	1	61
121	24	23	47	1104	1105	1	4	1	23	1	25
122	25	2	621	100	629	0	2	23	1	2	189
123	25	4	609	200	641	0	3	21	1	3	182
124	25	6	589	300	661	0	2	19	3	2	170
125	25	8	561	400	689	0	4	17	1	4	151
126	25	12	481	600	769	0	3	13	3	3	126
127	25	14	429	700	821	0	2	11	7	2	111
128	25	16	369	800	881	0	5	9	1	5	88
129	25	18	301	900	949	0	2	7	9	2	76
130	25	22	141	1100	1109	0	2	3	11	2	38
131	25	24	49	1200	1201	0	4	1	3	4	9
132	26	1	675	52	677	1	2	25	1	1	201
133	26	3	667	156	685	1	2	23	3	1	192
134	26	5	651	260	701	1	2	21	5	1	184
135	26	7	627	364	725	1	2	19	7	1	174
136	26	9	595	468	757	1	2	17	9	1	160
137	26	11	555	572	797	1	2	15	11	1	145
138	26	15	451	780	901	1	2	11	15	1	115
139	26	17	387	884	965	1	2	9	17	1	97
140	26	19	315	988	1037	1	2	7	19	1	82
141	26	21	235	1092	1117	1	2	5	21	1	62
142	26	23	147	1196	1205	1	2	3	23	1	42
143	26	25	51	1300	1301	1	2	1	25	1	26
144	27	2	725	108	733	0	2	25	1	2	202
145	27	4	713	216	745	0	3	23	1	3	190
146	27	8	665	432	793	0	4	19	1	4	167
147	27	10	629	540	829	0	2	17	5	2	157
148	27	14	533	756	925	0	2	13	7	2	131
149	27	16	473	864	985	0	5	11	1	5	103
150	27	20	329	1080	1129	0	3	7	5	3	74
151	27	22	245	1188	1213	0	2	5	11	2	58
152	27	26	53	1404	1405	0	2	1	13	2	20
153	28	1	783	56	785	1	3	27	1	1	210
154	28	3	775	168	793	1	3	25	3	1	205

Continued

155	28	5	759	280	809	1	3	23	5	1	195
156	28	9	703	504	865	1	3	19	9	1	176
157	28	11	663	616	905	1	3	17	11	1	161
158	28	13	615	728	953	1	3	15	13	1	146
159	28	15	559	840	1009	1	3	13	15	1	135
160	28	17	495	952	1073	1	3	11	17	1	116
161	28	19	423	1064	1145	1	3	9	19	1	98
162	28	23	255	1288	1313	1	3	5	23	1	63
163	28	25	159	1400	1409	1	3	3	25	1	43
164	29	2	837	116	845	0	2	27	1	2	211
165	29	4	825	232	857	0	3	25	1	3	203
166	29	6	805	348	877	0	2	23	3	2	193
167	29	8	777	464	905	0	4	21	1	4	183
168	29	10	741	580	941	0	2	19	5	2	173
169	29	12	697	696	985	0	3	17	3	3	155
170	29	14	645	812	1037	0	2	15	7	2	144
171	29	16	585	928	1097	0	5	13	1	5	123
172	29	18	517	1044	1165	0	2	11	9	2	113
173	29	20	441	1160	1241	0	3	9	5	3	91
174	29	22	357	1276	1325	0	2	7	11	2	78
175	29	24	265	1392	1417	0	4	5	3	4	52
176	30	1	899	60	901	1	2	29	1	1	217
177	30	7	851	420	949	1	2	23	7	1	197
178	30	11	779	660	1021	1	2	19	11	1	177
179	30	13	731	780	1069	1	2	17	13	1	162
180	30	17	611	1020	1189	1	2	13	17	1	136
181	30	19	539	1140	1261	1	2	11	19	1	117
182	30	23	371	1380	1429	1	2	7	23	1	83
183	31	2	957	124	965	0	2	29	1	2	218
184	31	4	945	248	977	0	3	27	1	3	212
185	31	6	925	372	997	0	2	25	3	2	206
186	31	8	897	496	1025	0	4	23	1	4	191
187	31	10	861	620	1061	0	2	21	5	2	185
188	31	12	817	744	1105	0	3	19	3	3	171
189	31	14	765	868	1157	0	2	17	7	2	159
190	31	16	705	992	1217	0	5	15	1	5	142
191	31	18	637	1116	1285	0	2	13	9	2	133

