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Some Classes of Bounded Sets in Quasi-Metric Spaces

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Abstract

This note deals with some classes of bounded subsets in a quasi-metric space. We study and compare the bounded sets, totally-bounded sets and the Bourbaki-bounded sets on quasi metric spaces. For example, we show that in a quasi-metric space, a set may be bounded but not totally bounded. In addition, we investigate their bornologies as well as their relationships with each other. For example, given a compatible quasi-metric, we intend to give some necessary and sufficient conditions for which a quasi metric bornology coincides with the bornology of totally bounded sets, the bornology of bourbaki bounded sets and bornology of bourbaki bounded subsets.

Keywords

Quasi-Metric-Boundedness, Totally Boundedness, Bourbaki Boundedness, Bornology

1. Introduction

The theory of bounded sets on metric spaces has been studied by many authors with different motivations. For instance, Kubrusly and Willard proved that a metric space (X,d) is totally bounded if and only if every sequence in X has a Cauchy subsequence. In 2012, Olela Otafudu investigated total boundedness of the u-injective hull of a totally bounded T_0-ultra-quasi-metric space. He first defined a set to be bounded if it is contained in a double ball and total bounded if it is contained in the union of finite number of $\tau(q^s)$-open balls. He then proved that total boundedness is preserved by the ultra-quasimetrically injective hull of a T_0-ultra-quasi-metric space (see ([1], Proposition 5.4.1)).

According to Cobzas ([2], p. 63), a quasi-pseudometric space (X,q) is said to be *totally bounded* if for each $\varepsilon > 0$ there exists a finite subset

$M_\varepsilon = \{x_1, x_2, x_3, \cdots, x_k\}$ of X such that $X \subseteq \bigcup_{j=1}^{k} B_{q^s}(x_j, \varepsilon)$. As it is known, in metric spaces precompactness and total boundedness are equivalent notions, a result that is not true in quasi-metric spaces (see ([2], Proposition 1.2.21)). In quasi metric spaces, Mukonda and Otafudu have defined a set to be *Bourbaki bounded* if for each $\varepsilon > 0$ and a nutural number n, there exists a finite subset $M_\varepsilon = \{x_1, x_2, x_3, \cdots, x_k\}$ of X such that $X \subseteq \bigcup_{j=1}^{k} B_q^n(x_j, \varepsilon)$.

Morever, our recent work [3] has extended the concept of bornology from metric settings to the framework of quasi-metrics. Naturally, this has led to the speculation of what is the relationship between the bornology of bounded sets and other types of bornologies on quasi-metric spaces. Toachieve this, a careful study of bornologyof bounded sets, bornology of totally bounded sets and bornolgies of bourbaki bounded sets in quasi-pseudometric spaces is required.

In this present work, we intend to generalize some classical bornological results of Garrido and Meroño [4] on classes of bounded sets from metric spaces to the category of quasi-metric spaces. For instance, given a compatible quasi-metric, we intend to give some necessary and sufficient conditions for which a bornology of totally bounded sets and bornology of bourbaki bounded sets coincide with our quasi-metric bornology studied in [5].

2. Preliminaries

This section recalls and introduces the terminology and notation for quasi-metric spaces we will use in the sequel. Further details about theory of asymmetric topology can be found in [2] [6] [7].

Definition 2.1. Let X be a set and let $q : X \times X \to [0, \infty)$ be a function mapping into the set $[0, \infty)$ of the nonnegative reals. Then, q is called a quasi-pseudometric on X if

1) $q(x, x) = 0$ whenever $x \in X$.
2) $q(x, z) \le q(x, y) + q(y, z)$ whenever $x, y, z \in X$.

We say q is a T_0-quasi-metric provided that q also satisfies the following condition:

$$q(x, y) = 0 = q(y, x) \text{ implies } x = y.$$

If q is a quasi-pseudometric on a set X, then $q^{-1} : X \times X \to [0, \infty)$ defined by $q^{-1}(x, y) = q(y, x)$ for every $x, y \in X$, often called the conjugate quasi-pseudometric, is also quasi-pseudometric on X. The quasi-pseudometric on a set X such that $q = q^{-1}$ is a pseudometric. Note that if (X, q) is a quasi-metric space, then $q^s = \max\{q, q^{-1}\} = q \vee q^{-1}$ is also a metric.

Remark 2.2. [2] Let (X, q) be a quasi-pseudometric space. The open ball of radius $\varepsilon > 0$ centred at $x \in X$ is the set $D_q(x, \varepsilon) = \{y \in X : q(x, y) < \varepsilon\}$. The collection of open balls yields a base for the topology $\tau(q)$ and it is called the topology induced by q on X. Similarly, the closed ball of radius $\varepsilon \ge 0$ centred at $x \in X$ is the set $D_q[x, \varepsilon] = \{y \in X : q(x, y) \le \varepsilon\}$. If (X, q) is a quasi-pseudometric space, then the pair $\{D_q[x, r]; D_{q^t}[x, s]\}$ where $x \in X$ and $r, s \in [0, \infty)$ is called a double ball. In general, $\left\{(D_q(x_i, r_i))_{i \in I}; (D_{q^t}(x_i, s_i))_{i \in I}\right\}$, with $x_i \in X$

and $r_i, s_i \in [0, \infty)$, is called the family of double balls.

Note that the set $D_q(x, \varepsilon) = \{y \in X : q(x, y) < \varepsilon\}$ is a $\tau(q^t)$-closed set, but not $\tau(q)$-closed in general. The following inclusions holds:

$$D_{q^s}(x, \varepsilon) \subset D_q(x, \varepsilon) \text{ and } D_{q^s}(x, \varepsilon) \subset D_{q^t}(x, \varepsilon).$$

Definition 2.3. ([3], Definition 4.1) Let (X, q) be a quasi-pseudometric. An arbitrary subset A is called q-bounded if only if there exists $x \in X$, $r > 0$ and $s > 0$ such that $A \subseteq D_q(x, r) \cap D_{q^{-1}}(x, s)$.

