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Uniform Convergence of Translation Operators

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Abstract

We denote $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ the sets of natural, real and complex numbers respectively. Let (λ_n), $n \in \mathbb{N}$ be an unbounded sequence of complex numbers. Costakis has proved the following result. There exists an entire function *f* with the following property: for every $x, y \in \mathbb{R}$ with $0 < x < y$, every $\theta \in (0,1)$ and every $a \in \mathbb{C}$ there is a subsequence of natural numbers (m_n), $n \in \mathbb{N}$ such that, for every compact subset $L \subseteq \mathbb{C}$, $\sup_{r \in [x,y]} \sup_{t \in [0,\theta]} \sup_{z \in L} \left| f\left(z + \lambda_{m_n} r e^{2\pi i t}\right) - a \right| \to 0$ as $n \to \infty$ (*). In the present paper we show that the constant function *a* cannot be replaced by any non-constant entire function *G*. This is so even if one demands the convergence in (*) only for a single radius *r* and a single positive number θ. This result is related with the problem of existence of common universal vectors for an uncountable family of sequences of translation operators.

Keywords

Hypercyclic Operator, Common Hypercyclic Vectors, Translation Operator

1. Introduction

We denote $\mathcal{H}(\mathbb{C})$ the set of entire functions endowed with the topology $\mathcal{T}_u$ of uniform convergence on compacta.

Let $a \in \mathbb{C}$. We denote $t_a : \mathbb{C} \to \mathbb{C}$ the translation function with the formula $t_a(z) = z + a$ for every $z \in \mathbb{C}$.

We consider the translation operator $T_a : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$ with the formula $T_a(f) = f \circ t_a$ for every $f \in \mathcal{H}(\mathbb{C})$. The operator T_a is a linear and continuous operator.

We write $T_a^1 = T_a$ and

$$T_a^{n+1} = T_a \circ T_a^n \text{ for } n = 1, 2, \cdots.$$

Birkhoff proved [1] that there is $f \in \mathcal{H}(\mathbb{C})$ so that

$$\overline{\{T_a^n(f), n \in \mathbb{N}\}} = \mathcal{H}(\mathbb{C}), \text{ where } a \in \mathbb{C} \setminus \{0\}.$$

In modern language this means that the operator T_a is hypercyclic, or in other words the sequence of operators (T_a^n), $n \in \mathbb{N}$ is hypercyclic.

His proof was constructive.

Let $(a_n)_{n\in\mathbb{N}}$ be an unbounded sequence of complex numbers. Luh proved [2] that there is $f \in \mathcal{H}(\mathbb{C})$ so that

$$\overline{\{T_{a_n}(f), n \in \mathbb{N}\}} = \mathcal{H}(\mathbb{C}).$$

Gethner and Shapiro [3] and Grosse-Erdmann [4] have also proved the above results by using the Baire's Category Theorem. In particular, we denote

$$\mathcal{U}(T_{a_n}) = \left\{ f \in \mathcal{H}(\mathbb{C}) \mid \overline{\{T_{a_n}(f) \mid n \in \mathbb{N}\}} = \mathcal{H}(\mathbb{C}) \right\}.$$

Then, the set $\mathcal{U}(T_{a_n})$ is a G_δ and dense subset of $\mathcal{H}(\mathbb{C})$. This means that the sequence of operators (T_{a_n}), $n \in \mathbb{N}$ is hypercyclic. Let $(b_m)_{m\in\mathbb{N}}$ be a sequence of non-zero complex numbers. Based on the previous result, the set $\bigcap_{m\in\mathbb{N}} \mathcal{U}(T_{b_m a_n})$ is a G_δ and dense subset of $\mathcal{H}(\mathbb{C})$. This is a simple consequence of Baire's Category Theorem.

Costakis and Sambarino [5] established a notable strengthening of Birkhoff's result. More specifically, they proved that the set

$$\bigcap_{a\in\mathbb{C}\setminus\{0\}} \left\{ f \in H(\mathbb{C}) \mid \overline{\{T_a^n(f), n \in \mathbb{N}\}} = H(\mathbb{C}) \right\}$$

contains a G_δ and dense subset of $\mathcal{H}(\mathbb{C})$.

