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Transformation Semigroup of Alternating Nonnegative Integers

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Abstract

Set of integers, Z_n is split into even-odd parts. The even part is arranged in $\begin{pmatrix} \frac{n}{2} \\ 2 \end{pmatrix}$ ways, while the odd part fixes one point at a time to compliment the even part thereby forming the semigroup, AZ_n. Thus, $\frac{n}{2}$-spaces are filled choosing maximum of two even points at a time. Green's relations have formed important structures that enhance the algebraic study of transformation semigroups. The semigroup of Alternating Nonnegative Integers for n-even ($AZ_{n\text{-even}}$) is shown to have only two D-classes, $|\mathcal{D}_k| = 2^{2^{\frac{n}{2}-1}}$ and there are $\frac{1}{k}|\mathcal{D}_k|\mathcal{H}$-classes for $n \geq 4$. The cardinality of L-classes is constant. Certain cardinalities and some other properties were derived. The coefficients of the zigzag triples obtained are 1, $\frac{n-3}{2}$ and $\frac{n-3}{2}\frac{n-1}{4}$. The second and third coefficients can be obtained by zigzag addition.

Keywords

Green's Relations, Partial Order Relation, Idempotents, Band, Generator

1. Introduction

Let X be a set of nonnegative integers of order n denoted by $Z_n = \{0,1,\cdots,n-1\}$ which is split into even nonnegative integers, $Z_{n\text{-even}}$ and odd positive integers, $Z_{n\text{-odd}}$ as $\{0,2,\cdots,2k\}$ and $\{1,3,\cdots,2k+1\}$ respectively. It is noted that $k = \left\{0,1,2,\cdots,\frac{n-2}{2}\right\}$ for both the even and odd parts of $Z_{n\text{-even}}, n \geq 2$ while

$k = \left\{0,1,2,\cdots,\frac{n-1}{2}\right\}$ for even part of $Z_{n\text{-odd}}$ and $k = \left\{0,1,2,\cdots,\frac{n-3}{2}\right\}$ for odd part of $Z_{n\text{-odd}}, n \geq 3$. The semigroup $AZ_n = AZ_{n\text{-even}} \cup AZ_{n\text{-odd}}$.

The domain of a map, α is denoted dom(α) and its image is im(α). The domain of the transformation remains Z_n while a map α is obtained for $i, j \in \text{dom}(\alpha)$ and $\alpha(i), \alpha(j) \in \text{im}(\alpha)$ such that $(i, \alpha(i)) \in \{0,2,\cdots,2k\}$, $(j, \alpha(j)) \in \{1,3,\cdots,2k+1\}$. In $AZ_{n\text{-even}}$, $(i, \alpha(i))$ assumes $\binom{\frac{n}{2}}{2}$ positions from $\{0,2,\cdots,2k\}$ and $(j, \alpha(j)) \in \{1,3,\cdots,2k+1\}$ occupies the remaining $\frac{n}{2}$ positions using only one odd element at a time. Also, in $AZ_{n\text{-odd}}$, $(i, \alpha(i))$ assumes $\binom{\frac{n+1}{2}}{2}$ positions from $\{0,2,\cdots,2k\}$ and $(j, \alpha(j)) \in \{1,3,\cdots,2k+1\}$ occupies the remaining $\frac{n-1}{2}$ positions using one odd point at a time.

Let $\alpha \in S$. An element $\alpha \in S$ is idempotent if $\alpha^2 = \alpha$. $E(S)$ denotes idempotent elements in the semigroup *S*. The element $x \in X$ satisfying $\alpha(x) = x$ is usually called a fixed point of α and is denoted by $f(\alpha)$. Given any transformation $\alpha \in AZ_n$, kernel of α is given by the relation $\ker \alpha = (x, y) \in Z_n \times Z_n : x\alpha = y\alpha$.