Definition 2.4. Let (X, q) be a quasi-pseudometric space and $F \subseteq X$. We say that F is totally bounded, if for any $\delta > 0$ there exists a finite subset $\{f_1, f_2, \cdots, f_k\}$ of X such that

$$F \subseteq \bigcup_{i=1}^{k} D_q(f_i, \delta).$$

Definition 2.5. Let (X, q) be a quasi-pseudometric space and $F \subseteq X$. We say that F is q-Bourbaki-bounded, if for any $\delta > 0$ there exists a finite subset $\{f_1, f_2, \cdots, f_k\}$ of X and for some positive integer n such that

$$F \subseteq \bigcup_{i=1}^{k} D_q^n(f_i, \delta).$$

Definition 2.6. A bornology on a set X is a collection $\mathscr{B}$ of subsets of X which satisfies the following conditions:

1) $\mathscr{B}$ forms a cover of X, *i.e.* $X = \bigcup \mathscr{B}$;

2) for any $B \in \mathscr{B}$, and $A \subseteq B$, then $A \in \mathscr{B}$;

3) $\mathscr{B}$ is stable under finite unions, *i.e.* if $X_1, X_2, \cdots, X_n \in \mathscr{B}$, then

$$\bigcup_{i=1}^{n} X_i \in \mathscr{B}.$$

If we take a nonempty set X and a bornology $\mathscr{B}$ on X, then the pair $(X, \mathscr{B})$ is called a bornological universe. For every nonempty set X, the family $\mathscr{B} = \{B \subset X : B \text{ is finite}\}$ is the smallest bornology on X.

Recall from [3] that the bornology of quasi-pseudometric bounded sets is denoted by $\mathscr{B}_q(X)$. However, in [8], the family of totally bounded subsets and boubark bounded sets their bornologies are denoted by $\mathscr{TB}_q(X)$ and $\mathscr{BB}_q(X)$ respectively. We will compare these bornologies in the next sections.

Let (X, q) be a T_0-quasi-metric space. Then (X, q) is called *bicomplete* provided that the metric space (X, q^s) is complete. A mapping f between two quasi-metric spaces (X, q) and (Y, ρ) is said to be *quasi-isometry* if $q(f(x), f(y)) = \rho(x, y)$ for all x, y in X.

A *bicompletion* of a quasi-metric space (X, q) is a bicomplete quasi-metric space $(\tilde{X}, \tilde{q})$ in which (X, q) can be quasi-isometrically embedded as a $\tau(\tilde{q}^s)$-dense subspace.

We recall the concepts of asymmetric norms and semi-Lipschitz functions in quasi-metric spaces.

Definition 2.7. [2] An asymmetric norm on a real vector space X is a function

$\| \cdot | : X \to [0,\infty)$ satisfying the conditions:

1) $\| x | = \| -x | = 0$ then $x = 0$;
2) $\| ax | = a \| x |$;
3) $\| x + y | \le \| x | + \| y |$,

for all $x, y \in X$ and $a \ge 0$. Then the pair $(X, \| \cdot |)$ is called *an asymmetric normed space*.

The *conjugate asymmetric norm* $| \cdot \|$ of $\| \cdot |$ and the *symmetrized norm* $\| \cdot \|$ of $\| \cdot |$ are defined respectively by

$$| x \| := \| -x | \text{ and } \|x\| := \max\{| x \|, \| x |\} \text{ for any } x \in X.$$

An asymmetric norm $\| \cdot |$ on X induces a quasi-metric $q_{\|\cdot|}$ on X defined by

$$q_{\|\cdot|}(x, y) = \|x - y\| \text{ for any } x, y \in X.$$

If $(X, \|\cdot\|)$ is a normed lattice space, then the function $\| x | := \|x^+\|$ with $x^+ = \max\{x, 0\}$ is an asymmetric norm on *X*.

Definition 2.8. Let (X, q) be a quasi-metric space and $(Y, \| \cdot |)$ be an asymmetric normed space. Then a function $\varphi : (X, q) \to (Y, \| \cdot |)$ is called *k-semi-Lipschitz* (or *semi-Lipschitz*) if there exists $k \ge 0$ such that

$$\|\varphi(x) - \varphi(y)| \le kq(x, y) \text{ for all } x, y \in X. \tag{1}$$

A number *k* satisfying inquality (1) is called *semi-Lipschitz constant* for φ.

3. Some Results of Boundedness in Quasi-Metric Spaces

This section is as a result of the distinction that we gave in [3] about the bornologies $\mathscr{B}_q(X)$ and $\mathscr{B}_{q^s}(X)$. We will investigate further the connection between the bornologies $\mathscr{B}_{q^s}(X)$, $\mathscr{B}_q(X)$, $\mathscr{TB}_q(X)$ and $\mathscr{BB}_q(X)$.

Lemma 3.1. *If* (X, q) *is a quasi-metric space. Then the following statement is true*:

$$\mathscr{B}_{q^s}(X) \subseteq \mathscr{B}_q(X) \tag{2}$$

and the quasi-metric bornologies $\mathscr{B}_q(X)$ *and* $\mathscr{B}_{q^t}(X)$ *are equivalent.*

Proof. Let $A \in \mathscr{B}_{q^s}(X)$, then *A* is q^s-bounded. By Remark 2.2, *A* is *q*-bounded too. Thus $A \in \mathscr{B}_q(X)$. The equivalence of $\mathscr{B}_q(X)$ and $\mathscr{B}_{q^t}(X)$ comes from the fact that any subset *A* of *X* is *q*-bounded if and only if it is q^t-bounded. □

The converse of Lemma 3.1 above does not holds. *i.e.*, a set on a quasi-metric can be *q*-bounded but not q^s-bounded (check ([3], Remark 4.2)).

Definition 3.2. ([6], p.85) Let (X, q) be a T_0-quasi-metric space. Then (X, q) is called *joincompact* provided that the metric space (X, q^s) is compact.

Theorem 3.3. (Compare ([9], Theorem 3.78).) Let (X, q) be a T_0-quasi-metric space. A set $B \subseteq X$ is joincompact if and only if *B* is both bicomplete and totally bounded.