The important element here is the uncountable range of *a*, because Baire's Category Theorem is not applied for the intersection of an uncountable family of sets.

Furthermore, Costakis [6] proved a more general result, that is, the set $\bigcap_{b\in C(0,1)} \mathcal{U}(T_{ba_n})$ contains a G_δ and dense subset of $\mathcal{H}(\mathbb{C})$, where a_n is an unbounded and specific sequence of complex numbers and $C(0,1) = \{z \in \mathbb{C} \mid |z| = 1\}$.

The proof of this result follows a similar method to the one used to prove a similar result in [5]. In the same article [6], Costakis examined a simpler and more specific case of the above result. In particular:

In the above set, the request is to find an entire function *f*, so that:

$$\overline{\{f(z+\lambda_n\alpha) : n \in \mathbb{N}\}} = \mathcal{H}(\mathbb{C}) \text{ for every } \alpha \in C(0,1).$$

Firstly, Costakis proved in [6] that there is some $f \in \mathcal{H}(\mathbb{C})$, so that the set of constant functions is a subset of $\overline{\{f(z+\lambda_n\alpha) : n \in \mathbb{N}\}}$ for every $\alpha \in C(0,1)$.

This result offered a different and notable proof. In fact, he proved something stronger in this case, by adding a stronger condition of convergence. More specifically, Costakis proved the following result:

Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of non-zero complex numbers, so that $\lambda_n \to \infty$.

We have the set:

$\mathcal{U}_{\mathbb{C}}(\lambda)=\{f\in\mathcal{H}(\mathbb{C})|$ for every $x,y\in\mathbb{R}$ such that $0<x<y$, for every $\theta\in(0,1)$ and for every $\alpha\in\mathbb{C}$ there is a sequence $m=(m_n)_{n\in\mathbb{N}}$, so that $m_n\in\{\lambda_n,n\in\mathbb{N}\}$, for every $n\in\mathbb{N}$, so that, for every compact subset $L\subseteq\mathbb{C}$ $\sup_{r\in[x,y]}\sup_{t\in[0,\theta]}\sup_{z\in L}\left|f\left(z+m_nre^{2\pi it}\right)-a\right|\to 0$ as $n\to+\infty\}$.

Costakis [6] proved that the above set $\mathcal{U}_{\mathbb{C}}(\lambda)$ is a G_δ and dense subset of $\mathcal{H}(\mathbb{C})$. However, he did not use this method in the general case and gave a completely different proof in the general case.

Therefore, it is reasonable to ask if we can deal with the general case by imitating the proof of the above specific case. In this paper, we shall prove that this cannot be done. More specifically, we will prove here the following result:

Let $\lambda=(\lambda_n)_{n\in\mathbb{N}}$ be a given sequence of non-zero complex numbers, so that $\lambda_n\to\infty$, $\theta\in(0,1)$ and $G\in\mathcal{H}(\mathbb{C})$, where θ is a given number and G is a given non constant function. We shall consider the set $\mathcal{U}(\lambda,\theta,G)=\{f\in\mathcal{H}(\mathbb{C})|$ there exists a sequence $m=(m_n)_{n\in\mathbb{N}}$, where $m_n\in\{\lambda_k,k\in\mathbb{N}\}$ for every $n\in\mathbb{N}$, so that for every compact set $L\subseteq\mathbb{C}$ $\sup_{(t,z)\in[0,\theta]\times L}\left|f\left(z+m_ne^{2\pi it}\right)-G(z)\right|\to 0$ as $n\to+\infty\}$.

Our main result is that $\mathcal{U}(\lambda,\theta,G)=\varnothing$, that confirms that we cannot achieve the general result of Costakis [6] by giving a proof similar to the proof of the specific case of constant functions.

The paper is organized as follows:

After the introduction in Section 1, we shall prove Proposition 2.1 that is a specific case of our main result, in the case that $G\in\mathcal{H}(\mathbb{C})$ is not a constant function, so that $G'(0)\neq 0$.

In order to prove Proposition 2.1, we shall use 3 lemmas.

In Section 2, we shall analyze the proofs of the 3 lemmas.

In Section 3, we shall give a helping corollary and the proof of our main result in Theorem 4.2.