Continued

192	31	20	561	1240	1361	0	3	11	5	3	109
193	31	22	477	1364	1445	0	2	9	11	2	95
194	32	1	1023	64	1025	1	6	31	1	1	225
195	32	3	1015	192	1033	1	6	29	3	1	221
196	32	5	999	320	1049	1	6	27	5	1	214
197	32	7	975	448	1073	1	6	25	7	1	207
198	32	9	943	576	1105	1	6	23	9	1	198
199	32	11	903	704	1145	1	6	21	11	1	186
200	32	13	855	832	1193	1	6	19	13	1	178
201	32	15	799	960	1249	1	6	17	15	1	163
202	32	17	735	1088	1313	1	6	15	17	1	147
203	32	19	663	1216	1385	1	6	13	19	1	137
204	32	21	583	1344	1465	1	6	11	21	1	118
205	33	2	1085	132	1093	0	2	31	1	2	226
206	33	4	1073	264	1105	0	3	29	1	3	219
207	33	8	1025	528	1153	0	4	25	1	4	204
208	33	10	989	660	1189	0	2	23	5	2	196
209	33	14	893	924	1285	0	2	19	7	2	175
210	33	16	833	1056	1345	0	5	17	1	5	152
211	33	20	689	1320	1489	0	3	13	5	3	129
212	34	1	1155	68	1157	1	2	33	1	1	232
213	34	3	1147	204	1165	1	2	31	3	1	228
214	34	5	1131	340	1181	1	2	29	5	1	223
215	34	7	1107	476	1205	1	2	27	7	1	216
216	34	9	1075	612	1237	1	2	25	9	1	208
217	34	11	1035	748	1277	1	2	23	11	1	199
218	34	13	987	884	1325	1	2	21	13	1	187
219	34	15	931	1020	1381	1	2	19	15	1	179
220	35	2	1221	140	1229	0	2	33	1	2	233
221	35	4	1209	280	1241	0	3	31	1	3	227
222	35	6	1189	420	1261	0	2	29	3	2	222
223	35	8	1161	560	1289	0	4	27	1	4	213
224	35	12	1081	840	1369	0	3	23	3	3	194
225	35	16	969	1120	1481	0	5	19	1	5	168
226	36	1	1295	72	1297	1	3	35	1	1	236
227	36	5	1271	360	1321	1	3	31	5	1	230
228	36	7	1247	504	1345	1	3	29	7	1	224

Continued

229	36	11	1175	792	1417	1	3	25	11	1	209
230	36	13	1127	936	1465	1	3	23	13	1	200
231	37	2	1365	148	1373	0	2	35	1	2	237
232	37	4	1353	296	1385	0	3	33	1	3	234
233	37	6	1333	444	1405	0	2	31	3	2	229
234	37	8	1305	592	1433	0	4	29	1	4	220
235	37	10	1269	740	1469	0	2	27	5	2	215
236	38	1	1443	76	1445	1	2	37	1	1	239
237	38	3	1435	228	1453	1	2	35	3	1	238
238	38	5	1419	380	1469	1	2	33	5	1	235
239	38	7	1395	532	1493	1	2	31	7	1	231

In **Table 3**, which is double entry too: This one gives the positive non-trivial primitive Pythagorician triplets:

$$(a,b,c)=\left(d^2+\left(2^S\bar{e}\right)d,\frac{\left(2^S\bar{e}\right)^2}{2}+\left(2^S\bar{e}\right)d,\frac{\left(2^S\bar{e}\right)^2}{2}+\left(2^S\bar{e}\right)d+d^2\right)\text{, where}$$

$(d,\bar{e},S)\in\left(2\mathbb{N}+1\underset{cop}{\times}2\mathbb{N}+1\right)\times\mathbb{N}^*$, note that:

$d=gcd(a,c-b)$, $\bar{e}=\frac{gcd(b,c-a)}{2^S}$, $b=2^s b_1$ and $S=s-\lambda(s-1)$, with $\lambda\in\{0,1\}/\frac{c-a}{2}\equiv\lambda(\text{mod}\,2)$.

We give as well the corresponding usual Pythagorician parameters: $u>v$; $u+v\equiv1(\text{mod}\,2)$ and $gcd(u,v)=1$; such that $a=u^2-v^2$; $b=2uv$; $c=u^2+v^2$.

Table 3. Table of positive primitive Pythagorician triplets of type (a,b,c) with $c<1500$.

N°	d	$\bar{e}$	S	a	b	c	λ	s	u	v	N°
1	1	1	1	3	4	5	1	2	2	1	1
2	1	1	2	5	12	13	0	2	3	2	2
3	1	1	3	9	40	41	0	3	5	4	6
4	1	1	4	17	144	145	0	4	9	8	18
5	1	1	5	33	544	545	0	5	17	16	63
6	1	3	1	7	24	25	1	3	4	3	4
7	1	3	2	13	84	85	0	2	7	6	11
8	1	3	3	25	312	313	0	3	13	12	37
9	1	3	4	49	1200	1201	0	4	25	24	131
10	1	5	1	11	60	61	1	2	6	5	8

Continued

11	1	5	2	21	220	221	0	2	11	10	27
12	1	5	3	41	840	841	0	3	21	20	92
13	1	7	1	15	112	113	1	4	8	7	15
14	1	7	2	29	420	421	0	2	15	14	47
15	1	9	1	19	180	181	1	2	10	9	22
16	1	9	2	37	684	685	0	2	19	18	78
17	1	11	1	23	264	265	1	3	12	11	31
18	1	11	2	45	1012	1013	0	2	23	22	113
19	1	13	1	27	364	365	1	2	14	13	43
20	1	13	2	53	1404	1405	0	2	27	26	152
21	1	15	1	31	480	481	1	5	16	15	55
22	1	17	1	35	612	613	1	2	18	17	69
23	1	19	1	39	760	761	1	3	20	19	86
24	1	21	1	43	924	925	1	2	22	21	102
25	1	23	1	47	1104	1105	1	4	24	23	121
26	1	25	1	51	1300	1301	1	2	26	25	143
27	3	1	1	15	8	17	1	3	4	1	3
28	3	1	2	21	20	29	0	2	5	2	5
29	3	1	3	33	56	65	0	3	7	4	10
30	3	1	4	57	176	185	0	4	11	8	26
31	3	1	5	105	608	617	0	5	19	16	77
32	3	5	1	39	80	89	1	4	8	5	14
33	3	5	2	69	260	269	0	2	13	10	36
34	3	5	3	129	920	929	0	3	23	20	112
35	3	7	1	51	140	149	1	2	10	7	21
36	3	7	2	93	476	485	0	2	17	14	62
37	3	11	1	75	308	317	1	2	14	11	42
38	3	11	2	141	1100	1109	0	2	25	22	130
39	3	13	1	87	416	425	1	5	16	13	54
40	3	17	1	111	680	689	1	3	20	17	85
41	3	19	1	123	836	845	1	2	22	19	101
42	3	23	1	147	1196	1205	1	2	26	23	142
43	3	25	1	159	1400	1409	1	3	28	25	163
44	5	1	1	35	12	37	1	2	6	1	7
45	5	1	2	45	28	53	0	2	7	2	9
46	5	1	3	65	72	97	0	3	9	4	17
47	5	1	4	105	208	233	0	4	13	8	35
48	5	1	5	185	672	697	0	5	21	16	91