Proof. We leave this proof to the reader. □

We rephrase the above theorem in the following Corrolary as proved by Fletcher and Lindgreen in quasi-uniform spaces (see ([7], p. 65)).

Corollary 3.4. ([7], Proposition 3.36) *Let* (X,q) *be a* T_0*-quasi-metric space. Then* (X,q) *is totally bounded if and only* $(\tilde{X},\tilde{q}^s)$ *is compact.*

Definition 3.5. ([2], Definition 1.44) Let (X,q) be a T_0-quasi-metric space. Then (X,q) is called supseparable provided that the metric space (X,q^s) is separable.

Proposition 3.6. (Compare ([9], Proposition 3.72)) *A totally bounded quasi-pseudometric space* (X,q) *is supseparable.*

Proof. Suppose (X,q) is totally bounded, for any positive interge *n*, we can find a finite set $A_n \subseteq X$ such that for all $x \in X$, $q^s(x,A_n) < \frac{1}{n}$. Now let $B = \bigcup_{n\in\mathbb{N}} A_n$. The set *B* is either finite or infinitely countable, thus countable. To show the $\tau(q^s)$-density of *B*, let us pick $x \in X$, then we have $q^s(x,B) \le q^s(x,A_n) < \frac{1}{n}$ implying that $q^s(x,B) = 0$ and $x \in \mathrm{cl}_{\tau(q^s)}(B)$. This proves that *x* is a q^s-limit point of *B* and hence *B* is a $\tau(q^s)$-dense subset of *X*. Consequently, (X,q^s) separable and by Definition 3.5, (X,q) is supseparable. □

The next example shows that for finite dimension spaces, total boundedness coincide with boundedness.

Example 3.7. If we equip a real unit interval $X=[0,1]$ with the T_0-quasi-metric $q(x,y)=\max\{x-y,0\}$, then the pair (X,q) is both *q*-bounded and totally bounded space.

Proof. It can be seen that *X* is *q*-bounded. Now If we pick $\{0,1\}$ to be a finite subset of $X=[0,1]$ and $\varepsilon = 1/2$, then

$$X \subset B_{q^s}(0,1/2) \cup B_{q^s}(1,1/2).$$ □

The next Lemma proves that for infinite dimension spaces, total boundedness and quasi-metric boundedness are two different notions.

Lemma 3.8. *Let* (X,q) *be a quasi-metric space, then* $\mathscr{TB}_q(X) \subseteq \mathscr{B}_q(X)$.

Proof. Let $B \in \mathscr{TB}_q(X)$. For $\varepsilon > 0$, there exists a finite subset $F_\varepsilon = \{x_1,x_2,x_3,\cdots,x_k\}$ of *B* such that $B \subseteq \bigcup_{j=1}^{k} B_{q^s}(x_j,\varepsilon)$. The set *B* is a finite family of q^s-bounded subsets thus its is q^s-bounded. Hence $B \in \mathscr{B}_q(X)$ by Lemma 3.1. □

The following example illustrates the converse of Lemma 3.8 above.

Example 3.9. Let us equip the set of natural numbers $\mathbb{N}$ with the T_0-quasi-metric

$$q(x,y) = \begin{cases} x-y & \text{if } x \ge y \\ 1 & \text{if } x < y \end{cases}$$

The T_0-quasi-metric space $(\mathbb{N},q)$ is *q*-bounded but not *q*-totally bounded.

Proof. For all $x,y \in \mathbb{N}$ we can find $k \ge 0$ such that $q(x,y) \le k$. But any finite set $\{x_1,x_2,x_3,\cdots,x_n\} \subset \mathbb{N}$ with the discrete metric q^s, the set $\mathbb{N}$ can not be covered by $D_{q^s}(x_i,\varepsilon)$ for $1 \le i \le n$. Hence, $(\mathbb{N},q)$ is not *q*-totally bounded. □

It is important to note that $\mathscr{TB}_{q^s}(X)$ is a metric bornology in the sense of

Beer *et al.* [10].

Definition 3.10. Let (X,q) be a quasi-pseudometric space and $\delta>0$. For any $\varnothing \neq F \subset X$, we define the δ*-enlargement* $D_q(F,\delta)$ of *F* by

$$D_q(F,\delta) := \{x \in X : \operatorname{dist}(F,x) < \delta\} = \bigcup_{f\in F} D_q(f,x)$$

and

$$D_{q^t}(F,\delta) := \{x \in X : \operatorname{dist}^t(F,x) < \delta\} = \bigcup_{f\in F} D_{q^t}(f,x).$$

Furthermore,

$$D_{q^s}(F,\delta) = \max\{D_q(F,\delta), D_{q^t}(F,\delta)\} = \bigcup_{f\in F} D_{q^s}(f,x).$$

Remark 3.11. For a given quasi-pseudometric space (X,q). For any $\delta>0$ and $x,y \in X$. It is easy to see that if $(x_i)_{i=0}^n$ is a δ-chain in (X,q^s) of length *n* from *x* to *y*, then $(x_i)_{i=0}^n$ is also a δ-chain in (X,q) and in (X,q^t) of length *n* from *x* to *y*. We have

$$D_{q^s}^n(x,\delta) \subseteq D_q^n(x,\delta) \tag{3}$$

and

$$D_{q^s}^n(x,\delta) \subseteq D_{q^t}^n(x,\delta). \tag{4}$$

Lemma 3.12. *Let* (X,q) *be a quasi-pseudometric space and for any* $\varepsilon,\delta>0$. *We have* $D_q(D_q(F,\varepsilon),\delta) \subset D_q(F,\varepsilon+\delta)$.

Lemma 3.13. *Let* (X,q) *be a quasi-pseudometric space and* $\delta>0$. *For any* $x\in X$ *and* $n=0,1,2,\cdots$, *we have* $D_q^n(x,\delta) \subseteq D_q^{n+1}(x,\delta)$.

Corollary 3.14. *Let* (X,q) *be a quasi-pseudometric space and* $\delta>0$. *If there exists a* δ*-chain of length n from x to y in* (X,q^t), *then there exists a* δ*-chain of length n from y to x in* (X,q) *whenever* $x,y\in X$.