There are several results concerning the existence or non-existence of common hypercyclic vectors for translation operators, see [5]-[11].

2. A Specific Case

We fix a positive number θ and a sequence $\lambda=(\lambda_n)_{n\in\mathbb{N}}$ from complex numbers.

We define $\mathcal{H}(\mathbb{C})$ for the set of entire functions. We fix $G\in\mathcal{H}(\mathbb{C})$.

Now we state and prove a specific case of our main result.

Proposition 2.1. *Let $\lambda=(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of non-zero complex numbers, so that $\lambda_n\to\infty$. We assume also that $\theta\in(0,1)$ and $G\in\mathcal{H}(\mathbb{C})$, so that $G'(0)\neq 0$. Then we have: $\mathcal{U}(\lambda,\theta,G)=\varnothing$.*

Proof. So, as to provide a proof by contradiction, we suppose that $\mathcal{U}(\lambda,\theta,G)\neq\varnothing$. Let $F\in\mathcal{U}(\lambda,\theta,G)$. Then by Lemma 3.1 there is a subsequence $m=(m_n)_{n\in\mathbb{N}}$ of λ, from different terms so that for every compact subset L of

$\mathbb{C}$

$$\sup_{(t,z)\in[0,\theta]\times L}\left|F\left(z+m_n e^{2\pi it}\right)-G(z)\right|\to 0 \text{ as } n\to+\infty. \tag{1}$$

Now, we use (1) with $L=\{0\}$ and $t=0$ and we get

$$F(m_n)\to G(0) \text{ as } n\to\infty. \tag{2}$$

If we use (1) with $L=\{0\}$ and $t=\theta$ we get

$$F\left(m_n e^{2\pi i\theta}\right)\to G(0) \text{ as } n\to+\infty. \tag{3}$$

Subtracting the above two convergence (2) and (3) we get that

$$F\left(m_n e^{2\pi i\theta}\right)-F(m_n)\to 0 \text{ as } n\to\infty. \tag{4}$$

We have by complex analysis

$$\int_{\left[m_n, m_n e^{2\pi i\theta}\right]}F'(z)\,dz = F\left(m_n e^{2\pi i\theta}\right)-F(m_n) \text{ for every } n=1,2,\cdots. \tag{5}$$

By (4), (5) we take:

$$\int_{\left[m_n, m_n e^{2\pi i\theta}\right]}F'(z)\,dz\to 0 \text{ as } n\to\infty. \tag{6}$$

Based on Lemma 3.2 and (1) we get that for every compact subset L of $\mathbb{C}$

$$\sup_{(t,z)\in[0,\theta]\times L}\left|F'\left(z+m_n e^{2\pi it}\right)-G'(z)\right|\to 0 \text{ as } n\to\infty. \tag{7}$$

The above convergence (7) for $L=\{0\}$ gives

$$\sup_{t\in[0,\theta)}\left|F'\left(m_n e^{2\pi i\theta}\right)-G'(0)\right|\to 0 \text{ as } n\to\infty. \tag{8}$$

Now setting $f=F'$, to the above convergence (6), (8) and Lemma 3.3 we take a contradiction and the proposition follows. ∎

In the following pages, we shall prove the lemmas we have used in the above Proposition 2.1.

3. Proofs of 3 Lemmas

Lemma 3.1. *Let* $\lambda=(\lambda_n)_{n\in\mathbb{N}}$ *be a sequence of non-zero complex number and* θ *be a positive number. We suppose that* $f\in\mathcal{U}(\lambda,\theta,G)$, *where* G *is an entire function so that* $G'(0)\neq 0$. *Then, the sequence* $m=(m_n)_{n\in\mathbb{N}}$, *which satisfies the condition of* $\mathcal{U}(\lambda,\theta,G)$, *that is for every compact subset* $L\subseteq G$, $\sup_{(t,z)\in[0,\theta]\times L}\left|f\left(z+m_n e^{2\pi it}\right)-G(z)\right|\to 0$ as $n\to+\infty$, is an infinite subset of $\mathbb{C}$ and can be chosen to be a subsequence of λ from different terms.