Continued

49	5	3	1	55	48	73	1	4	8	3	13
50	5	3	2	85	132	157	0	2	11	6	25
51	5	3	3	145	408	433	0	3	17	12	61
52	5	3	4	265	1392	1417	0	4	29	24	175
53	5	7	1	95	168	193	1	3	12	7	30
54	5	7	2	165	532	557	0	2	19	14	76
55	5	9	1	115	252	277	1	2	14	9	41
56	5	9	2	205	828	853	0	2	23	18	111
57	5	11	1	135	352	377	1	5	16	11	53
58	5	11	2	245	1188	1213	0	2	27	22	151
59	5	13	1	155	468	493	1	2	18	13	68
60	5	17	1	195	748	773	1	2	22	17	100
61	5	19	1	215	912	937	1	4	24	19	120
62	5	21	1	235	1092	1117	1	2	26	21	141
63	5	23	1	255	1288	1313	1	3	28	23	162
64	7	1	1	63	16	65	1	4	8	1	12
65	7	1	2	77	36	85	0	2	9	2	16
66	7	1	3	105	88	137	0	3	11	4	24
67	7	1	4	161	240	289	0	4	15	8	46
68	7	1	5	273	736	785	0	5	23	16	110
69	7	3	1	91	60	109	1	2	10	3	20
70	7	3	2	133	156	205	0	2	13	6	34
71	7	3	3	217	456	505	0	3	19	12	75
72	7	5	1	119	120	169	1	3	12	5	29
73	7	5	2	189	340	389	0	2	17	10	60
74	7	5	3	329	1080	1129	0	3	27	20	150
75	7	9	1	175	288	337	1	5	16	9	52
76	7	9	2	301	900	949	0	2	25	18	129
77	7	11	1	203	396	445	1	2	18	11	67
78	7	11	2	357	1276	1325	0	2	29	22	174
79	7	13	1	231	520	569	1	3	20	13	84
80	7	15	1	259	660	709	1	2	22	15	99
81	7	17	1	287	816	865	1	4	24	17	119
82	7	19	1	315	988	1037	1	2	26	19	140
83	7	23	1	371	1380	1429	1	2	30	23	182
84	9	1	1	99	20	101	1	2	10	1	19
85	9	1	2	117	44	125	0	2	11	2	23
86	9	1	3	153	104	185	0	3	13	4	33

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87	9	1	4	225	272	353	0	4	17	8	59
88	9	1	5	369	800	881	0	5	25	16	128
89	9	5	1	171	140	221	1	2	14	5	40
90	9	5	2	261	380	461	0	2	19	10	74
91	9	5	3	441	1160	1241	0	3	29	20	173
92	9	7	1	207	224	305	1	5	16	7	51
93	9	7	2	333	644	725	0	2	23	14	109
94	9	11	1	279	440	521	1	3	20	11	83
95	9	11	2	477	1364	1445	0	2	31	22	193
96	9	13	1	315	572	653	1	2	22	13	98
97	9	17	1	387	884	965	1	2	26	17	139
98	9	19	1	423	1064	1145	1	3	28	19	161
99	11	1	1	143	24	145	1	3	12	1	28
100	11	1	2	165	52	173	0	2	13	2	32
101	11	1	3	209	120	241	0	3	15	4	45
102	11	1	4	297	304	425	0	4	19	8	73
103	11	1	5	473	864	985	0	5	27	16	149
104	11	3	1	187	84	205	1	2	14	3	39
105	11	3	2	253	204	325	0	2	17	6	58
106	11	3	3	385	552	673	0	3	23	12	108
107	11	5	1	231	160	281	1	5	16	5	50
108	11	5	2	341	420	541	0	2	21	10	90
109	11	5	3	561	1240	1361	0	3	31	20	192
110	11	7	1	275	252	373	1	2	18	7	66
111	11	7	2	429	700	821	0	2	25	14	127
112	11	9	1	319	360	481	1	3	20	9	82
113	11	9	2	517	1044	1165	0	2	29	18	172
114	11	13	1	407	624	745	1	4	24	13	118
115	11	15	1	451	780	901	1	2	26	15	138
116	11	17	1	495	952	1073	1	3	28	17	160
117	11	19	1	539	1140	1261	1	2	30	19	181
118	11	21	1	583	1344	1465	1	6	32	21	204
119	13	1	1	195	28	197	1	2	14	1	38
120	13	1	2	221	60	229	0	2	15	2	44
121	13	1	3	273	136	305	0	3	17	4	57
122	13	1	4	377	336	505	0	4	21	8	89
123	13	1	5	585	928	1097	0	5	29	16	171
124	13	3	1	247	96	265	1	5	16	3	49