Lemma 3.15. *If* (X,q) *is a quasi-metric space. Then the following statement is true*:

$$\mathscr{B}\mathscr{B}_{q^s}(X) \subseteq \mathscr{B}\mathscr{B}_q(X) \tag{5}$$

and the quasi-metric bornologies $\mathscr{B}\mathscr{B}_q(X)$ *and* $\mathscr{B}\mathscr{B}_{q^t}(X)$ *are equivalent.*

Proof. Let $\delta>0$. Suppose that $F \in \mathscr{B}\mathscr{B}_{q^s}(X)$. Then there exists a finite set $\{f_1,f_2,\cdots,f_k\} \subset X$ such that

$$F \subseteq \bigcup_{i=1}^k D_{q^s}^n(f_i,\delta)$$

for some positive integer *n*. By inclusion (3) we have $F \subseteq \bigcup_{i=1}^k D_q^n(f_i,\delta)$ for some positive integer *n*. Hence $F \in \mathscr{B}\mathscr{B}_q(X)$. Note that Corollary 3.14 confirms the equivalence of $\mathscr{B}\mathscr{B}_q(X)$ and $\mathscr{B}\mathscr{B}_{q^t}(X)$. □

The converse of the above lemma does not always hold. Let us determine this from the following example.

Example 3.16. Consider the four point set $X=\{1,2,3,4\}$. If we equip *X* with T_0-quasi-metric *q* defined by the distance matrix

$$Q = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix}$$

that is, $q(i,j) = q_{i,j}$ whenever $i, j \in X$. The symmetrized metric q^s of q is induced by the matrix

$$Q^s = \begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 2 & 2 & 1 & 0 \end{pmatrix}$$

Let $\delta = 1,5 > 0$. If we consider the sequence $(f_i)_{i=0}^2 := (4,2,1)$. Then we have

$$q(f_0, f_1) = (4,2) = 1 = q(f_1, f_2) = q(2,1) < \delta.$$

Hence the sequence $(f_i)_{i=0}^2 := (4,2,1)$ is a δ-chain in (X,q) of length 2 from 4 to 1. But the same sequence $(f_i)_{i=0}^2 := (4,2,1)$ is not a δ-chain in (X,q^s) of length 2 from 4 to 1 because $q^s(f_0, f_1) = q^s(4,2) = 2 > \delta$.

We state the following lemma that we will use in our next proposition.

Lemma 3.17. *Let* (X,q) *be a quasi-pseudometric space. For some positive integer n,* $\delta > 0$ *and* $x \in X$, *we have*

$$\bigcup_{i=1}^{k} D_q^n(x_i, \delta) \subseteq \bigcup_{i=1}^{k} D_q(x_i, n\delta).$$

Proof. Let $y \in \bigcup_{i=1}^{k} D_q^n(x_i, \delta)$, then for some *j* with $1 \le j \le k$, $y \in D_q^n(x_j, \delta)$. Moreover, for some *j* with $1 \le j \le k$, there exists $\{f_0, f_1, \cdots, f_n\}$ a δ-chain of length *n* from x_j to *y* such that $f_0 = x_j$, $f_n = y$ and $q(f_{i-1}, f_i) < \delta$ for all *i* with $1 \le i \le n$. Furthermore, we have

$$\begin{aligned} q(x_j, y) = q(f_0, f_n) &\le q(f_0, f_1) + q(f_1, f_2) + \cdots + q(f_{n-1}, f_n) \\ &< \delta + \delta + \cdots + \delta < n\delta. \end{aligned}$$

Thus, for some *j* with $1 \le j \le k$, $y \in D_q(x_j, n\delta)$. Hence, $y \in \bigcup_{i=i}^{k} D_q(x_i, n\delta)$. □

Proposition 3.18. *Given a quasi-pseudometric space* (X,q). *If F is a subset of X and* $\delta > 0$, *then we have the following conditions*:

1) $\mathcal{TB}_q(X) \subseteq \mathcal{BB}_q(X)$.

2) $\mathcal{BB}_q(X) \subseteq \mathcal{B}_q(X)$.

Proof.

1) Let $\delta > 0$. Suppose $F \in \mathcal{TB}_q(X)$ then there exists a set $\{f_1, f_2, \cdots, f_k\} \subseteq X$ such that

$$F \subseteq \bigcup_{i=1}^{k} D_{q^s}(f_i, \delta) = \bigcup_{i=1}^{k} D_{q^s}^1(f_i, \delta) \subseteq \bigcup_{i=1}^{k} D_q^1(f_i, \delta)$$

for some positive integer $n = 1$. Therefore, $F \in \mathcal{BB}_q(X)$.

2) Since $F \in \mathcal{BB}_q(X)$ there exists a set $\{x_1, x_2, \cdots, x_k\} \subseteq X$ and some positive integer *n* such that for $\delta > 0$ we have $F \subseteq \bigcup_{i=1}^{k} D_q^n(x_i, \delta)$. By Lemma 3.17, $F \subseteq \bigcup_{i=1}^{k} D_q^n(x_i, \delta) \subseteq \bigcup_{i=1}^{k} D_q(x_i, n\delta)$. Hence, $F \in \mathcal{B}_q(X)$. □

Let us provide the summary of the connections between these bornologies in the following remark.

Remark 3.19. If (X,q) is a quasi-pseudometric space, then we have the following inlusions:

$$\mathcal{TB}_q(X) \subseteq \mathcal{BB}_{q^s}(X) \subseteq \mathcal{BB}_q(X) \subseteq \mathcal{B}_q(X)$$

But if $(X,\|\cdot|)$ is an asymmetric normed space, then we have

$$\mathcal{BB}_{q_{\|\cdot|}}(X) = \mathcal{B}_{q_{\|\cdot|}}(X).$$

We have provided the proof in Proposition 4.1.

4. Main Results on Bornologies

One would still wonder, if is it indeed posible to find a quasi-metric metric q' equivalent to q such that $\mathcal{B}_{q'}(X) = \mathcal{BB}_q(X)$ or $\mathcal{BB}_{q'}(X) = \mathcal{TB}_q(X)$.