Many articles have been published on different kinds of transformation semigroups like full, partial, partial one to one, order-preserving, order-decreasing, identity-difference, signed, orientation preserving transformation semigroups, just to mention a few. Various topics covering both combinatorial results and algebraic properties ranging from Green's relations, regularity, idempotent depth, centralizers, generators, congruence, ideals, variants, idempotents, ranks etc. have been studied by authors like Adeniji [1], Higgins [2], Howie [3], Laradji & Umar [4], Mogbonju *et al.* [5] and Umar [6].

In 1952, Vagner [7] defined a natural partial order on inverse semigroups as $a \leq b$ if and only if $a = eb$ for some $e \in E(S)$.

Partial order on regular semigroup *S* is defined by Hartwig [8] and Nambooripad [9] as the relation $a \leq b$ if and only if $a = eb = bf$ for some $e, f \in E(S)$ and it extends the usual ordering of the set $E(S)$.

In 1986, Mitsch [10] defined the natural partial order on semigroups as any semigroup $(S,.)$ on which multiplication is defined by the relation $a \leq b$ if and only if $a = xb = by, xa = a$ for some $x, y \in S^1$.

Natural partial order on idempotents in this work is as defined by Higgins [11], that $e \leq f$ if and only if $ef = fe = e$ the result of which is seen in theorem (7).

An idempotent *e* is defined as a right identity [right zero] of *S* if $ae = a[ae = e]$ for all $a \in S$. A band is a semigroup *S* of which every element is an idempotent. Thus $S = E(S)$ if *S* is a band and so the natural partial ordering ($a \leq b$ if and only if $ab = ba = a$) applies to all *S*.

The cardinality $|\mathrm{im}(\alpha)|$ of the image of a transformation $\alpha \in AZ_n$ is called the rank of this transformation and is denoted by $\mathrm{rank}(\alpha)$. The number $\mathrm{def}(\alpha) = n - \mathrm{rank}(\alpha)$ is called the defect of the transformation α.

These and other standard definitions are found in Clifford and Preston [12], Higgins [11] and Howie [3].

The set of elements for each of $AZ_{n\text{-even}}$ and $AZ_{n\text{-odd}}$ in this study is generated using even-even points which are combinatorially arranged while one odd point at a time completes the combination of even points to form a nonempty set presented as:

$$\left\{ \mathrm{dom}(\alpha) = Z_n, (i, \alpha(i)) \subseteq \{0, 2, \cdots, 2k\} \mid \alpha(i) \in \begin{pmatrix} \frac{n}{2} \\ 2 \end{pmatrix} \right\}$$

and $(j, \alpha(j)) \in \{1, 3, \cdots, 2k+1\}$ for both $AZ_{n\text{-odd}}$ and $AZ_{n\text{-even}}$.

For example, some of the elements of AZ_6 are

$$\alpha_1 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\alpha_2 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & 1 & 2 & 1 \end{pmatrix}$$

$$\alpha_3 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 1 & 0 & 1 \end{pmatrix}$$

2. The Semigroup of $AZ_{n\text{-odd}}$

The semigroup formed from $Z_{n\text{-odd}}$ is denoted by $AZ_{n\text{-odd}}$.

An example of the semigroup with its elements is as follows:

Example 1. *Each of the following sets of elements is a semigroup of* AZ_5.

$$\left\{ \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 1 & 2 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 1 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 1 & 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 1 & 2 \end{pmatrix} \right\}$$

$$\left\{ \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 3 & 0 & 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 3 & 0 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 3 & 2 & 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 3 & 2 & 3 & 2 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 0 & 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 0 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 2 & 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 2 & 3 & 2 \end{pmatrix} \right\}$$

Theorem 1. *The Zigzag Triple Coefficients and cardinality of* $AZ_{n\text{-odd}}$

$$|AZ_{n\text{-odd}}| = \frac{n-1}{2}\left[2^{\frac{n+1}{2}} + \frac{n-3}{2}\left(2^{\frac{n+1}{2}} - 1\right) + \frac{n-3}{2}\frac{n-1}{4}\left(2^{\frac{n+1}{2}} - 2\right) \right], \forall n.$$

Proof. Each $Z_{n\text{-odd}}$ has $\frac{n-1}{2}$ odd numbers and $\begin{pmatrix} \frac{n+1}{2} \\ 2 \end{pmatrix}$ pairs of even numbers with each pair taken as image together with an odd number at a time to complete the number of elements forming a semigroup. The cardinality of $AZ_{n\text{-odd}}$ has three different significant terms. The coefficients of the terms are called "The zigzag triples". The first term is obtained from using the first pair of even consecutive numbers. Since repetition is not allowed, the second and third terms are derived from the first.