Continued

125	13	3	2	325	228	397	0	2	19	6	72
126	13	3	3	481	600	769	0	3	25	12	126
127	13	5	1	299	180	349	1	2	18	5	65
128	13	5	2	429	460	629	0	2	23	10	107
129	13	5	3	689	1320	1489	0	3	33	20	211
130	13	7	1	351	280	449	1	3	20	7	81
131	13	7	2	533	756	925	0	2	27	14	148
132	13	9	1	403	396	565	1	2	22	9	97
133	13	9	2	637	1116	1285	0	2	31	18	191
134	13	11	1	455	528	697	1	4	24	11	117
135	13	15	1	559	840	1009	1	3	28	15	159
136	13	17	1	611	1020	1189	1	2	30	17	180
137	13	19	1	663	1216	1385	1	6	32	19	203
138	15	1	1	255	32	257	1	5	16	1	48
139	15	1	2	285	68	293	0	2	17	2	56
140	15	1	3	345	152	377	0	3	19	4	71
141	15	1	4	465	368	593	0	4	23	8	106
142	15	1	5	705	992	1217	0	5	31	16	190
143	15	7	1	435	308	533	1	2	22	7	96
144	15	7	2	645	812	1037	0	2	29	14	170
145	15	11	1	555	572	797	1	2	26	11	137
146	15	13	1	615	728	953	1	3	28	13	158
147	15	17	1	735	1088	1313	1	6	32	17	202
148	17	1	1	323	36	325	1	2	18	1	64
149	17	1	2	357	76	365	0	2	19	2	70
150	17	1	3	425	168	457	0	3	21	4	88
151	17	1	4	561	400	689	0	4	25	8	125
152	17	1	5	833	1056	1345	0	5	33	16	210
153	17	3	1	391	120	409	1	3	20	3	80
154	17	3	2	493	276	565	0	2	23	6	105
155	17	3	3	697	696	985	0	3	29	12	169
156	17	5	1	459	220	509	1	2	22	5	95
157	17	5	2	629	540	829	0	2	27	10	147
158	17	7	1	527	336	625	1	4	24	7	116
159	17	7	2	765	868	1157	0	2	31	14	189
160	17	9	1	595	468	757	1	2	26	9	136
161	17	11	1	663	616	905	1	3	28	11	157
162	17	13	1	731	780	1069	1	2	30	13	179
163	17	15	1	799	960	1249	1	6	32	15	201

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164	19	1	1	399	40	401	1	3	20	1	79
165	19	1	2	437	84	445	0	2	21	2	87
166	19	1	3	513	184	545	0	3	23	4	104
167	19	1	4	665	432	793	0	4	27	8	146
168	19	1	5	969	1120	1481	0	5	35	16	225
169	19	3	1	475	132	493	1	2	22	3	94
170	19	3	2	589	300	661	0	2	25	6	124
171	19	3	3	817	744	1105	0	3	31	12	188
172	19	5	1	551	240	601	1	4	24	5	115
173	19	5	2	741	580	941	0	2	29	10	168
174	19	7	1	627	364	725	1	2	26	7	135
175	19	7	2	893	924	1285	0	2	33	14	209
176	19	9	1	703	504	865	1	3	28	9	156
177	19	11	1	779	660	1021	1	2	30	11	178
178	19	13	1	855	832	1193	1	6	32	13	200
179	19	15	1	931	1020	1381	1	2	34	15	219
180	21	1	1	483	44	485	1	2	22	1	93
181	21	1	2	525	92	533	0	2	23	2	103
182	21	1	3	609	200	641	0	3	25	4	123
183	21	1	4	777	464	905	0	4	29	8	167
184	21	5	1	651	260	701	1	2	26	5	134
185	21	5	2	861	620	1061	0	2	31	10	187
186	21	11	1	903	704	1145	1	6	32	11	199
187	21	13	1	987	884	1325	1	2	34	13	218
188	23	1	1	575	48	577	1	4	24	1	114
189	23	1	2	621	100	629	0	2	25	2	122
190	23	1	3	713	216	745	0	3	27	4	145
191	23	1	4	897	496	1025	0	4	31	8	186
192	23	3	1	667	156	685	1	2	26	3	133
193	23	3	2	805	348	877	0	2	29	6	166
194	23	3	3	1081	840	1369	0	3	35	12	224
195	23	5	1	759	280	809	1	3	28	5	155
196	23	5	2	989	660	1189	0	2	33	10	208
197	23	7	1	851	420	949	1	2	30	7	177
198	23	9	1	943	576	1105	1	6	32	9	198
199	23	11	1	1035	748	1277	1	2	34	11	217
200	23	13	1	1127	936	1465	1	3	36	13	230
201	25	1	1	675	52	677	1	2	26	1	132
202	25	1	2	725	108	733	0	2	27	2	144