Proposition 4.1. *Suppose that* $(X,\|\cdot|)$ *is an asymmetric normed space. Then we have the following*:

$$\mathcal{BB}_{q_{\|\cdot|}}(X) = \mathcal{B}_{q_{\|\cdot|}}(X).$$

Proof. For $\mathcal{BB}_{q_{\|\cdot|}}(X) \subseteq \mathcal{B}_{q_{\|\cdot|}}(X)$ follows from Proposition 3.18 (b).

For $\mathcal{BB}_{q_{\|\cdot|}}(X) \supseteq \mathcal{B}_{q_{\|\cdot|}}(X)$, suppose that F is $q_{\|\cdot|}$-bounded then $F \subseteq D_{q_{\|\cdot|}}(x_0,\varepsilon)$ for some $x_0 \in X$ and $\varepsilon > 0$. For any $\delta > 0$, there exists $n \in \mathbb{N}$ such that $\frac{\varepsilon}{n} < \delta$.

Let $f \in F$. We define $z_i := x_0 + \frac{i}{n}(f - x_0)$ whenever i with $1 \le i \le n$ and $z_0 = x_0$. Then

$$\begin{aligned}
\| q_{\|\cdot|}(z_{i-1},z_i)| &= \| z_{i-1} - z_i | \\
&= \left\| \left[x_0 + \frac{i-1}{n}(f - x_0) \right] - \left[x_0 + \frac{i}{n}(f - x_0) \right] \right| \\
&= \left\| \frac{x_0}{n} - \frac{f}{n} \right| = \left\| \frac{1}{n}(x_0 - f) \right| < \frac{\varepsilon}{n} < \delta.
\end{aligned}$$

Thus, for any $f \in F$ we have obtained a δ-chain of length n on $(X,q_{\|\cdot|})$ from z_0 to f. Therefore, $f \in \bigcup_{k=0}^{n} D^1_{q_{\|\cdot|}}(z_k,\delta)$. □

Definition 4.2. [11] Given a Hilbert cube $H = [0,1]^{\mathbb{N}}$, the product topology is defined in a usual way by a quasi-pseudometric

$$\rho_q(x,y) = \sum_{n=1}^{\infty} \frac{u(x_n,y_n)}{2^n}$$

where $u(x_n,y_n) = \max\{x_n - y_n, 0\}$.

Theorem 4.3. ([11], Theorem 3.10) Every supseparable quasi-metric space is embeddable as subspace of the Hilbert cube $H = [0,1]^{\mathbb{N}}$.

Theorem 4.4 (Tychonoff's Theorem). The topological product of a family of compact spaces is compact.

Theorem 4.5. (Compare ([10], Theorem 3.1).) Let (X,q) be a quasi-metric space and let $x_0 \in X$. The following conditions are equivalent:

1) There exists an equivalent quasi-metric ρ such that $\mathscr{B}_q(X)=\mathscr{TB}_\rho(X)$.

2) The quasi-metric space (X,q) is supseparable.

3) There is an embedding Φ of X into some quasi-metrizable space Y such that the family $\left\{\mathrm{cl}_{\tau\left(q_Y^s\right)}\left[\Phi\left(C_q(x_0,n)\cap C_{q^{-1}}(x_0,s)\right)\right]:n,s\in\mathbb{N}\right\}$ is cofinal in $\mathscr{K}_0(Y)$.

4) There exists an equivalent quasi-metric ρ with $\mathscr{B}_q(X)=\mathscr{TB}_\rho(X)=\mathscr{B}_\rho(X)$.

Proof.

$1\Rightarrow 2$: If there exists an equivalent quasi-metric space ρ such that $\mathscr{B}_q(X)=\mathscr{TB}_\rho(X)$, then $X=\bigcup_{i=1}^n B_i$ where B_i are ρ-totally bounded subsets. This means that X is a countable union of ρ-totally bounded sets, thus its ρ-totally bounded and by Proposition 3.6, the quasi-metric space (X,q) is supseparable.

$2\Rightarrow 3$: First case: If q is bounded, then by Theorem 4.3, we can find an embedding $\Phi:(X,q)\to\left([0,1]^{\mathbb{N}},\rho_q\right)$. Let $Y=\mathrm{cl}_{\tau\left(\rho_q^s\right)}(\Phi(X))$ and choose $n,s\in\mathbb{N}$ so that $Y=\mathrm{cl}_{\tau\left(\rho_q^s\right)}\left[\Phi\left(C_q(x_0,n)\cap C_{q^{-1}}(x_0,s)\right)\right]$. Since $[0,1]^{\mathbb{N}}$ is joincompact with respect to product topology, its subset Y is joincompact and confinal in $\mathscr{K}_0(Y)$.

Second case: If q is unbounded, consider $\{x_i:i\in\mathbb{N}\}$ as a $\tau\left(q^s\right)$-dense subset in X. For each i in $\mathbb{N}$, Let us define $f_i:X\Rightarrow\mathbb{R}$ by $f_i(x)=q(x,x_i)$. Now if A is a nonempty $\tau\left(q^s\right)$-closed subset of X and $x\notin A$ then we can choose x_i with $q^s(x,x_i)<q^s(A,x_i)$ and $f_i(x)\notin\mathrm{cl}_{\tau(q)}\left(f_i(C)\right)$. From the choice of x_i, the set $\{f_i:i\in\mathbb{N}\}$ separates points from $\tau\left(q^s\right)$-closed sets and we can define an embedding $\Phi:X\to\mathbb{R}^{\mathbb{N}}$ by $\Phi(x)=\left\{f_i(x)\right\}_{i=1}^{\infty}$ equipped with the product topology.