Proof. *We set*

$$a_n=\sup_{(t,z)\in[0,\theta]\times L}\left|f\left(z+m_n e^{2\pi it}\right)-G(z)\right|,\quad n=0,1,2,\cdots,$$

for some specific compact subset $L\subseteq\mathbb{C}$.

We suppose that $a_n=0$ for some $n\in\mathbb{N}$. Let $z_0\in L$. Because of $a_n=0$, we have $f\left(z_0+m_n e^{2\pi it}\right)=G(z_0)$ for every $t\in[0,\theta]$. Because of $m_n\neq 0$, based on our hypothesis, we take that the function f is a constant by the principle of analytical continuation, so we have $f(z)=G(z_0)$ for every $z\in\mathbb{C}$. As a result, $\left|f\left(z+m_n e^{2\pi it}\right)-G(z)\right|=\left|G(z)-G(z_0)\right|$ for every $z\in\mathbb{C}, n\in\mathbb{N}, t\in[0,\theta]$ and

$$a_n=\sup_{(t,z)\in[0,\theta]\times L}\left|f\left(z+m_n e^{2\pi it}\right)-G(z)\right|=\sup_{z\in L}\left|G(z)-G(z_0)\right|$$

for every $n\in\mathbb{N}$, $t\in[0,\theta]$ and compact set $L\subseteq\mathbb{C}$.

So, for specific compact set $L\subseteq\mathbb{C}$ we have $G(z)=G(z_0)$ for every $z\in L$ thus function G is a constant function $G(z)=G(z_0)$, for every $z\in\mathbb{C}$, which is false because $G'(0)\neq 0$, according to our hypothesis.

So, we have $a_n\neq 0$ for every $n\in\mathbb{N}$. We suppose that the set $\{m_n, n\in\mathbb{N}\}$ is finite. Then, we have that the set $\{a_n, n\in\mathbb{N}\}$ is finite and because $a_n\to 0$ we get that there is some $\nu_0\in\mathbb{N}$, so that $a_n=a_{\nu_0}=0$ for every $n\in\mathbb{N}$, $n\geq\nu_0$, that is false. Thus, the set $\{m_n, n\in\mathbb{N}\}$ is infinite and this implies that there is a subset $\{m'_n, n\in\mathbb{N}\}\subseteq\{m_n, n\in\mathbb{N}\}$, so that m'_n, $n\in\mathbb{N}$ to be a sequence of λ, from different terms. ∎

Lemma 3.2. *Let* $m=(m_n)_{n\in\mathbb{N}}$ *be a sequence of complex numbers,* θ *be a positive number and* f,g *be two entire functions. We suppose that for every compact subset L of* $\mathbb{C}$ *we have:*

$$\sup_{(t,z)\in[0,\theta]\times L}\left|f\left(z+m_n e^{2\pi it}\right)-g(z)\right|\to 0 \text{ as } n\to+\infty.$$

Then, for every compact subset L of $\mathbb{C}$:

$$\sup_{(t,z)\in[0,\theta]\times L}\left|f'\left(z+m_n e^{2\pi it}\right)-g'(z)\right|\to 0 \text{ as } n\to\infty.$$

Proof. We fix some compact subset L of $\mathbb{C}$. Let $n_0\in\mathbb{N}$, so that L is a subset of $\bar{D}(0,n_0)$, where $D(0,p)=\{z\in\mathbb{C}\mid |z|<p\}$ for every $p>0$.

It is easy to see that

$$\bigcup_{z\in\bar{D}(0,n_0)}\bar{D}\left(z,\frac{n_0}{3}\right)\subseteq\bar{D}(0,2n_0). \tag{1}$$

We shall consider the sequence of functions F_n, $n=1,2,\cdots$, $F_n:[0,\theta]\times\mathbb{C}\to\mathbb{C}$, so that $F_n((t,z))=f\left(z+m_n e^{2\pi it}\right)-g(z)$, for every $(t,z)\in[0,\theta]\times\mathbb{C}$, $n\in\mathbb{N}$ and their partial functions

$$F_n^t(z)=F_n((t,z)) \text{ for every } (t,z)\in[0,\theta]\times\mathbb{C}.$$

Based on the hypothesis, we conclude that for every compact subset $K\subseteq\mathbb{C}$ we have:

$$\|F_n\|_{[0,\theta]\times K}\to 0 \text{ as } n\to+\infty$$

where

$$\|F_n\|_{[0,\theta]\times K}=\sup_{w\in[0,\theta]\times K}|F_n(w)| \text{ for } n=1,2,\cdots.$$

Let some $\varepsilon_0>0$. We set $\varepsilon_1=\frac{n_0\varepsilon_0}{3}$.