Continued

203	25	1	3	825	232	857	0	3	29	4	165
204	25	1	4	1025	528	1153	0	4	33	8	207
205	25	3	1	775	168	793	1	3	28	3	154
206	25	3	2	925	372	997	0	2	31	6	185
207	25	7	1	975	448	1073	1	6	32	7	197
208	25	9	1	1075	612	1237	1	2	34	9	216
209	25	11	1	1175	792	1417	1	3	36	11	229
210	27	1	1	783	56	785	1	3	28	1	153
211	27	1	2	837	116	845	0	2	29	2	164
212	27	1	3	945	248	977	0	3	31	4	184
213	27	1	4	1161	560	1289	0	4	35	8	223
214	27	5	1	999	320	1049	1	6	32	5	196
215	27	5	2	1269	740	1469	0	2	37	10	235
216	27	7	1	1107	476	1205	1	2	34	7	215
217	29	1	1	899	60	901	1	2	30	1	176
218	29	1	2	957	124	965	0	2	31	2	183
219	29	1	3	1073	264	1105	0	3	33	4	206
220	29	1	4	1305	592	1433	0	4	37	8	234
221	29	3	1	1015	192	1033	1	6	32	3	195
222	29	3	2	1189	420	1261	0	2	35	6	222
223	29	5	1	1131	340	1181	1	2	34	5	214
224	29	7	1	1247	504	1345	1	3	36	7	228
225	31	1	1	1023	64	1025	1	6	32	1	194
226	31	1	2	1085	132	1093	0	2	33	2	205
227	31	1	3	1209	280	1241	0	3	35	4	221
228	31	3	1	1147	204	1165	1	2	34	3	213
229	31	3	2	1333	444	1405	0	2	37	6	233
230	31	5	1	1271	360	1321	1	3	36	5	227
231	31	7	1	1395	532	1493	1	2	38	7	239
232	33	1	1	1155	68	1157	1	2	34	1	212
233	33	1	2	1221	140	1229	0	2	35	2	220
234	33	1	3	1353	296	1385	0	3	37	4	232
235	33	5	1	1419	380	1469	1	2	38	5	238
236	35	1	1	1295	72	1297	1	3	36	1	226
237	35	1	2	1365	148	1373	0	2	37	2	231
238	35	3	1	1435	228	1453	1	2	38	3	237
239	37	1	1	1443	76	1445	1	2	38	1	236

1.4. Application to the Resolution of the Equations $x^2 + y^2 = 2z^2$ and $x^2 + 2y^2 = z^2$

1.4.1. Resolution of the Diophantine Equation $x^2 + y^2 = 2z^2$

- Consider the equation: $x^2 + y^2 = 2z^2$ with x, y, z, positive, coprime in pairs and $x > y$ (cf. [9], for another type of resolution). We suppose that $(x, y, z) \neq (1,1,1)$ that is to say that it is non-trivial. It is clear that: $x \equiv y = z \equiv 1 \pmod 2$. But then:

$$x^2 + y^2 = 2z^2 \Leftrightarrow \left(\frac{x-y}{2}\right)^2 + \left(\frac{x+y}{2}\right)^2 = z^2 \tag{1.6}$$

(with $\frac{x-y}{2}$, $\frac{x+y}{2}$, z coprime in pairs and $\frac{x-y}{2}$, $\frac{x+y}{2}$ of opposite parity).

Let $\beta \in \{0, 1\}$ defined by:

$$\frac{x+(-1)^{\beta} y}{2} \equiv 0 \pmod 2 \Leftrightarrow \frac{x-(-1)^{\beta} y}{2} \equiv 1 \pmod 2; \tag{1.7}$$

Then, the Pythagoras Equation (1.6) is written as:

$$\left(\frac{x-(-1)^{\beta} y}{2}\right)^2 + \left(\frac{x+(-1)^{\beta} y}{2}\right)^2 = z^2 \tag{1.8}$$

That is: $\left(\frac{x-(-1)^{\beta} y}{2}, \frac{x+(-1)^{\beta} y}{2} = 2^s b_1, z\right) \in \overline{T^+}$, (where $s \geq 2$; b_1 odd).

For such a triplet, we know (cf. Definition 1.2.) That there exists $\lambda \in \{0,1\}$, such that:

$$\frac{z - \frac{x-(-1)^{\beta} y}{2}}{2} \equiv \lambda \pmod 2 \tag{1.9}$$

As in the previous section, and if necessary, we state:

$$S = s - \lambda(s-1) \text{ and } S' = S - 1 = (s-1)(1-\lambda). \tag{1.10}$$

Then, $\exists \overline{e}$ odd, such that we have the expression of the Pythagorician divisors associated with (1.8):

$$\begin{cases} (d, d'') = \left(gcd\left(\frac{x-(-1)^{\beta} y}{2}, z - \frac{x+(-1)^{\beta} y}{2}\right), \frac{\frac{x-(-1)^{\beta} y}{2}}{d} \right) \text{ and } \frac{x-(-1)^{\beta} y}{2} = dd''; \\ \begin{cases} e = gcd\left(\frac{x+(-1)^{\beta} y}{2}, z - \frac{x-(-1)^{\beta} y}{2}\right) = 2^S \overline{e}; \\ e'' = \frac{\frac{x+(-1)^{\beta} y}{2}}{e} = 2^{s-S} \frac{b_1}{\overline{e}}. \end{cases} \text{ And } \frac{x+(-1)^{\beta} y}{2} = ee''. \end{cases}$$

In addition we have: $\begin{cases} d'' = d + e = d + 2^S \overline{e}; \\ e'' = \frac{e}{2} + d = 2^{S-1} \overline{e} + d. \end{cases}$

The notations being the same, we then apply the Pythagorician divisors theorem to (1.8), from which we deduce the following corollary:

Corollary 1.4 *There is an equivalence between the following propositions* (*the* *solutions are supposed to be primitive and positive non-trivial*).