Now let p be a quasi-metric compatible with the product topology on $\mathbb{R}^{\mathbb{N}}$, we now prove that $Y=\mathrm{cl}_{\tau\left(p^s\right)}\Phi(X)\subseteq\mathbb{R}^{\mathbb{N}}$ equipped with the relative topology is cofinal in $\mathscr{K}_0(Y)$. If $n\in\mathbb{N}$ is chosen arbitrary then for each $i\in\mathbb{N}$, $f_i\left(C_q(x_0,n)\cap C_{q^{-1}}(x_0,s)\right)$ is q-bounded, so by the Theorem 4.4, $Y=\mathrm{cl}_{\tau\left(p^s\right)}\left[\Phi\left(C_q(x_0,n)\cap C_{q^{-1}}(x_0,s)\right)\right]$ is joincompact as it is contained in a product $\mathbb{R}^{\mathbb{N}}$. Suppose $Y=\mathrm{cl}_{\tau\left(p^s\right)}\left[\Phi\left(C_q(x_0,n)\cap C_{q^{-1}}(x_0,s)\right)\right]$ is not confinal in $\mathscr{K}_0(Y)$. Let $B\in\mathscr{K}_0(Y)\setminus Y$ then for each $n\in\mathbb{N}$, take $y_n\in B$ and pick $x_n\in X$ with $q(x_n,x_0)>n$ and $p^s\left(y_n;\Phi(x_n)\right)<\dfrac{1}{n}$.

By the joincompactness of B and the quasi-metrizability of Y, we can find some q^s-subsequence $\left\{y_{n_k}\right\}_{k=1}^{\infty}$ of $\{y_n\}_{n=1}^{\infty}$ such that $p^s\left(y_{n_k},y_0\right)=0$. This implies that $q\left(\Phi\left(x_{n_k}\right),y_0\right)=0$. But this is not possible, since q is unbounded.

$3\Rightarrow 4$: If (X,ρ) is an quasi-metric equivalent to q then $\mathscr{B}_q(X)=\mathscr{B}_\rho(X)$ by ([3], Theorem 5.4). To prove that $\mathscr{B}_q(X)=\mathscr{TB}_q(X)$, let $B\in\mathscr{TB}_\rho(X)$ and $(Y,\tilde{\rho})$ be a bicompletion of ρ. Since $\tilde{\rho}$ is bicomplete by, the set $\mathrm{cl}_{\tau\left(\tilde{\rho}^s\right)}(B)$

is compact. Given the cofinality of $\mathscr{K}_0(Y)$, let us choose $n \in \mathbb{N}$ with $\mathrm{cl}_{\tau(\tilde{\rho}^s)}(B) \subseteq \mathrm{cl}_{\tau(\tilde{\rho}^s)}\left(C_q(x_0,n) \cap C_{q^{-1}}(x_0,s)\right)$. But this means that

$$B \subseteq \mathrm{cl}_{q^s}(B) \subseteq \mathrm{cl}_{q^s}\left(C_q(x_0,n) \cap C_{q^{-1}}(x_0,s)\right) = C_q(x_0,n) \cap C_{q^{-1}}(x_0,s).$$

Thus $B \in \mathscr{B}_q(X)$ and it follows that $\mathscr{TB}_\rho(X) \subseteq \mathscr{B}_q(X)$. For the reverse inclusion. If $B \in \mathscr{B}_q(X)$, we can choose $n \in \mathbb{N}$ with $B \subseteq C_q(x_0,n) \cap C_{q^{-1}}(x_0,s)$. The $B \subseteq \mathrm{cl}_{\tau(\tilde{\rho}^s)}\left(C_q(x_0,n) \cap C_{q^{-1}}(x_0,s)\right)$ is compact and $\tilde{\rho}$-totally bounded. Therefore, $B \in \mathscr{TB}_\rho(X)$. The equivalence $4 \Rightarrow 1$ follows from ([3], Theorem 5.4). □

Definition 4.6. (Compare ([10], Definition 3)). Let (X,q) be a T_0-quasi-metric space. Given the point $p \notin X$ and a quasi-metric bornology $\mathscr{B}_q(X)$ on X we can form the one-point extension of X associated with $\mathscr{B}_q(X)$ by a $X' = X \cup \{p\}$.

If $\tau(q)$ is the topology X, then the corresponding topology on X' is defined by

$$\tau(q) \cup \left\{\{p\} \cup X \setminus B : B = \mathrm{cl}_{\tau(q)}(B) \in \mathscr{B}_q(X)\right\}.$$

The quasi-metric bornology associated with X' is denoted by $\mathscr{B}_q(X')$.

Remark 4.7. If $\mathcal{B}_0$ is a $\tau(q)$-closed base of the bornology then $\{\{p\} \cup X \setminus B : B \in \mathcal{B}_0\}$ forms a $\tau(q)$-neighbourhood base at the point *p*.

Lemma 4.8. *Let (X,q) be a T_0-quasi-metric space. If the bornology $B_q(X)$ is quasi-metrizable then the associated bornolgy $B_q(X')$ on X' is quasi-metrizable.*

Theorem 4.9. (Compare ([10], Theorem 3.4)) Let (X,q) be a quasi-metric space The following conditions are equivalent:

1) $\mathscr{TB}_q(X)$ has a countable base;

2) There exists an equivalent quasi-metric q' such that $\mathscr{TB}_q(X) = \mathscr{B}_{q'}(X)$

3) The one-point extension of X associated with $\mathscr{TB}_q(X)$ is quasi-metrizable.

4) The one-point extension of X associated with $\mathscr{TB}_q(X)$ has a $\tau(q)$-neighborhood base at the ideal point.

Proof.

$1 \Rightarrow 2$: Since $\mathscr{TB}_q(X)$ has a countable base by Hu's theorem (see ([3], Theorem 4.18)) there exists an equivalent quasi-metric q' such that $\mathscr{TB}_q(X) = \mathscr{B}_{q'}(X)$.

$2 \Rightarrow 3$: By (2), $\mathscr{TB}_q(X) = \mathscr{B}_{q'}(X)$. From Lemma 4.8 $\mathscr{B}_q(X')$ on X' is quasi- metrizable thus $\mathscr{TB}_q(X')$ is quasi-metrizable.

$3 \Rightarrow 4$ Since the bornology $\mathscr{TB}_q(X')$ has a $\tau(q^s)$-closed base, thus by the Remark 4.7 $\mathscr{TB}_q(X')$ has a $\tau(q^s)$-neighborhood base at the ideal point.