The coefficients are 1, $\frac{n-3}{2}$ and $\frac{n-3}{2}\frac{n-1}{4}$ respectively. Second and third coefficients can be obtained without the formula by performing simple zigzag addition. □

Remark 1. *Analogous proof goes for the zigzag triples of* $AZ_{n\text{-even}}$ *except that each* $Z_{n\text{-even}}$ *has* $\frac{n}{2}$ *odd numbers and* $\binom{\frac{n}{2}}{2}$ *set of pairs of even numbers. The coefficients of the three terms involved in the cardinality of* $AZ_{n\text{-even}}$ *are* 1, $\frac{n-4}{2}$ *and* $\frac{n-4}{2}\frac{n-2}{4}$ *respectively. Second and third coefficients can also be obtained without the formular by performing simple zigzag addition.*

3. The Cardinality of $AZ_{n\text{-even}}$ and Its Idempotents

The following theorem shows that $AZ_{n\text{-even}}$ is a semigroup.

Theorem 2. *Let* $X = Z_n$ *and* $S = AZ_{n\text{-even}}$ (or $AZ_{n\text{-odd}}$). *Then* S *is a semigroup of transformation.*

Proof. Let $Z_n = \{2k, 2k+1\}$, for $k = 0,1,2,\frac{n-2}{2}$ and $\alpha,\beta,\gamma \in S$ such that

$$\alpha = \begin{pmatrix} 0 & 1 & \cdots & n-1 \\ \alpha(0) & \alpha(1) & \cdots & \alpha(n-1) \end{pmatrix}$$

$$\beta = \begin{pmatrix} 0 & 1 & \cdots & n-1 \\ \beta(0) & \beta(1) & \cdots & \beta(n-1) \end{pmatrix}$$

$$\gamma = \begin{pmatrix} 0 & 1 & \cdots & n-1 \\ \gamma(0) & \gamma(1) & \cdots & \gamma(n-1) \end{pmatrix}$$

$$\begin{aligned} \alpha(\beta\gamma) &= \begin{pmatrix} 0 & 1 & \cdots & n-1 \\ \alpha(0) & \alpha(1) & \cdots & \alpha(n-1) \end{pmatrix}\left(\begin{pmatrix} 0 & 1 & \cdots & n-1 \\ \beta(0) & \beta(1) & \cdots & \beta(n-1) \end{pmatrix}\right. \\ &\quad \left.\begin{pmatrix} 0 & 1 & \cdots & n-1 \\ \gamma(0) & \gamma(1) & \cdots & \gamma(n-1) \end{pmatrix}\right) \\ &= \begin{pmatrix} 0 & 1 & \cdots & n-1 \\ \alpha(\beta\gamma(0)) & \alpha(\beta\gamma(1)) & \cdots & \alpha(\beta\gamma(n-1)) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & \cdots & n-1 \\ (\alpha\beta)\gamma(0) & (\alpha\beta)\gamma(1) & \cdots & (\alpha\beta)\gamma(n-1) \end{pmatrix} \\ &= (\alpha\beta)\gamma \end{aligned}$$

$\therefore$ S is a semigroup. □

Theorem 3. $\left|AZ_{n\text{-even}}\right| = 2^{\frac{n}{2}}$.