(i) $x^2+y^2=2z^2$ *is solvable*; (ii) $\begin{cases} x=e''^2-\left(\frac{e}{2}\right)^2+2\left(\frac{e}{2}\right)e''; \\ y=(-1)^\beta\left[-e''^2+\left(\frac{e}{2}\right)^2+2\left(\frac{e}{2}\right)e''\right]; \\ z=e''^2+\left(\frac{e}{2}\right)^2. \end{cases}$

(iii) $$\begin{cases} x=\left(\frac{d''+d}{2}\right)^2-\left(\frac{d''-d}{2}\right)^2+2\left(\frac{d''-d}{2}\right)\left(\frac{d''+d}{2}\right); \\ y=(-1)^\beta\left[-\left(\frac{d''+d}{2}\right)^2+\left(\frac{d''-d}{2}\right)^2+2\left(\frac{d''-d}{2}\right)\left(\frac{d''+d}{2}\right)\right]; \\ z=\left(\frac{d''+d}{2}\right)^2+\left(\frac{d''-d}{2}\right)^2. \end{cases}$$

(iv) $$\begin{cases} x=2\times 2^{2S'}\bar{e}^2+4\times 2^{S'}\bar{e}d+d^2; \\ y=(-1)^\beta\left[2\times 2^{2S'}\bar{e}^2-d^2\right]; \\ z=2\times 2^{2S'}\bar{e}^2+2\times 2^{S'}\bar{e}d+d^2. \end{cases}$$

Proof 4 *It suffices to apply Theorem* 1.1., *and Corollary* 1.1. & 1.2., *with respect to* (1.8).

Remark 1.4 1) *We have*: $\beta\in\{0,1\}$ *such that*: $(-1)^\beta=sign\left(2\times 2^{2S'}\bar{e}^2-d\right)^2$ *i.e.* $y\geq 0$.

2) *Note that the non-trivial, primitive and positive solutions of the equation* $x^2+y^2=2z^2$ *are given by unique expressions of the type*: $(x,y,z)=\left(u^2-v^2+2uv,u^2-v^2-2uv,u^2+v^2\right)$ *when* $\beta=1$, *and* $\left(u^2-v^2+2uv,v^2-u^2+2uv,u^2+v^2\right)$ *when* $\beta=0$, *with* $(u,v)=\left(e'',\frac{e}{2}\right)=\left(\frac{d''+d}{2},\frac{d''-d}{2}\right)$, *such that*: $u>v$, $gcd(u,v)=1$, $u+v\equiv 1(\mathrm{mod}\,2)$.

In other words by: $(x,y,z)=\left(u^2-v^2+2uv,\left|u^2-v^2-2uv\right|,u^2+v^2\right)$.

Let us keep the same notations, then on the model of Corollary 1.3., We have:

Corollary 1.5 *The set of non-trivial, primitive and positive solutions of the equation* $x^2+y^2=2z^2$ *is in bijection with the set* $\left(2\mathbb{N}+1\underset{cop}{\times}2\mathbb{N}+1\right)\times\mathbb{N}$, *as follows*:

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\overset{f}{\mapsto}\begin{pmatrix} d=gcd\left(\frac{x-(-1)^\beta y}{2},z-\frac{x+(-1)^\beta y}{2}\right) \\ \bar{e}=\frac{gcd\left(\frac{x+(-1)^\beta y}{2},z-\frac{x-(-1)^\beta y}{2}\right)}{2\times 2^{S'}} \\ S' \end{pmatrix}$$, *where* β *and* S' *are defined in* (1.7), (1.9) (1.10).

Whose reciprocal bijection is:

$$\begin{pmatrix} d \\ \overline{e} \\ S' \end{pmatrix} \overset{f^{-1}}{\mapsto} \begin{pmatrix} x = 2\times 2^{2S'}\overline{e}^2 + 4\times 2^{S'}\overline{e}d + d^2 \\ y = (-1)^{\beta}\left[2\times 2^{2S'}\overline{e}^2 - d^2\right] \\ z = 2\times 2^{2S'}\overline{e}^2 + 2\times 2^{S'}\overline{e}d + d^2 \end{pmatrix}$$, where $\beta \in \{0,1\}$ and $(-1)^{\beta} = sign\left(2\times 2^{2S'}\overline{e}^2 - d^2\right)$.

Proof 5 *The proof is similar to that made in Corollary* 1.3.

Remark 1.5 *By reasoning similar to that already done in the paragraph* 1.3, *if we consider* Σ, *the set of non-trivial, primitive and positive solutions of* $x^2 + y^2 = 2z^2$, *then* $(x_1, y_1, z_1) \cong (x_2, y_2, z_2) \Leftrightarrow (d_1, \overline{e}_1) = (d_2, \overline{e}_2)$ *define an equivalence relation on* Σ, *and each class* $\overline{(x, y, z)}$ *contains a unique solution* (x_0, y_0, z_0) *for which* $S_0' = (s_0 - 1)(1 - \lambda_0) = 0$ *i.e.* $\lambda_0 = 1$ (*cf.* (1.10)), *and which will be its canonical representative. These solutions characterized by* $S' = 0$, *are "rare", they are of the type*: $\begin{pmatrix} x = 2\overline{e}^2 + 4\overline{e}d + d^2 \\ y = (-1)^{\beta}\left[2\overline{e}^2 - d^2\right] \\ z = 2\overline{e}^2 + 2\overline{e}d + d^2 \end{pmatrix}$; *their cardinality is equal to*: $Card\left(2\mathbb{N}+1 \underset{cop}{\times} 2\mathbb{N}+1\right)$, *while the solutions characterized by* $S' \geq 1$, *have for cardinality*: $Card\left(2\mathbb{N}+1 \underset{cop}{\times} 2\mathbb{N}+1\right) \times Card\left(\mathbb{N}^*\right)$.