$4 \Rightarrow 1$: If $\mathscr{TB}_q(X')$ have a $\tau(q^s)$-neighborhood base at each point, then $\mathscr{TB}_q(X)$ has countable base. □

Definition 4.10. Let (X,q) be a quasi-metric space and $(Y,\|\cdot|)$ be an asymmetric normed space. A function $\varphi:(X,q) \to (Y,\|\cdot|)$ is called *semi-Lipschitz in* *the small* if there exists $\delta > 0$ and $k \geq 0$ such that if $q(x,y) < \delta$ then $\|\varphi(x) - \varphi(y)| \leq kq(x,y)$.

The following lemma follows directly from the definitions of semi-Lipschitz in the small function and uniformly continuous.

Lemma 4.11. *Let* (X,q) *be a quasi-metric space and* $(Y,\|\cdots|)$ *be an asymmetric normed space. If a function* $\varphi:(X,q) \to (Y,\|\cdot|)$ *is semi-Lipschitz in the small, then* $\varphi:(X,q) \to (Y,\|\cdot|)$ *is uniformly continuous.*

Theorem 4.12. (Compare ([12], Theorem 3.4)) Let (X,q) be a quasi-metric space and $\varnothing \neq F \subseteq X$. Then the following conditions are equivalent:

1) $F \in \mathscr{BB}_q(X)$;

2) if $(Y,\|\cdot|)$ is an asymmetric normed space and $\varphi:(X,q) \to (Y,\|\cdot|)$ is uniformly continuous, then $\varphi(F) \in \mathscr{B}_{q_{\|\cdot|}}(Y)$;

3) if $(Y,\|\cdot|)$ is an asymmetric normed space and $\varphi:(X,q) \to (Y,\|\cdot|)$ is semi-Lipschitz in the small function, then $\varphi(F) \in \mathscr{B}_{q_{\|\cdot|}}(Y)$;

4) if $\varphi:(X,q) \to (\mathbb{R},u)$ is semi-Lipschitz in the small function, then $\varphi(F) \in \mathscr{B}_u(\mathbb{R})$.

Proof.

$(1) \Rightarrow (2)$ If $\varphi:(X,q) \to (Y,\|\cdot|)$ is uniformly continuous then there exists $\delta > 0$ such that whenever $x,y \in X$ with $q(x,y) < \delta$, we have

$$q_{\|\cdot|}(\varphi(x),\varphi(y)) = \|\varphi(x) - \varphi(y)| < 1. \tag{6}$$

By the q-Bourbaki-boundedness of F, there exists $A := \{a_1, a_2, \cdots, a_m\} \subseteq X$ such that

$$F \subseteq \bigcup_{i=1}^{m} D_q^n(a_i,\delta)$$

for some positive integer n. If we take f artbitrary in F, then there exists k with $1 \leq k \leq m$ such that $f \in D_q^n(a_k,\delta)$. Then for some k with $1 \leq k \leq m$, there exists a δ-chain $\{f_0, f_1, \cdots, f_n\}$ with $f_0 = a_k$, $f_n = f$ and

$$q(f_{i-1}, f_i) < \delta \text{ whenever } i \text{ with } 1 \leq i \leq m. \tag{7}$$

It follows from the uniform continuity of φ and inequality (6) that

$$q_{\|\cdot|}(\varphi(f_{i-1}),\varphi(f_i)) < 1 \text{ whenever } i \text{ with } 1 \leq i \leq m. \tag{8}$$

Hence, for some k with $1 \leq k \leq m$, we have

$$q_{\|\cdot|}(\varphi(a_k),\varphi(f)) = q_{\|\cdot|}(f_0, f_n) \leq q_{\|\cdot|}(f_0, f_1) + q_{\|\cdot|}(f_1, f_2) + \cdots + q_{\|\cdot|}(f_{n-1}, f_n) < n.$$

Thus, $\varphi(f) \in \bigcup_{i=1}^{m} D_{q_{\|\cdot|}}(\varphi(a_i),n)$ for any $f \in F$ and $\varphi(F) \subseteq D_q(\varphi(A),n)$. Therefore, $\varphi(F)$ is $q_{\|\cdot|}$-bounded.

$(2) \Rightarrow (3)$ Follows from Lemma 4.11.

$(3) \Rightarrow (4)$ Follows directly by replacing $(Y,\|\cdot|)$ with $(\mathbb{R},u)$ in (3).

$(4) \Rightarrow (1)$. Suppose that F is not q-Bourbaki-bounded. Then there exists a $\delta > 0$ such that if $\{f_1, f_2, \cdots, f_k\} \subseteq X$ and a positive integer n, we have $F \not\subseteq \bigcup_{i=1}^{k} D_q^n(f_i,\delta)$. We have two cases on the structure of F.

Case 1: If $f \in F$, then there exists a positive integer n such that for all $j \in \mathbb{N}$

$$F \cap D_q^n(f,\delta) = F \cap f_{\asymp_\delta}.$$

Let f_1 be an arbitrary point of *F*. We choose a positive integer n_1 such that

$$F \cap D_q^{n_1}(f_1,\delta) = F \cap f_{1\asymp_\delta}.$$

Since *F* is not *q*-Bourbaki-bounded, there exists $f_2 \in F$ such that $f_2 \notin D_q^{n_1}(f_1,\delta)$. It follows that $f_{1\asymp_\delta} \neq f_{2\asymp_\delta}$ by the choice of n_1.

One chooses another $n_2 \in \mathbb{Z}^+$ such that $n_2 > n_1$ and $F \cap D_q^{n_2}(f_2,\delta) = F \cap f_{2\asymp_\delta}$. Moreover, since $F \not\subseteq \bigcup_{j=1}^{2} D_q^{n_2}(f_j,\delta)$, we can find $f_3 \in F \setminus \left(f_{3\asymp_\delta} \cup f_{2\asymp_\delta}\right)$. Continuing this procedure by induction, we can find a sequence (f_j) with distinct terms in *F* such that for any $i \neq j$ we have $f_{i\asymp_\delta} \neq f_{j\asymp_\delta}$. Therefore, we define a function $:(X,q) \to (\mathbb{R},u)$ by

$$\varphi(x) = \begin{cases} j & \text{if } x \asymp_\delta f_j \text{ for some } j \\ 0 & \text{otherwise.} \end{cases}$$

It follows that the function φ is constant on $D_q(x,\delta)$ and it is unbounded on *F* since $\varphi(f_j) = j$. Therefore, the function φ is semi-Lipschitz in the small function.