Based on our hypothesis, there is a natural number $\nu_0\in\mathbb{N}$, so that for every

$n \in \mathbb{N}$, $n \geq \nu_0$, $\|F_n\|_{[0,\theta]\times \bar{D}_{2n_0}} < \varepsilon_1$, where we applied our hypothesis for

$$K = \bar{D}_{2n_0}. \tag{2}$$

We fix some $t_0 \in [0,\theta]$. Then, function $F_n^{t_0}$ is entire for every $n \in \mathbb{N}$.

Let some $z \in \bar{D}_{n_0}$.

Based on Canchy's estimates we have:

$$\left|\left(F_n^{t_0}\right)'(z)\right| \leq \frac{3}{n_0} \sup_{w \in \bar{D}\left(z,\frac{n_0}{3}\right)} \left|F_n^{t_0}(w)\right| \leq \frac{3}{n_0}\|F_n\|_{[0,\theta]\times \bar{D}_{2n_0}}, \tag{3}$$

where for the second inequality we used relation (1).

Based on inequality (3), we have:

$$\sup_{(t,z)\in[0,\theta]\times L}\left|f'\left(z+m_n e^{2\pi it}\right)-g'(z)\right| \leq \frac{3}{n_0}\|F_n\|_{[0,\theta]\times \bar{D}_{2n_0}}, \text{ for every } n \in \mathbb{N}. \tag{4}$$

Based on (2) and (4) we have that for every $n \in \mathbb{N}$, $n \geq \nu_0$:

$$\sup_{(t,z)\in[0,\theta]\times L}\left|f'\left(z+m_n e^{2\pi it}\right)-g'(z)\right| < \varepsilon_0. \tag{5}$$

This gives that:

$$\sup_{(t,z)\in[0,\theta]\times L}\left|f'\left(z+m_n e^{2\pi it}\right)-g'(z)\right| \to 0 \text{ as } n \to +\infty,$$

that implies the desired result for every compact subset L of $\mathbb{C}$. ■

Lemma 3.3. *Let* $(m_n)_{n\in\mathbb{N}}$ *be a sequence of complex numbers, so that* $m_n \to \infty$, $\theta \in (0,1)$ *and* $a \in \mathbb{C}$, $a \neq 0$.

Then, there is no entire function *f*, so that:

$$\sup_{t\in[0,\theta]}\left|f\left(m_n e^{2\pi it}\right)-a\right| \to 0 \text{ and}$$

$$\int_{\left[m_n, m_n e^{2\pi i\theta}\right]} f(z)\mathrm{d}z \to 0 \text{ as } n \to +\infty.$$

Proof. To take a contradiction we suppose that there exists an entire function *f* that satisfies the above two convergence.

We have the curves $\gamma_n : [0,\theta] \to \mathbb{C}$, where $\gamma_n(t) = m_n e^{2\pi it}$ for every $t \in [0,\theta]$, $n \in \mathbb{N}$. We also have $\gamma_n^* := \gamma_n([0,\theta])$ for $n = 1,2,\cdots$.

Because $m_n \to \infty$, we use only the terms m_n, $n \in \mathbb{N}$, such that $m_n \neq 0$, for some *n* big enough.

Based on Cauchy's Theorem we have:

$$\int_{\left[m_n, m_n e^{2\pi i\theta}\right]} f(z)\mathrm{d}z = \int_{\gamma_n^*} f(z)\mathrm{d}(z) \text{ for every } n = 1,2,\cdots. \tag{1}$$

Let $A : \mathbb{C} \to \mathbb{C}$ be the constant function, so that $A(z) = a$ for every $z \in \mathbb{C}$.