Exemples 1.1 *Let's take the minimum possible value* $(d, \overline{e}) = (1,1)$, *we get the minimal solution of* $x^2 + y^2 = 2z^2$, *that is* $(7,1,5) = f^{-1}(1,1,0)$, *whose class is* $\left\{f^{-1}(1,1,S'), S' \in \mathbb{N}\right\}$, *i.e.*:

$$\overline{(7,1,5)} = \{(\mathbf{7,1,5}), (17,7,13), (49,31,41), (161,127,145), (577,511,545), \cdots\}$$

1.4.2. Resolution of the Diophantine Equation $x^2 + 2y^2 = z^2$

- We finish this paragraph, using methods similar to that of Pythagorician divisors, to solve the equation $x^2 + 2y^2 = z^2$ (One can compare our method with those applied to the resolution of similar equations cf. [10] and [11]). One can see also ref. [12] for details and applications to congruent numbers problem.

We suppose that x, y, z are positive and primitive, therefore necessarily x, z are odd, $y \neq 0$ is even and x, y, z are pairwise coprime. Let's define:

$$\begin{aligned} &\beta \in \{0,1\} \text{ by}: \frac{z + (-1)^{\beta} x}{2} \equiv 0 (\bmod 2), \\ &\text{and therefore } \frac{z - (-1)^{\beta} x}{2} \equiv 1 (\bmod 2). \end{aligned} \quad (1.11)$$

Recall (cf. Reminder 1.1.) That $gcd\left(\frac{z + (-1)^{\beta} x}{2}, \frac{z - (-1)^{\beta} x}{2}\right) = 1$.

Then: $y^2 = 2 \times \frac{z + (-1)^{\beta} x}{2} \times \frac{z - (-1)^{\beta} x}{2}$.

Hence $\exists\, a_1$ even and a_2 odd, such that $y = a_1 a_2$, $gcd(a_1, a_2) = 1$ and:

$$\begin{cases} \dfrac{z+(-1)^\beta x}{2} = \dfrac{a_1^2}{2}; \\ \dfrac{z-(-1)^\beta x}{2} = a_2^2. \end{cases} \Rightarrow gcd\left(y,\ \frac{z+(-1)^\beta x}{2}\right) = gcd\left(a_1 a_2,\ \frac{a_1^2}{2}\right) = a_1;$$

So, if we set:

$$e = gcd\left(y, \frac{z+(-1)^\beta x}{2}\right) = a_1 \text{ and } e'' = \frac{y}{e} = a_2 \tag{1.12}$$

it follows that:

$$\begin{cases} \dfrac{z+(-1)^\beta x}{2} = \dfrac{e^2}{2}; \\ \dfrac{z-(-1)^\beta x}{2} = e''^2. \end{cases} \Rightarrow \begin{cases} x = (-1)^\beta \left(\dfrac{e^2}{2} - e''^2\right) = \begin{cases} \dfrac{e^2}{2} - e''^2, \text{ if } \beta = 0; \\ -\dfrac{e^2}{2} + e''^2, \text{ if } \beta = 1. \end{cases} \\ y = ee''; \\ z = e''^2 + \dfrac{e^2}{2}. \end{cases}$$

To sum up:

If $\beta \in \{0,\ 1\}$ is defined by: $\dfrac{z+(-1)^\beta x}{2} \equiv 0 \pmod 2 \Leftrightarrow \dfrac{z-(-1)^\beta x}{2} \equiv 1 \pmod 2$, and that: $gcd\left(y,\ \dfrac{z+(-1)^\beta x}{2}\right) = e$, and $\dfrac{y}{e} = e''$.

We just come to show the following result:

Proposition 1.8 *There is an equivalence between the following propositions* (*the solutions are supposed to be non-trivial, primitive and positive*).

(i) $x^2 + 2y^2 = z^2$ is solvable;

(ii) $$\begin{cases} x = (-1)^\beta \left(2\left(\dfrac{e}{2}\right)^2 - e''^2\right) = \begin{cases} 2\left(\dfrac{e}{2}\right)^2 - e''^2, \text{ if } \beta = 0; \\ e''^2 - 2\left(\dfrac{e}{2}\right)^2, \text{ if } \beta = 1. \end{cases} \\ y = 2e''\left(\dfrac{e}{2}\right); \\ z = e''^2 + 2\left(\dfrac{e}{2}\right)^2. \end{cases}$$

Remark 1.6 *We have* $(x, y, z) = \left(\left|u^2 - 2v^2\right|,\ 2uv,\ u^2 + 2v^2\right)$ *where* $(u, v) = \left(e'', \dfrac{e}{2}\right)$.

Let (x, y, z) be a positive non-trivial and primitive solution $x^2 + 2y^2 = z^2$, then we have:

$$y = 2^s y_1 \text{ and } s \geq 1, y_1 \text{ odd. Let's put } S'' = s - 1 \in \mathbb{N}. \tag{1.13}$$

Then since e'' is odd, there exists odd $\overline{e}$ such that: $e = 2^s \overline{e}$ with $\overline{e}e'' = y_1$ from where $y = 2^s \overline{e}e''$.

We have the following Proposition:

Proposition 1.9 *The set of non-trivial, primitive and positive solutions of the equation* $x^2+2y^2=z^2$ *is in bijection with the set* $\left(2\mathbb{N}+1 \underset{cop}{\times} 2\mathbb{N}+1\right)\times\mathbb{N}$, *as follows*:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \overset{g}{\mapsto} \begin{pmatrix} e''=\dfrac{y}{gcd\left(y,\dfrac{z+(-1)^{\beta}x}{2}\right)} \\ \overline{e}=\dfrac{gcd\left(y,\dfrac{z+(-1)^{\beta}x}{2}\right)}{2\times 2^{S''}} \\ S'' \end{pmatrix}$$

, *where* β *and* S'' *are defined in* (1.11) *and* (1.13).