Case 2: If there exists $f \in F$ and for all positive integer *n*, there exists $j \in \mathbb{N}$ such that

$$F \cap D_q^n(f,\delta) \subset F \cap D_q^{n+j}(f,\delta).$$

For $x \asymp_\delta f$, let $n(x)$ be the smallest positive integer *n* such that

$$x \in F \cap D_q^n(f,\delta). \tag{9}$$

We then define the function $\varphi:(X,q) \to (\mathbb{R},u)$ by

$$\varphi(x) = \begin{cases} \left(n(x)-1\right)\delta + \operatorname{dist}_q\left(x, D_q^{n(x)-1}(f,\delta)\right) & \text{if } x \neq f \text{ and } x \asymp_\delta f \\ 0 & \text{otherwise.} \end{cases}$$

By definition, the function φ is unbounded on *F*. We now have to show that if $x \neq y$ and $q(x,y) < \delta$, then for $k = 2$

$$u\left(\varphi(x),\varphi(y)\right) \le kq(x,y).$$

If either *x* or *y* is not related to *f* with respect to $\asymp_\delta$, then since $x \neq y$, both *x* and *y* are not related to *f* with respect to $\asymp_\delta$ and

$$u\left(\varphi(x),\varphi(y)\right) = 0 < 2q(x,y).$$

If $x \asymp_\delta f$ and $y \asymp_\delta f$, then we have some cases on $n(x)$ and $n(y)$:

If $n(x) > n(y)$. Suppose that $n(y) = 0$ then $y = f$ and $0 < q(x,y) < \delta$ which implies that $y \in D_q(x,\delta)$ hence $n(x) = 1$.

Furthermore,

$$\begin{aligned} u\left(\varphi(x),\varphi(y)\right) &= u\left[(1-1)\delta + \operatorname{dist}_q\left(x, D_q^0(f,\delta)\right), 0\right] \\ &= \operatorname{dist}_q\left(x,\{y\}\right) \\ &= q(x,y) < 2q(x,y). \end{aligned}$$

If $n(y) \ge 1$ and $n(x) = n(y)$, then

$$u(\varphi(x),\varphi(y)) = \max\left\{\left[\operatorname{dist}_q\left(x, D_q^{n(x)-1}(f,\delta)\right) - \operatorname{dist}_q\left(y, D_q^{n(x)-1}(f,\delta)\right)\right], 0\right\}$$
$$\leq q(x,y) < 2q(x,y).$$

If $n(y) \geq 1$ and $n(x) > n(y)$ (*i.e.*, $n(x) = n(y)+1$) with $\varphi(x) \leq \varphi(y)$, then there is nothing to prove since $u(\varphi(x),\varphi(y)) = 0 < 2q(x,y)$.

If $\varphi(x) > \varphi(y)$, then

$$\begin{aligned} u(\varphi(x),\varphi(y)) &= \varphi(x) - \varphi(y) \\ &= \left[(n(x)-1)\delta + \operatorname{dist}_q\left(x, D_q^{n(x)-1}(f,\delta)\right)\right] \\ &\quad - \left[(n(y)-1)\delta + \operatorname{dist}_q\left(y, D_q^{n(y)-1}(f,\delta)\right)\right] \\ &= (n(y)+1-1)\delta - (n(y)-1)\delta - \operatorname{dist}_q\left(x, D_q^{n(y)+1-1}(f,\delta)\right) \\ &\quad - \left[(n(y)-1)\delta - \operatorname{dist}_q\left(y, D_q^{n(y)-1}(f,\delta)\right)\right]. \end{aligned}$$

Furthermore,

$$\begin{aligned} & u(\varphi(x),\varphi(y)) \\ & = \delta + \operatorname{dist}_q\left(x, D_q^{n(y)}(f,\delta)\right) - \left[(n(y)-1)\delta - \operatorname{dist}_q\left(y, D_q^{n(y)-1}(f,\delta)\right)\right] \\ & \leq \delta + q(x,y) + \operatorname{dist}_q\left(y, D_q^{n(y)}(f,\delta)\right) - \operatorname{dist}_q\left(y, D_q^{n(y)-1}(f,\delta)\right). \end{aligned}$$

Since $n(w)$ is the smallest n such that $y \in F \cap D_q^n(f,\delta)$, it therefore means

$$\operatorname{dist}_q\left(y, D_q^{n(y)}(f,\delta)\right) = 0.$$

Thus, we have

$$u(\varphi(x),\varphi(y)) \leq \delta + q(x,y) - \operatorname{dist}_q\left(y, D_q^{n(y)-1}(f,\delta)\right). \tag{10}$$

We claim that,

$$\delta - q(x,y) \leq \operatorname{dist}_q\left(y, D_q^{n(y)-1}(f,\delta)\right). \tag{11}$$

Suppose otherwise, *i.e.*, $\operatorname{dist}_q\left(y, D_q^{n(y)-1}(f,\delta)\right) < \delta - q(x,y)$, then

$$\begin{aligned} \operatorname{dist}_q\left(x, D_q^{n(y)-1}(f,\delta)\right) &\leq q(x,y) + \operatorname{dist}_q\left(y, D_q^{n(y)-1}(f,\delta)\right) \\ &< q(x,y) + \delta - q(x,w) \\ &< \delta. \end{aligned}$$

So $x \in D_q^{n(y)-1}(f,\delta)$ which implies that $n(x) \leq n(y)-1+1$ but this is a contradiction since $n(x) > n(y)$.

Combining (10) and (11) we have

$$u(\varphi(x),\varphi(y)) \leq \delta + q(x,y) - \delta + q(x,y) \leq 2q(x,y).$$

Therefore, the proof is complete. □

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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