Proof. Since Z_n is split into even-odd parts, the even part is arranged in $\binom{\frac{n}{2}}{2}$ ways; while the odd part fixes one point at a time to compliment the even part in order to form a complete semigroup. Thus $\frac{n}{2}$ spaces are filled choosing maximum of two points at a time. □

Remark 2. *The superset of maps,* $CAZ_{n\text{-even}}$ (or $CAZ_{n\text{-odd}}$) *is a semigroup following theorem* 2, *which is a combination of odd points to complete the combinatorially arranged even points. It can be stated that supersemigroup* $CAZ_{n\text{-even}}$ *is obtained by combining semigroups* $AZ_{n\text{-even}}$ *formed from all the odd points and even points of* $Z_{n\text{-even}}$.

Example 2. *Each of the following sets of elements is a semigroup of* AZ_4.

$$\left\{\begin{pmatrix}0&1&2&3\\0&1&0&1\end{pmatrix},\begin{pmatrix}0&1&2&3\\0&1&2&1\end{pmatrix},\begin{pmatrix}0&1&2&3\\2&1&0&1\end{pmatrix},\begin{pmatrix}0&1&2&3\\2&1&2&1\end{pmatrix}\right\}$$

$$\left\{\begin{pmatrix}0&1&2&3\\0&3&0&3\end{pmatrix},\begin{pmatrix}0&1&2&3\\0&3&2&3\end{pmatrix},\begin{pmatrix}0&1&2&3\\2&3&0&3\end{pmatrix},\begin{pmatrix}0&1&2&3\\2&3&2&3\end{pmatrix}\right\}$$

Theorem 4. *The cardinality of the supersemigroup,* $CAZ_{n\text{-even}}$ *of the combined odd points to complete the combinatorially arranged even points is* $p\binom{p}{2}2^p$ for $n = 2p$ in $Z_n, p = 2,3,\cdots,\frac{n}{2}$ *for* $n \ge 4$ *while the closed form for all n-even is* $\frac{n}{2}\left[2^{\frac{n}{2}} + \frac{n-4}{2}\left(2^{\frac{n}{2}} - 1\right) + \frac{n-4}{2}\frac{n-2}{4}\left(2^{\frac{n}{2}} - 2\right)\right]$.

Proof. Let $CAZ_{n\text{-even}}$ denote the combined semigroups formed by using all the odd points together with even points of Z_n as described in the introduction, thereby forming a bigger semigroup. The even points is arranged as $\binom{\frac{n}{2}}{2}$ in $2^{\frac{n}{2}}$ ways. Repeated elements are deducted by default. Theorem 3 shows the cardinality for each odd point. Combining the cardinality of each semigroup AZ_n formed from each odd point occurs in $\frac{n}{2}$ ways. □

Remark 3. *Examples* 1 *and* 2 *clearly explain theorem* 4.

Theorem 5. *Let* $S = AZ_{n\text{-even}}$, *then* $\left|E(S)\right| = 2^{\frac{n}{2}-2} + 2, n \ge 4$.

Proof. $E(AZ_2) = 1$. Idempotents in $AZ_{n\text{-even}}, n \ge 4$ is known by the points that are fixed. There is a direct relationship between fixed points and idempotency. There are two right zero elements for each n and the elements having the equivalence $\left|\operatorname{Im}(\alpha)\right| = \left|F(\alpha)\right|$ is $\left|E(S)\right|$ for $n \ge 4$. □

Theorem 6. $E(AZ_{n\text{-even}})$ *is a band.*

Proof. Let S be the semigroup $AZ_{n\text{-even}}$. Following the proof of theorem 2, if

$a,b,c \in S$ and composition of map is defined on S, then $a(bc)=(ab)c \in E(AZ_n)$. Hence $E(S)$ is a semigroup. □

Theorem 7.

$$E(CAZ_{n\text{-even}})=\frac{n}{2}\left[\left(2^{\frac{n-4}{2}}+2\right)+\frac{n-4}{2}\left(2^{\frac{n-4}{2}}+1\right)+\frac{n-4}{2}\frac{n-2}{4}\left(2^{\frac{n-4}{2}}\right)\right],\forall n.$$