Whose reciprocal bijection is:

$$\begin{pmatrix} e'' \\ \overline{e} \\ S'' \end{pmatrix} \overset{g^{-1}}{\mapsto} \begin{pmatrix} x=(-1)^{\beta}\left(2\left(2^{S''}\overline{e}\right)^2-e''^2\right) \\ y=2e''\left(2^{S''}\overline{e}\right) \\ z=e''^2+2\left(2^{S''}\overline{e}\right)^2 \end{pmatrix}$$

, *where* $\beta\in\{0,1\}$ *and* $(-1)^{\beta}=sign\left(2\left(2^{S''}\overline{e}\right)^2-e''^2\right)$.

Remark 1.7 *In the same way as the previous equations it is possible to classify the positive primitive solutions of this equation, in terms of equivalence classes*:

$$(x,y,z)\cong(x_1,y_1,z_1)\Leftrightarrow\begin{cases} e''=\dfrac{y}{gcd\left(y,\dfrac{z+(-1)^{\beta}x}{2}\right)}=\dfrac{y_1}{gcd\left(y_1,\dfrac{z_1+(-1)^{\beta_1}x_1}{2}\right)}=e''_1 \\ \overline{e}=\dfrac{gcd\left(y,\dfrac{z+(-1)^{\beta}x}{2}\right)}{2\times 2^{S''}}=\dfrac{gcd\left(y_1,\dfrac{z_1+(-1)^{\beta_1}x_1}{2}\right)}{2\times 2^{S''_1}}=\overline{e}_1 \end{cases}$$

As an example, consider the smallest positive and primitive solution $(1,2,3)$ *of this equation, obtained for* $(e'',\overline{e},S'')=(1,1,0)$, *i.e.* $(1,2,3)=g^{-1}(1,1,0)$, *then*:

$$\overline{(1,2,3)}=\left\{g^{-1}(1,1,S''),S''\in\mathbb{N}\right\}=\left\{(\mathbf{1},\mathbf{2},\mathbf{3}),(7,4,9),(31,8,33),(127,16,129),\cdots\right\}$$

2. Conclusions and Perspectives

This new method exposed here, made it possible to solve the Pythagoras equation from its Pythagorician divisors, and to make a classification of the solutions of this one, highlighting the importance of the quantity $\lambda=0$ or 1, defined by $\dfrac{c-a}{2}\equiv\lambda(\text{mod}\,2)$, as well as other quantities coming directly from the Pythagorician divisors.

In addition, this method opens interesting prospects for resolution, concerning the Pythagoras equation and those which proceed from it, as we have exposed it in this paper, but probably also, concerning the quadratic equations, in particular those of Pell-Fermat of the type: $a^2 - db^2 = 1$. This is how the well-known problem of the existence of the square area of the right-angled triangle could also be considered from this point of view, and the same applies to Fermat's equations of the type $a^4 + b^4 = c^4$, $a^{2p} + b^{2p} = c^{2p}$ or more generally those of the type $a^p + b^p = c^p$.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Rimbeboim, P. (1999) Fermat's Last Theorem for Amateurs. Springer-Verlag New-York Inc., New York.

[2] Terjanian, G. (1977) Sur l'équation $x^{2p} + y^{2p} = z^{2p}$. *Comptes Rendus de l'Académie des Sciences*, **285**, 973-975.

[3] Wiles, A. (1995) Modular Elliptic Curves and Fermat's Last Theorem. *Annals of Mathematics*, **141**, 443-551. https://doi.org/10.2307/2118559

[4] Carmichael, R.D. (1959) The Theory of Numbers and Diophantine Analysis. Dover

Publications, Inc., New York.

[5] Clagett, M. (1999) Ancient Egyptian Science a Book Source, Vol. 3, Ancient Egyptian Mathematics, Memoirs Series, Volume 232, American Philosophical Society, Independance Square, Philadelphia.

[6] Andreescu, T., Andrica, D. and Cucurezeanu, I. (2010) An Introduction to Diophantine Equations. Birkhäuser Boston, MA. https://doi.org/10.1007/978-0-8176-4549-6

[7] Mordell, L.J. (1969) Diophantine Equations. Academic Press, London and New York.

[8] Mouanda, J. (2022) On Fermat's Last Theorem and Galaxies of Sequences of Positive Integers. *American Journal of Computational Mathematics*, **12**, 162-189. https://doi.org/10.4236/ajcm.2022.121009

[9] Abdelalim, S. and Dyani, H. (2015) Caracterization of the Solution of Diophantine Equation $x^2 + y^2 = 2z^2$. *Gulf Journal of Mathematics*, **3**, 1-4. https://doi.org/10.56947/gjom.v3i2.154

[10] Abdelalim, S. and Dyani, H. (2014) The Solution of Diophantine Equation $x^2 + 3y^2 = z^2$. *International Journal of Algebra*, **8**, 729-723. https://doi.org/10.12988/ija.2014.4779

[11] Mangalaeng, A.H. (2022) The Primitive-Solutions of Diophantine Equation $x^2 + pqy^2 = z^2$, for Primes *p*, *q*. *Jurnal Matematika*, *Statidstika & Komputasi*, **18**, 308-314. https://doi.org/10.20956/j.v18i2.19018

[12] Keumean, D.L. (2021) Diviseurs Pythagoriciens appliqués à la résolution du problème de certains nombres congruent. Master Thesis, N 453, UFR de Mathématiques et Informatique, Université Houphouët-Boigny, Abidjan, Côte d'Ivoire.