We also have:

$$\int_{\left[m_n, m_n e^{2\pi i\theta}\right]} A(z)\mathrm{d}z = a\left(m_n e^{2\pi i\theta} - m_n\right) \text{ for every } n = 1,2,\cdots. \tag{2}$$

We fix $\varepsilon_0 \in \left(0, \frac{|a|\cdot\left|e^{2\pi i\theta}-1\right|}{2\pi}\right)$.

We can write down the first of the two convergences of hypothesis as follows:

$$\sup_{z\in\gamma_n^*}\left|f(z)-a\right|\to 0 \text{ as } n\to+\infty. \tag{3}$$

By our hypothesis (1) and (3) we take that there is some $\nu_0\in\mathbb{N}$, so that for every $n\in\mathbb{N}$, $n\ge\nu_0$ has as follows:

$$\sup_{z\in\gamma_n^*}\left|f(z)-a\right|<\varepsilon_0 \text{ and} \tag{4}$$

$$\left|\int_{\gamma_n^*}f(z)\mathrm{d}z\right|<\varepsilon_0,\ m_n\ne 0. \tag{5}$$

Based on (2), Cauchy's Theorem, (4) and the simple properties of the complex integrals, we have:

$$\left|\int_{\gamma_n^*}f(z)\mathrm{d}z-a\left(m_n\mathrm{e}^{2\pi i\theta}-m_n\right)\right|<2\pi\left|m_n\right|\varepsilon_0. \tag{6}$$

Based on (4), (5), (6), triangle inequality and the specific of ε_0 we assume that for every $n\in\mathbb{N}$, $n\ge\nu_0$, the following applies:

$$\left|m_n\right|<\frac{\varepsilon_0}{|a|\left|\mathrm{e}^{2\pi i\theta}-1\right|-2\pi\varepsilon_0} \tag{7}$$

(where $|a|\left|\mathrm{e}^{2\pi i\theta}-1\right|-2\pi\varepsilon_0>0$ from the certain choice of ε_0).

Inequality (7) and the fact that $m_n\to\infty$ gives a contradiction and this completes the proof of this lemma. ■

4. The Main Result

In order to prove the main result, we also need the following corollary of Lemma 3.2.

Corollary 4.1. *Let $(m_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers, θ be a positive number and f, g be two entire functions.*

We suppose that for every compact subset L of $\mathbb{C}$ the following shall apply:

$$\sup_{(t,z)\in[0,\theta]\times L}\left|f\left(z+m_n\mathrm{e}^{2\pi it}\right)-g(z)\right|\to 0 \text{ as } n\to+\infty.$$

Then, for every compact subset L of $\mathbb{C}$ and $\nu\in\mathbb{N}$

$$\sup_{(t,z)\in[0,\theta]\times L}\left|f^{(\nu)}\left(z+m_n\mathrm{e}^{2\pi it}\right)-g^{(\nu)}(z)\right|\to 0 \text{ as } n\to+\infty.$$

Proof. It is simple implication of Lemma 3.2 by induction.

Now, we are ready to prove the main result of this article. ■

Theorem 4.2. *Let $\lambda=(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of non-zero complex numbers, so that $\lambda_n\to\infty$, $\theta\in(0,1)$ and $G\in\mathcal{H}(\mathbb{C})$, where G is not a constant function.*

Then, we have: $\mathcal{U}(\lambda,\theta,G)=\varnothing$.

Proof. *We shall prove the Theorem by distinguishing two cases.*

- **Case 1**

  $G'(0)\ne 0$.

  The result is supported by Proposition 1.

- **Case 2**

$G'(0)=0$.

We shall distinguish two cases here:

1) $G^{(\nu)}(0)=0$ for every $\nu\in\mathbb{N}$.

Provided that $G\in\mathcal{H}(\mathbb{C})$ we have $G(z)=\sum_{\nu=0}^{+\infty}\frac{G^{(\nu)}(0)}{\nu!}z^{\nu}$ for every $z\in\mathbb{C}$, so we have $G(z)=0$ for every $z\in\mathbb{C}$, which is false because G is not a constant function in our hypothesis.

2) There is a $\nu\in\mathbb{N}$, $\nu\geq 2$ so that $G^{(\nu)}(0)\neq 0$.