Proof. The idempotents of the combined semigroup over $AZ_{n\text{-even}}$ is obtained using theorem 5 and zigzag triple coefficients. □

Theorem 8. *Let* $S=AZ_{n\text{-even}}$, *where a and b are right zero elements of* $E(S)$, *then* $E(S)\setminus\{a,b\}$ *is a left zero semigroup.*

Proof. The semigroup $E(AZ_{n\text{-even}})$ has two right zero elements. Each of these elements is the right identity of all other elements except the second right zero element. That is: $ab=b$ and $ba=a$. Then for all $c \in E(AZ_n)\setminus\{b\}$, $ac=a$ and $bc=b \in E(AZ_n)\setminus\{a\}$. Hence $c_1c_2=c_1$ and $c_2c_1=c_2$ for $c_1,c_2 \in E(AZ_n)\setminus\{a,b\}$.

Thus, $E(S)\setminus\{a,b\}$ is a left zero semigroup. □

Theorem 9. *Commutativity of Noncommuting Idempotents.*

Proof. The two right zero elements, a and b as seen in theorem (8) are noncommuting. Taking $c \in E(S)$, $ac=ca=a$ $\forall c \in E(S)\setminus b$ and also $bc=cb=b$ $\forall c \in E(S)\setminus a$. Thus $a \le c$ and $b \le c$. □

4. Green's Relations

Green's relations are important equivalences in describing and decomposing semigroups. The relations have been studied on various semigroups and subsemigroups by many authors like Ganyushkin & Mazorchuk [13], Howie [3], Magill & Subbiah [14], Sun & Pei [15] and Zhao & Yang [16].

Ganyushkin & Mazorchuk [13] defined a left (resp. right or two-sided) ideal I of S as principal provided that there exists $x \in S$ such that $I=S^1x$ (resp. $I=xS^1$, $I=S^1xS^1$). The element x is called the generator of the ideal I with $x \in S^1x$, $x \in xS^1$ and $x \in S^1xS^1$.

Let S (or S^1 if an identity is adjoined) be a semigroup. Let the semigroup be $AZ_{n\text{-even}}$. Let $x,y \in S$, if $S=S^1$.

When x and y generate the same principal left ideal then xLy if $S^1x=S^1y$. Equivalently, xLy if and only if, $x=ay, y=bx$ for some $a,b \in S^1$.

$\mathcal{R}$-relation is defined as $x\mathcal{R}y$ if $xS^1=yS^1$ and equivalently, $x\mathcal{R}y$ if and only if $x=ya, y=xb$ for some $a,b \in S^1$ implying that x and y generate the same principal right ideal.

Two-sided principal ideal generated by x and y is called $\mathcal{J}$-relation, that is, $x\mathcal{J}y$ if $S^1xS^1=S^1yS^1$. Equivalently, $x\mathcal{J}y$ if and only if $x=ayb, y=cxd$ for some $a,b,c,d \in S^1$.

Also $\mathcal{H}=\mathcal{L}\cap\mathcal{R}$ and $\mathcal{D}=\mathcal{L}\circ\mathcal{R}$.

The notations $\mathcal{L}_x$ $(\mathcal{R}_x,\mathcal{H}_x,\mathcal{D}_x,\mathcal{J}_x)$ denote the set of all elements of AZ_n which are $\mathcal{L}$-related ($\mathcal{R}$-related, $\mathcal{H}$-related, $\mathcal{D}$-related, $\mathcal{L}$-related) to x, where $x \in AZ_n$.

These five equivalences are known as Green's relations, first introduced by Green [17] in 1951.

The following are some results obtained with respect to Green's relations.

Theorem 10. $\mathcal{L}_x$ *has a generator.*

Proof. Since x is in $\mathcal{L}_x$, $x^n \in \mathcal{L}_x$, where $n \in Z^+$. Hence the proof. □

Remark 4: *It should be noted that theorem* 8 *also holds for* $\mathcal{R}_x, \mathcal{H}_x, \mathcal{D}_x$ *and* $\mathcal{J}_x$.

Theorem 11. *The semigroup* $AZ_{n\text{-even}}$ *has exactly* $2^{\frac{n}{2}-1}$ *different principal right ideals* (*p.r.i*).

Proof. Let $S = AZ_{n\text{-even}}$ and $\alpha, \beta \in S$, from the definition of kernel of α that $x\alpha = y\alpha$, then $S\alpha = S\beta$. The fact that only two elements can be R-related implies there are $\frac{1}{2}\left|AZ_{n\text{-even}}\right|$ p.r.i. □

There are three $\mathcal{L}$-classes in the semigroup herein discussed, two (which are $\mathcal{R}$-related) of which generate only one element (right zero) each. The third class is such that for each $x \in \mathcal{L}_x$, $Sx = S$. The following theorem explains $\mathcal{L}$-classes.

Theorem 12. *If a is a right zero element, then* $\forall x \in S\left|\mathcal{L}_{x\backslash a}\right| = \frac{1}{2}\left|AZ_{n\text{-even}}\right|, n > 2$.

Proof. The $\mathcal{L}$-related class of a right zero element *a* generate a. That is, $x \in \mathcal{L}_a$ implies that $Sx = a$ and $x^n = a, n \in Z^+$. But for $x \backslash a \in S, S(x \backslash a)S$. Thus the semigroup in terms of $\mathcal{L}$-relation is split into right zero and non-right zero elements. $\left|\mathcal{L}_a\right| = \frac{1}{4}\left|AZ_n\right| = 2^{\frac{n}{2}-2}$. □

Proposition 1. *Let* $S = AZ_{n\text{-even}}$, *then* $\left|\mathcal{D}_k\right| = 2^{\frac{n}{2}-1}$ *where* $k = \{1,2\}$.

Proof. There are $2^{\frac{n}{2}-1}$ different $\mathcal{R}$-related classes of elements as in Theorem 11 where each class have two elements with equal kernel. Considering natural ordering of maps, the first half of $\mathcal{R}$-related classes is a $\mathcal{D}$-class, so also is the second half. This implies that there are only two $\mathcal{D}$-classes in AZ_n. □

Lemma 1. *There exists a* $\mathcal{D}$*-class of equal cardinality with a* $\mathcal{L}$*-class.*

Theorem 13. *Each* $\mathcal{D}$*-class in* $\mathcal{D}_k$ *has* $\frac{1}{k}\left(2^{\frac{n}{2}-1}\right)H$*-different classes in S. Let* $H \in \mathcal{H}$ *of a particular* $\mathcal{D}$*-class as described in Lemma* 1, *then the following are equivalent:*

(i) H is a group.

(ii) H contains an idempotent.

(iii) There exist $x, y \in H$ such that $xy = yx \in H$.

Proof. There are only two $\mathcal{D}$-classes as shown in Proposition 1. Hence the first part of the proof of this theorem depends on the proposition. This implies that the semigroup has nonempty $\mathcal{D}$-classes. Each $\mathcal{D}$-class is clearly distinguished in relation to both $\mathcal{L}$ and $\mathcal{R}$ classes as seen in **Table 1** and **Table 2**. One of the classes, denoted by $\mathcal{D}_2$ has $\frac{1}{2}\left(2^{\frac{n}{2}-1}\right)\mathcal{H}$-different classes.

Table 1. $\mathcal{D}_1$ -class.