Let $\nu_0=\min\left\{\nu\in\mathbb{N}\mid G^{(\nu)}(0)\neq 0\right\}$, that is ν_0 is the smallest natural number, so that $G^{(\nu_0)}(0)\neq 0$. Of course, $\nu_0\geq 2$.

We suppose that $\mathcal{U}(\lambda,\theta,G)\neq\varnothing$. Let $f\in\mathcal{U}(\lambda,\theta,G)$. Then, there is a sequence $(m_n)_{n\in\mathbb{N}}$, so that $m_n\in\{\lambda_n,n\in\mathbb{N}\}$ for every $n\in\mathbb{N}$, where for every compact subset $L\subseteq\mathbb{C}$

$$\sup_{(t,z)\in[0,\theta]\times L}\left|f\left(z+m_n e^{2\pi it}\right)-G(z)\right|\to 0 \text{ as } n\to+\infty.$$

Based on the above Corollary 4.1, we take that for every compact subset $L\subseteq\mathbb{C}$

$$\sup_{(t,z)\in[0,\theta]\times L}\left|f^{(\nu_0-1)}\left(z+m_n e^{2\pi it}\right)-G^{(\nu_0-1)}(z)\right|\to 0 \text{ as } n\to+\infty. \tag{1}$$

Because $G^{(\nu_0)}(0)\neq 0$ we take that the function $G^{(\nu_0-1)}\in\mathcal{H}(\mathbb{C})$ is not a constant function. Of course, the function $f^{(\nu_0-1)}\in\mathcal{H}(\mathbb{C})$.

Based on (1) and Proposition 2.1 we have a contradiction, because, according to (1), we have $f^{(\nu_0-1)}\in\mathcal{U}\left(\lambda,\theta,G^{(\nu_0-1)}\right)$, that is $\mathcal{U}\left(\lambda,\theta,G^{(\nu_0-1)}\right)\neq\varnothing$ that is false by Proposition 2.1.

The proof of our main result is complete now. ∎

Let us compare now the main result of this article, that is Theorem 4.2, with the result of Costakis and Sambarino [5], in order to see what is new in thew present paper. As we said in the introduction in [5] the authors proved that the intersection

$$A=\bigcap_{a\in\mathbb{C}\setminus\{0\}}\left\{f\in\mathcal{H}(\mathbb{C})\mid\overline{\left\{T_a^n(f),n\in\mathbb{N}\right\}}=\mathcal{H}(\mathbb{C})\right\}$$

contains a G_δ and dense subset of $\mathcal{H}(\mathbb{C})$ and so it is non-empty. Let $f\in A$. Then by the definition of A if we choose a non-zero complex number a and an entire function g then there exists a subsequence $(m_n)_{n\in\mathbb{N}}$ of natural numbers so that

$$T_a^{m_n}(f)\to g \text{ uniformly on compacta as } n\to+\infty.$$

Therefore, for every pair $(a,g)\in(\mathbb{C}\setminus\{0\})\times\mathcal{H}(\mathbb{C})$ as above the sequence $(m_n)_{n\in\mathbb{N}}$ depends on this pair.

However, the convergence in set $\mathcal{U}_{\mathbb{C}}(\lambda)$ in the introduction [6] does not depend on the specific complex number $re^{2\pi it}$ where $r\in[x,y]$ and $t\in[0,\theta]$ and it is achieved simultaneously for all these numbers.

In this direction an open question for us, up to now, is the following

Question:

Can we achieve the convergence in the set $\mathcal{H}_{\mathbb{C}}(\lambda)$ simultaneously for all numbers $re^{2\pi it}$ where $r \in [x, y]$, $t \in [0, \theta]$, for specific $x, y \in \mathbb{R}$, $x < y$ and $\theta \in (0,1)$, but not uniformly?

More formal, the question is the following: "We choose a non constant entire function g and a positive number $\theta \in (0,1)$. Does there exist an entire function f and a sequence of natural $(m_n)_{n\in\mathbb{N}}$, so that for every $t \in [0, \theta]$ and compact set $L \subseteq \mathbb{C}$ $\sup_{z\in L} \left| f\left(z + m_n e^{2\pi it}\right) - g \right| \to 0$ as $n \to +\infty$?"

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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