	$\mathcal{L}_1$	$\mathcal{L}_2$
R_1	$\begin{pmatrix} 0&1&2&3&4&5&6&7\\ 0&1&0&1&0&1&0&1 \end{pmatrix}$	$\begin{pmatrix} 0&1&2&3&4&5&6&7\\ 2&1&2&1&2&1&2&1 \end{pmatrix}$
R_2	$\begin{pmatrix} 0&1&2&3&4&5&6&7\\ 0&1&0&1&0&1&2&1 \end{pmatrix}$	$\begin{pmatrix} 0&1&2&3&4&5&6&7\\ 2&1&2&1&2&1&0&1 \end{pmatrix}$
R_3	$\begin{pmatrix} 0&1&2&3&4&5&6&7\\ 0&1&0&1&2&1&0&1 \end{pmatrix}$	$\begin{pmatrix} 0&1&2&3&4&5&6&7\\ 2&1&2&1&0&1&2&1 \end{pmatrix}$
R_4	$\begin{pmatrix} 0&1&2&3&4&5&6&7\\ 0&1&0&1&2&1&2&1 \end{pmatrix}$	$\begin{pmatrix} 0&1&2&3&4&5&6&7\\ 2&1&2&1&0&1&0&1 \end{pmatrix}$

Table 2. $\mathcal{D}_2$ -class $\equiv$ $\mathcal{L}_3$ -class.

R_1	$\begin{pmatrix} 0&1&2&3&4&5&6&7\\ 0&1&2&1&0&1&0&1 \end{pmatrix}$	$\begin{pmatrix} 0&1&2&3&4&5&6&7\\ 2&1&0&1&2&1&2&1 \end{pmatrix}$
$\mathcal{R}_2$	$\begin{pmatrix} 0&1&2&3&4&5&6&7\\ 0&1&2&1&0&1&2&1 \end{pmatrix}$	$\begin{pmatrix} 0&1&2&3&4&5&6&7\\ 2&1&0&1&2&1&0&1 \end{pmatrix}$
$\mathcal{R}_3$	$\begin{pmatrix} 0&1&2&3&4&5&6&7\\ 0&1&2&1&2&1&0&1 \end{pmatrix}$	$\begin{pmatrix} 0&1&2&3&4&5&6&7\\ 2&1&0&1&0&1&2&1 \end{pmatrix}$
$\mathcal{R}_4$	$\begin{pmatrix} 0&1&2&3&4&5&6&7\\ 0&1&2&1&2&1&2&1 \end{pmatrix}$	$\begin{pmatrix} 0&1&2&3&4&5&6&7\\ 2&1&0&1&0&1&0&1 \end{pmatrix}$

(i) $\Rightarrow$ (ii)

Let *x*, *y* be elements in each $\mathcal{H}$ -class and there exist $e \in \mathcal{H}$ such that $x \cdot y = e \Rightarrow x = y^{-1}$ and vice-versa. Therefore $e \cdot e = e$.

(ii) $\Rightarrow$ (iii)

For $x, y \in H$, $x \cdot y = y \cdot x \in H$ as evident in (i).

Hence H is a commutative group showing that (iii) $\Rightarrow$ (i). □

Remark 5. *The relationship between* $\mathcal{L}, \mathcal{R}$ *and* $\mathcal{D}$ *-classes is embedded in* $\mathcal{R}$ *-classes. Splitting the* $\mathcal{R}$ *-classes into two in the natural ordering of maps makes the later half a* $\mathcal{D}$ *-class and at the same time an* $\mathcal{L}$ *-class. The former half is a* $\mathcal{D}$ *-class which parts into two* $\mathcal{L}$ *-classes.*

An example is seen below.

Example 3. $\mathcal{L}$, $\mathcal{R}$, $\mathcal{D}$ *and* $\mathcal{H}$ *-Classes of* AZ_8.

Remark 6. *All* $\mathcal{H}$ *-classes in the same* $\mathcal{D}$ *-class have the same cardinality* (*Lemma* 1).

5. Conclusions

AZ_n has been shown to be a semigroup interacting between elements of the set of nonnegative integers *Z*. Some of the results obtained are on Green's relations, principal right ideals, band and certain cardinalities.

Further studies could be carried out verifying other algebraic properties like variants and centralizers on $AZ_{n\text{-even}}$ or $AZ_{n\text{-odd}}$ and $CAZ_{n\text{-even}}$ or $CAZ_{n\text{-odd}}$.

The results obtained on Green's relations in this work can be generalised.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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